Cup and Cap Products and Symmetric Signatures in Intersection (Co)homology

Greg Friedman

Texas Christian University
Fort Worth, TX

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Contents

Poincaré Duality

Let M be an n-dimensional connected oriented closed manifold

Then M has a fundamental class $\Gamma \in H_n(M) \cong \mathbb{Z}$

Theorem (Poincaré Duality)

$$H^i(M) \cong H_{n-i}(M)$$

$$\alpha \to \alpha \cap \Gamma$$

A reformulation

If we use coefficients in $\mathbb{Q}$ (or any field),

Theorem (Universal Coefficient Theorem)

$$H^i(M;\mathbb{Q}) \cong Hom(H_i(M;\mathbb{Q}),\mathbb{Q})$$

So Poincaré duality

$$H^i(M;\mathbb{Q}) \cong H_{n-i}(M;\mathbb{Q})$$

becomes

$$\mathrm{Hom}(H_i(M;\mathbb{Q}),\mathbb{Q}) \cong H_{n-i}(M;\mathbb{Q})$$

Corollary

There is a nonsingular pairing

$$H_i(M;\mathbb{Q}) \otimes H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}$$

More familiar versions of this pairing

Nonsingular Poincaré "intersection pairing"

$$H_i(M;\mathbb{Q}) \otimes H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}$$

More common (equivalent) versions:

$$H^i(M;\mathbb{Q}) \otimes H^{n-i}(M;\mathbb{Q}) \to \mathbb{Q}$$

$$\alpha \otimes \beta \to (\alpha \cup \beta)(\Gamma)$$

If M is smooth, we also have:

$$H^i_{DR}(M;\mathbb{R}) \otimes H^{n-i}_{DR}(M;\mathbb{R}) \to \mathbb{R}$$

$$\eta \otimes \omega \to \int_M \eta \wedge \omega$$

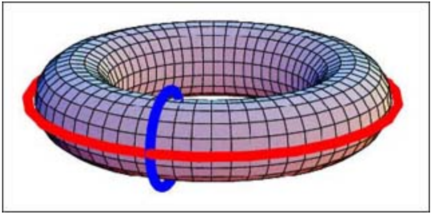
Intersection pairing

The homology version of the pairing

$$H_i(M;\mathbb{Q}) \otimes H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}$$

has a nice geometric interpretation as an intersection pairing

$$x \otimes y \to x \pitchfork y \in \mathbb{Q}$$



The signature of a manifold

If $\dim(M) = 4k$,

$$\pitchfork\colon H_{2k}(M;\mathbb{Q}) \otimes H_{2k}(M;\mathbb{Q}) \to \mathbb{Q}$$

is a symmetric pairing with a symmetric matrix (so real eigenvalues)

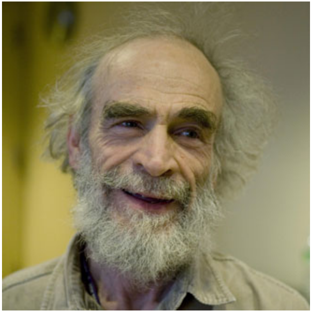
Definition (Signature of M^{4k})

$$\begin{aligned}\sigma(M) &= \text{signature}(\pitchfork)\\ &= \#\{\text{eigenvalues} > 0\} - \#\{\text{eigenvalues} < 0\}\end{aligned}$$

The signature is a bordism invariant and related to L-classes, surgery theory, Novikov conjecture, Hodge theory, index theory of differential operators (Atiyah-Singer),...

Signatures

The signature "is not just 'an invariant' but the invariant which can be matched in beauty and power only by the Euler characteristic."
– Mikhail Gromov



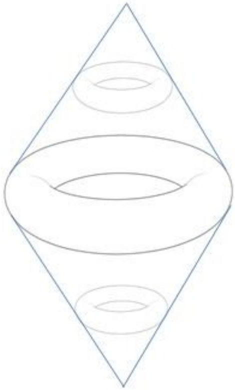
Singular spaces

Question: How much of this can we do for spaces that aren't manifolds (but are still "nice" in some way)?

especially singular algebraic varieties (irreducible)?

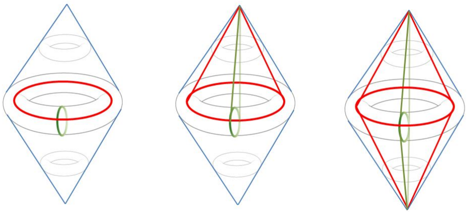
or quotient spaces of manifolds under "nice" group actions?

An example



$X^3 = ST^2$ is a manifold except at two points

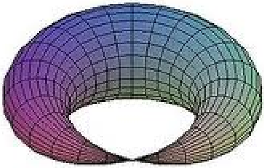
Example continued



$$H_3(X;\mathbb{Q}) \cong \mathbb{Q}$$
$$H_2(X;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$$
$$H_1(X;\mathbb{Q}) \cong 0$$
$$H_0(X;\mathbb{Q}) \cong \mathbb{Q}$$

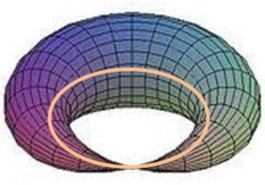
$H_1(X;\mathbb{Q}) \ncong H_2(X;\mathbb{Q})$ so there can be no Poincaré duality

Another example



X^2 is a manifold except at one point

Another example



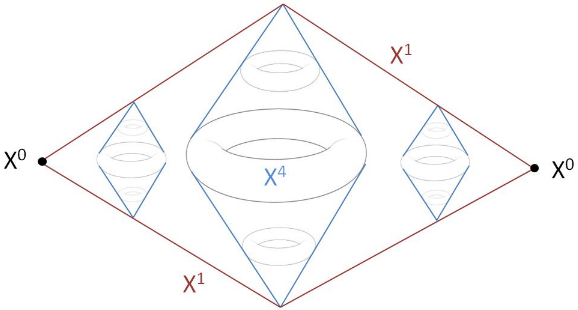
X^2 is a manifold except at one point

$$H_1(X;\mathbb{Q}) \cong \mathbb{Q}$$

But how to define $\pitchfork$??

TRANSVERSALITY PROBLEMS!!

Manifold Stratified Spaces



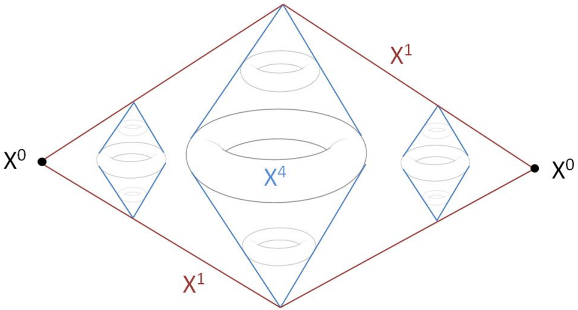
Manifold Stratified Spaces

Definition (Manifold stratified space)

- $X = X^n \supset X^{n-1} \supset X^{n-2} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$
- $X_k = X^k - X^{k-1}$ is a k-manifold (or empty); each component of X_k is a stratum of X
- $X - X^{n-1}$ is dense in X
- local normality conditions

- pseudomanifolds: cone bundle neighborhoods $\mathbb{R}^{n-k} \times cL^{k-1}$
- Quinn's homotopically stratified spaces: local homotopy conditions

A pseudomanifold



Examples

Examples of Stratified Spaces

- Irreducible algebraic and analytic varieties (with Whitney stratifications)
- PL and topological pseudomanifolds
- Orbit spaces of "nice" group actions on manifolds
- Manifolds, either unstratified or stratified by subsets
  - Submanifolds
  - Knots
  - Hypersurfaces

Pseudomanifolds

For the rest of today, we'll stick with pseudomanifolds (which often include the other examples, e.g. varieties)

In particular, every point x in every stratum of codimension k has a neighborhood

$$\mathbb{R}^{n-k} \times cL^{k-1}$$

L is called the link of the stratum

Intersection homology

Need to adapt homology to better suit this framework:

INTERSECTION HOMOLOGY

Due to Mark Goresky and Robert MacPherson:

Towards intersection homology [Goresky-MacPherson]

A perversity is a function

$$\bar{p} : \{\text{singular strata of } X\} \to \mathbb{Z}$$

Idea: assign numbers to strata

These numbers will determine the allowable degree of failure of transversality of intersections of chains with strata

Note: Goresky-MacPherson had other requirements on perversities. These can be avoided, but the definition of $I^{\bar{p}}C_*(X;\mathbb{Q})$ becomes a bit more complicated. We won't get into this here.

Intersection Homology [Goresky-MacPherson]

Intersection chain complex

$$I^{\bar{p}}C_*(X;\mathbb{Q}) \subset C_*(X;\mathbb{Q}),$$

where $C_*(X;\mathbb{Q})$ can be simplicial or singular chain complex

$\xi \in I^{\bar{p}}C_i(X;\mathbb{Q})$ if for each stratum Z,

1. $\dim|\xi \cap Z| \leq i - \operatorname{codim}(Z) + \bar{p}(Z)$
2. $\dim|\partial\xi \cap Z| \leq i - 1 - \operatorname{codim}(Z) + \bar{p}(Z)$

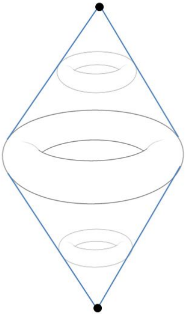
IDEA:

1. Condition 1 is about transversality: $\dim|\xi \cap Z| = i - \operatorname{codim}(Z)$ would be exactly the condition that ξ and Z are in general position
2. Condition 2 just makes $I^{\bar{p}}C_*(X;\mathbb{Q})$ a chain complex

Definition

$I^{\bar{p}}H_*(X;\mathbb{Q}) := H_*(I^{\bar{p}}C_*(X;\mathbb{Q}))$

Revisit suspended torus



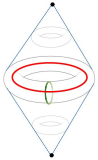
Stratify X as $X^0 = \{N, S\} \subset X$

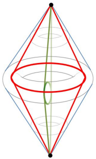
We'll consider two perversities

1. $\bar{0}(N) = \bar{0}(S) = 0$
2. $\bar{1}(N) = \bar{1}(S) = 1$

IH of the suspended torus

Rough idea for computing IH:
$\bar{0}$ isn't very permissive, so low dimension chains avoid X^0
$\bar{1}$ is more permissive, so low dimension chains can touch X^0





$I^{\bar{0}}H_3(X;\mathbb{Q}) \cong \mathbb{Q}$ $\qquad$ $I^{\bar{1}}H_3(X;\mathbb{Q}) \cong \mathbb{Q}$

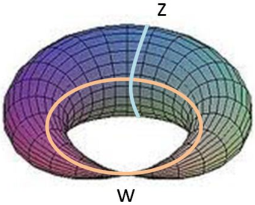
$I^{\bar{0}}H_2(X;\mathbb{Q}) \cong 0$ $\qquad$ $I^{\bar{1}}H_2(X;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$

$I^{\bar{0}}H_1(X;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ $\qquad$ $I^{\bar{1}}H_1(X;\mathbb{Q}) \cong 0$

$I^{\bar{0}}H_0(X;\mathbb{Q}) \cong \mathbb{Q}$ $\qquad$ $I^{\bar{1}}H_0(X;\mathbb{Q}) \cong \mathbb{Q}$

THESE LOOK DUAL!

IH of pinched torus



$$I^{\overline{-1}}H_1(X;\mathbb{Q}) \cong \mathbb{Q} \cong \langle z \rangle$$

$$I^{\bar{1}}H_1(X;\mathbb{Q}) \cong \mathbb{Q} \cong \langle w \rangle$$

$$w \pitchfork z = 1$$

Intersection homology duality

The previous examples look artificial. But:

Theorem (Goresky-MacPherson-Poincaré Duality)

Suppose

- X^n *is a connected oriented closed pseudomanifold*
- $\bar{p}(Z) + \bar{q}(Z) = codim(Z) - 2$ *for all singular strata* Z

Then there is a nondegenerate intersection pairing

$$\pitchfork: I^{\bar{p}}H_i(X;\mathbb{Q}) \otimes I^{\bar{q}}H_{n-i}(X;\mathbb{Q}) \to \mathbb{Q}$$

Note: theorem pairs complementary dimensions and complementary perversities

Goresky-MacPherson-Poincaré duality

Proofs:

- Simplicial pseudomanifolds: using combinatorics of simplicial complexes and stratified general position of Clint McCrory [Goresky-MacPherson - 1980]

Clint McCrory

- Topological pseudomanifolds: using sheaf theory [Goresky-MacPherson - 1983]
- Quinn's homotopically stratified spaces: using sheaf theory and singular chains [F. - 2009]
- Topological pseudomanifolds and general perversities: using sheaf theory [F. - 2010]
- Topological pseudomanifolds and general perversities: cup and cap products [F.-McClure - preprint]

Topological invariance

For many perversities (those satisfying the Goresky-MacPherson conditions), $I^{\bar{p}}H_*(X)$ depends only on X, not on the stratification.

Then $I^{\bar{p}}H_*$ is a topological invariant

In particular if M is a manifold

$$I^{\bar{p}}H_*(M;\mathbb{Q}) = H_*(M;\mathbb{Q}),$$

independent of choice of stratification.

(Though interesting things can still be done with local coefficient systems off the singular set.)

Signatures

What about signatures?

Even though we have pairings

$$\pitchfork : I^{\bar p}H_{2k}(X^{4k};\mathbb{Q}) \otimes I^{\bar q}H_{2k}(X^{4k};\mathbb{Q}) \to \mathbb{Q},$$

we can't define a signature because these are not the same group, so we don't have a symmetric self-pairing

In general $I^{\bar p}H_{2k}(X;\mathbb{Q}) \not\cong I^{\bar q}H_{2k}(X;\mathbb{Q})$

Middle perversities

There are two complementary Goresky-MacPherson perversities that are as close to each other as possible:

Definition

$$\bar{m}(Z) = \left\lfloor \frac{\operatorname{codim}(Z) - 2}{2} \right\rfloor \qquad \bar{n}(Z) = \left\lfloor \frac{\operatorname{codim}(Z) - 1}{2} \right\rfloor$$

These are the lower and upper middle perversities

$\bar{m}$ and $\bar{n}$ differ only on strata of odd codimension

So if X has only even codimension strata (e.g. complex algebraic varieties),

$$I^{\bar{m}}H_*(X;\mathbb{Q}) = I^{\bar{n}}H_*(X;\mathbb{Q})$$

Witt spaces

Witt spaces, due to Paul Siegel

Definition

X is Witt if it is an oriented simplicial pseudomanifold such that for each $x \in X_{n-(2k+1)}$,

$$I^{\bar{m}}H_k(L;\mathbb{Q}) = 0,$$

where L is the link of x.

Lemma

If X is Witt, $I^{\bar{m}}H_*(X;\mathbb{Q}) \cong I^{\bar{n}}H_*(X;\mathbb{Q})$.

Witt signature

If X^{4k} is Witt, there is a nondegenerate symmetric pairing

$$\pitchfork\colon I^{\bar{m}}H_{2k}(X;\mathbb{Q}) \otimes I^{\bar{m}}H_{2k}(X;\mathbb{Q}) \to \mathbb{Q}$$

The signature of this pairing is called the Witt signature,

$$\sigma^{\text{Witt}}(X)$$

$\sigma^{\text{Witt}}(X)$ is an invariant of Witt bordism:

Theorem

If Y^{4k+1} is Witt and $\partial Y = X \amalg -X'$, then

$$\sigma^{Witt}(X) = \sigma^{Witt}(X')$$

Witt bordism

Let Ω_n^{Witt} be set of the equivalence classes of n-dimensional Witt spaces under the bordism relation. The operation of disjoint union makes this a group. A Witt space represents 0 if it is a boundary.

Theorem (Siegel)

- *If* $n \neq 4k$, $\Omega_n^{Witt} = 0$
- $\Omega_0^{Witt} = \mathbb{Z}$
- *If* $n = 4k > 0$, $\Omega_{4k}^{Witt} = W(\mathbb{Q})$, *the Witt group of rational nondegenerate symmetric pairings*

Witt bordism II

Using work of Dennis Sullivan, Siegel showed the Witt bordism computation implies:

Theorem

For any Y,

$$\Omega_*^{Witt}(Y) \otimes \mathbb{Z}[1/2] \cong ko_*(Y) \otimes \mathbb{Z}[1/2]$$

$\mathbb{Z}_p$-Witt [F.]

Let $\Omega_n^{\mathbb{Z}_p\text{-Witt}}$ be the equivalence class of n-dimensional $\mathbb{Z}_p$-Witt spaces under the bordism relation, p prime, $p \neq 2$.

Theorem (F.)

- *If* $n \neq 4k$, $\Omega_n^{\mathbb{Z}_p\text{-}Witt} = 0$
- $\Omega_0^{\mathbb{Z}_p\text{-}Witt} = \mathbb{Z}$
- *If* $n = 4k > 0$, $\Omega_{4k}^{\mathbb{Z}_p\text{-}Witt} = W(\mathbb{Z}_p)$

Corollary (F.)

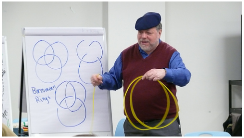
For any Y,

$$\Omega_n^{\mathbb{Z}_p\text{-}Witt}(Y) \cong \bigoplus_{r+s=n} H_r\left(Y; \Omega_s^{\mathbb{Z}_p-Witt}\right)$$

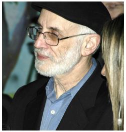
Signatures

Modern work on applications of signature invariants of Witt (and non-Witt!) spaces continues:

Some of my mathematical family working in these areas:

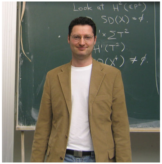


Sylvain Cappell



Julius Shaneson

Shmuel Weinberger



Markus Banagl

Laurentiu Maxim

Eugénie Hunsicker

Applications of intersection homology

Application of IH:

The generalization of the Kähler package from nonsingular complex varieties to singular complex varieties

The Kähler Package:

- Poincaré duality
- Lefschetz hyperplane theorem
- Hard Lefschetz theorem
- Hodge decomposition/Hodge signature theorem

Erich Kähler

Solomon Lefschetz

W.V.D. Hodge

Other applications/results/related work

- Hard Lefschetz/Hodge Decomposition/Hodge Signature Theorem [Saito]
- L^2 cohomology [Cheeger]
- Stratified Morse theory [Goresky-MacPherson]
- Perverse sheaves [Beilinson-Bernstein-Deligne-Gabber]
- Beilinson-Bernstein-Deligne-Gabber Decomposition Theorem
- the Weil conjecture for singular varieties
- Mixed Hodge modules
- The Kazhdan-Lusztig Conjecture, concerning representations of Weil groups [Beilinson-Bernstein, Brylinski-Kashiwara]
- D-modules and the Riemann-Hilbert correspondence

Applications

Some of the most famous applications of intersection homology are to

- Algebraic geometry
- Algebraic or analytic complex geometry
- Representation theory
- Analysis
- Combinatorics
- Number theory

Proofs are mostly via sheaf theory

Much work continues in all of these areas

Topology

There has been comparatively little work done on applications of intersection homology within algebraic topology and/or using singular chain techniques

F.-McClure

Recent work of F.-Jim McClure

F.-McClure

Recent work of F.-McClure

- Define intersection cochains $I_{\bar{p}}C^*(X;\mathbb{Q})$ and intersection cohomology $I_{\bar{p}}H^*(X;\mathbb{Q})$
- There are cup and cap products

$$\cup : I_{\bar{p}}H^i(X;\mathbb{Q}) \otimes I_{\bar{q}}H^j(X;\mathbb{Q}) \to I_{\bar{r}}H^{i+j}(X;\mathbb{Q})$$
$$\cap : I_{\bar{p}}H^i(X;\mathbb{Q}) \otimes I^{\bar{r}}H_{i+j}(X;\mathbb{Q}) \to I^{\bar{q}}H_j(X;\mathbb{Q})$$

for appropriate $\bar{p}, \bar{q}, \bar{r}$
The front-face/back-face construction of cup/cap products doesn't work in this context!

IH Künneth Theorem

Main tool in constructing cup/cap products

Theorem

Künneth theorem [F.] There is a perversity $Q_{\bar{p},\bar{q}}$ on $X \times Y$ such that

$$I^{Q_{\bar{p},\bar{q}}}H_*(X \times Y;\mathbb{Q}) \cong I^{\bar{p}}H_*(X;\mathbb{Q}) \otimes I^{\bar{q}}H_*(Y;\mathbb{Q})$$

Generalizes earlier theorem of Cohen-Goresky-Ji

IH Cup Product

Then define the IH cup product using

$$\cup : I_{\bar{p}}H^*(X;\mathbb{Q}) \otimes I_{\bar{q}}H^*(X;\mathbb{Q}) \xrightarrow[\text{Künneth}]{\cong} I_{Q_{\bar{p},\bar{q}}}H^*(X \times X;\mathbb{Q}) \xrightarrow{d^*} I_{\bar{r}}H^*(X;\mathbb{Q}),$$

where $d : X \to X \times X$ is the diagonal map $x \to (x, x)$

d is allowable (with respect to the perversities) if $\bar{p}, \bar{q}, \bar{r}$ satisfy

$$\bar{r}(Z) \leq \bar{p}(Z) + \bar{q}(Z) - \operatorname{codim}(Z) + 2$$

(i.e. $D\bar{r} \geq D\bar{p} + D\bar{q}$)

IH Cap Product

Cap product is define similarly.

Suppose $D\bar{r} \geq D\bar{p} + D\bar{q}$ and let

$$\bar{d} : I^{\bar{r}}H_*(X;\mathbb{Q}) \to I^{Q_{\bar{p},\bar{q}}}H_*(X \times X;\mathbb{Q}) \xrightarrow[\text{Künneth}]{\cong} I^{\bar{p}}H_*(X;\mathbb{Q}) \otimes I^{\bar{q}}H_*(X;\mathbb{Q})$$

Then

$$\cap : I_{\bar{q}}H^i(X;\mathbb{Q}) \otimes I^{\bar{r}}H_j(X;\mathbb{Q}) \to I^{\bar{p}}H_{j-i}(X;\mathbb{Q})$$

$$\alpha \cap x = (1 \otimes \alpha)\bar{d}(x)$$

Intersection Poincaré Duality

Lemma (F.-McClure)

If X^n is connected, closed, oriented, there is a fundamental class $\Gamma \in I^{\bar{0}}H_n(X;\mathbb{Q})$

Theorem (F.-McClure)

If $\bar{p}, \bar{q}$ are complementary perversities, cap product induces the Poincaré duality isomorphism:

$$\cap\Gamma : I_{\bar{p}}H^i(X;\mathbb{Q}) \to I^{\bar{q}}H_{n-i}(X;\mathbb{Q})$$

Further work shows compatibility between this duality and Goresky-MacPherson/sheaf theoretic dualities

An Application - Symmetric Signatures

A stratified homotopy invariant Mishchenko-Ranicki symmetric signature for Witt spaces

$$\sigma^*_{\mathrm{Witt}}(X) \in L^n(F[\pi_1(X)])$$

Alexander Mishchenko

Andrew Ranicki

This is a generalization of the signature invariant related to "universal" Poincaré duality relating the $F[\pi_1(X)]$-module homology and cohomology of the universal cover $\tilde{X}$:

$$\cap\Gamma : I_{\bar{p}}\tilde{H}^i(\tilde{X};\mathbb{Q}) \to I^{\bar{q}}H_{n-i}(\tilde{X};\mathbb{Q})$$

Symmetric Witt Signatures

Work on symmetric Witt signatures:

- Cappell-Shaneson-Weinberger (1991) - details not provided
- Banagl (2011) - based on work of Eppelmann on L-orientations of pseudomanifolds
- Albin-Leichtnam-Mazzeo-Piazza (preprint) - analytic construction
- F.-McClure (preprint) - topological singular (co)chain construction (based on a question of Piazza)

Symmetric L groups

Let R be a ring with involution.
$L^m(R)$ is the group of algebraic bordism classes of m-dimensional symmetric Poincaré complexes over R.

Elements of $L^m(R)$ are pairs (C, ϕ), where

- C is a homotopy finite R-module chain complex
- $\phi : W \to C^t \otimes_R C$ is a $\mathbb{Z}/2$-equivariant degree n chain map (W is the free $\mathbb{Z}[\mathbb{Z}/2]$ resolution of $\mathbb{Z}$)
- If $\iota \in H_0(W)$ is the generator, slant product with $\phi(\iota)$ induces an isomorphism

$$\backslash\phi(\iota) : H^*(\mathrm{Hom}_R(C, R)) \to H_{n-*}(C^t)$$

Example: Manifolds

Let M^m be a closed F-oriented m-dimensional manifold.

$$\sigma^*(M) = (C, \phi) \in L^m(F[\pi_1(X)])$$

- $C = C_*(\tilde{M}; F)$
-

$$\begin{aligned}\phi' : W \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\Gamma} C_*(M) &\cong F \otimes_{F[\pi]} C_*(\tilde{M}; F) \\ &\xrightarrow{1 \otimes d} F \otimes_{F[\pi]} C_*(\tilde{M} \times \tilde{M}; F)\end{aligned}$$

- $\phi = \Upsilon^{-1}(\phi')$,

$$\begin{aligned}\Upsilon : H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, C^t \otimes_{F[\pi]} C)) &\cong H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, F \otimes_{F[\pi]} (C \otimes_F C))) \\ &\cong H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, F \otimes_{F[\pi]} C_*(\tilde{M} \times \tilde{M})))\end{aligned}$$

Example: Manifolds (continued)

Roughly speaking:

$\phi(\iota)$ is the image of the fundamental class of M in $C_*(\tilde{M}) \otimes C_*(\tilde{M})$ (after lifting it to the cover and taking the diagonal image)

Then the slant product with $C^*(\tilde{M})$ is really just the universal Poincaré duality cap product.

Witt Spaces

Let X^m be a closed F-oriented m-dimensional Witt space

$$\sigma^*_{\text{Witt}}(X) \in L^m(F[\pi_1(X)])$$

- $C = I^{\bar n}C_*(\tilde X; F)$ is homotopy finite over $F[\pi]$
-

$$\phi' : W \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\zeta} F \otimes_{F[\pi]} I^{\bar 0}C_*(\tilde X; F) \xrightarrow{1 \otimes d} F \otimes_{F[\pi]} I^{Q_{\bar n, \bar n}}C_*(\tilde X \times \tilde X; F)$$

- $\phi = \Upsilon^{-1}(\phi')$,

$$\begin{aligned}\Upsilon : H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, C^t \otimes_{F[\pi]} C)) &\cong H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, F \otimes_{F[\pi]} (C \otimes_F C))) \\ &\cong H_*(\mathrm{Hom}_{\mathbb{Z}/2}(W, F \otimes_{F[\pi]} I^{Q_{\bar n, \bar n}}C_*(\tilde X \times \tilde X)))\end{aligned}$$

Witt Spaces (continued)

Roughly speaking:

$\phi(\iota)$ is the image of the fundamental class of X in $I^{\bar{n}}C_*(\tilde{X}) \otimes I^{\bar{n}}C_*(\tilde{M})$ (after lifting to the cover (which is trickier here!) and taking the diagonal image)

Then the slant product with $I_{\bar{n}}C^*(\tilde{X})$ is really just the universal intersection Poincaré duality cap product.

Properties

Properties of $\sigma^*_{\text{Witt}}(X)$

- If M is a manifold, $\sigma^*_{\text{Witt}}(M) = \sigma^*(M)$
- $L^{4k}(F[\pi]) \to L^{4k}(F) \to W(F)$ takes $\sigma^*_{\text{Witt}}(X)$ to the Witt class of the intersection pairing
- Additivity and multiplicativity
- PL homeomorphism and stratified homotopy invariance
- Bordism invariance
- For X smoothly stratified $\mathbb{Q}$-Witt: The image of $\sigma^*_{\text{Witt}}(X)$ in $K_*(C^*\pi_1(X)) \otimes \mathbb{Q}$ equals $\text{Ind}(\tilde{\eth}_{\text{sign}})_{\mathbb{Q}}$, the rational signature index class of Albin-Leichtnam-Mazzeo-Piazza

Future projects F.-McClure

- Algebraic structures of intersection cochain complex
- Intersection rational homotopy theory
- Intersection (co)homology operations
- Duality issues over non-fields
- Stratified homotopy theories
- Quillen model structures