Hsiang-Pati coordinates

Edward Bierstone

University of Toronto

Cortona, 20 June 2014

Collaborators

André Belotto

Vincent Grandjean

Pierre Milman

Franklin Vera Pacheco

Hsiang–Pati problem

Given a complex variety $X_0 \subset Z_0$ (Z_0 smooth, e.g., $P^N(\mathbb{C})$), can we find a resolution of singularities

$$\sigma : X \to X_0 \hookrightarrow Z_0$$

such that the pull-back cotangent sheaf is locally generated by

$$d(u^{\alpha_i}),\ i = 1, \ldots, s, \quad d(u^{\beta_j} v_j),\ j = 1, \ldots, n - s$$

where

$n = \dim X_0$

$(u, v) = (u_1, \ldots, u_s, v_1, \ldots, v_{n-s}))$ local coordinates on X,
$E = (u_1 \cdots u_s = 0)$ exceptional divisor

α_i linearly independent over $\mathbb{Q}$

$\{\alpha_i, \beta_j\}$ totally ordered ?

Consequence

Pull-back to X of induced Fubini-Study metric on $X_0 \backslash \mathrm{Sing}\, X_0$ locally quasi-isometric to

$$\sum_{i=1}^{s} d(u^{\alpha_i}) \otimes \overline{d(u^{\alpha_i})} + \sum_{j=1}^{n-s} d(u^{\beta_j} v_j) \otimes \overline{d(u^{\beta_j} v_j)}$$

Proved in case X_0 surface with isolated singularities by
Wu-Chung Hsiang, Vishwambkar Pati (1985)
William Pardon, Mark Stern (2001)

Formulation of HP problem due to Boris Youssin (1998)

(u, v) called Hsiang-Pati coordinates

Interest ?

Applications to L_2-cohomology (following Cheeger)

Hsiang-Pati: Intersection cohomology (with middle perversity) of a surface $X_0 \; = \; L_2$-cohomology of $X_0 \backslash \mathrm{Sing}\, X_0$

(cf. Cheeger-Goresky-Macpherson conjecture)

Melrose: Extension of b-calculus to singular varieties

Local HP problem. Is there a semiproper locally finite covering

$$\{\sigma_j : X_j \to X_0\}$$

of X_0 such that each σ_j is a finite composite of local blowings-up satisfying HP ?

Theorem

HP holds for X_0 of dimension ≤ 3 (at least locally)

Exercises

(1)
$$\begin{aligned} y_1 &= x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}}\, \delta_1 \\ &\vdots \\ y_n &= x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}}\, \delta_n \end{aligned}$$

(δ_i units). We can absorb units, i.e., $y_i = \overline{x}^{\alpha_i}$ after coordinate change $\overline{x}_j = \delta^{\epsilon_j} x_j$, where $\epsilon_j \in \mathbb{Q}^n$, provided that $\{\alpha_i\}$ linearly independent

(2) HP coordinates induce HP coordinates at nearby points

Problem. Does HP $\Rightarrow$ toroidalization (monomialization) of morphisms? (cf. Cutkosky)

Regularization of the Gauss mapping

We can reduce HP problem to the case that $\sigma^*(\Omega^1_{M_0})$ is locally free of rank n (i.e., defines a vector bundle)

by regularization of the Gauss mapping

$$\begin{aligned} G_{X_0} : X_0 \backslash \mathrm{Sing}\, X_0 &\to \mathrm{Grass}(n, TM_0) \\ a &\mapsto T_a X_0 \subset T_a M_0 \end{aligned}$$

A Gauss-regular resolution of singularities of X_0 can be obtained by taking the Nash blow-up of X_0, followed by resolution of singularities.

The Nash blow-up is the closure in $X_0 \times \mathrm{Grass}(n, TM_0)$ of the graph of G_{X_0} .

Log Fitting ideals

Given (X, E), where X smooth, E exceptional divisor

$\Omega^1_X(\log E)$ denotes sheaf of log 1-forms on X

i.e., in local coordinates $(u, v) = (u_1, \ldots, u_s, v_1, \ldots, v_{n-s})$ such that $E = (u_1 \cdots u_s = 0)$, generated by

$$\frac{du_i}{u_i}, \quad dv_j$$

Given resolution $\sigma : (X, E) \to (M_0, X_0, \mathrm{Sing}\, X_0)$, consider

$$\sigma^*(\Omega^1_{M_0}) \xrightarrow{\Sigma} \Omega^1_X(\log E) \longrightarrow \mathrm{Coker}\,\Sigma \longrightarrow 0$$

If σ Gauss-regular, then Σ has a presentation given by

$$\log \operatorname{Jac} \sigma = \begin{pmatrix} u_1 \frac{\partial \sigma_1}{\partial u_1} & \cdots & u_s \frac{\partial \sigma_1}{\partial u_s} & \frac{\partial \sigma_1}{\partial v_1} & \cdots & \frac{\partial \sigma_1}{\partial v_{n-s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_1 \frac{\partial \sigma_n}{\partial u_1} & \cdots & u_s \frac{\partial \sigma_n}{\partial u_s} & \frac{\partial \sigma_n}{\partial v_1} & \cdots & \frac{\partial \sigma_n}{\partial v_{n-s}} \end{pmatrix}$$

Fitting ideal

$$F_{n-k} = F_{n-k}(\sigma) \subset \mathcal{O}_X$$

generated by $k \times k$ minors of $\log \operatorname{Jac} \sigma$
(independent of presentation of Σ)

Log rank

$$\begin{aligned} \log \operatorname{rk}_a \sigma &:= \operatorname{rk}_a \log \operatorname{Jac} \sigma \\ &= \operatorname{rk}_a \sigma|_{E(a)}, \quad \text{where } E(a) = \text{ stratum of } a \text{ in } E \end{aligned}$$

Let
$$\begin{aligned} p &:= \max \log \operatorname{rk} \sigma \quad \text{(at points of } E) \\ &= \dim \operatorname{Sing} X_0 \\ \Sigma_k &:= \{a \in E : \log \operatorname{rk}_a \sigma \leq p - k\} \\ Y_k &:= \sigma(\Sigma_k) \quad (\text{so } Y_0 = \operatorname{Sing} X_0) \end{aligned}$$

Clearly $\Sigma_p \subset \Sigma_{p-1} \subset \cdots \subset \Sigma_0$

Theorem

HP is equivalent to the following conditions:

(1) $\sigma^*\mathcal{I}_{Y_j}$ principal (generated by monomial in E), $j = 0, \dots, p$

(2) fitting ideal $F_{n-k}(\sigma)$ principal, $k = 1, \dots, n$

Neither condition behaves well with respect to blowing up:

(1) is not stable after an admissible blowing up β but,
given σ, we can principalize the $\sigma^*\mathcal{I}_{Y_j}$ by further blowings up

(2) $F_{n-k}(\sigma \circ \beta) \subset \beta^* F_{n-k}(\sigma)$,
though $F_0(\sigma \circ \beta) = \mathrm{exc}^\ell\, \beta^* F_0(\sigma)$

If $\log \mathrm{rk}_a\, \sigma = r$ and $\sigma^*\mathcal{I}_{Y_{p-r}}$ is principle, then we can assume

$$\sigma_1 = v_1,\ \dots,\ \sigma_r = v_r \quad \text{and} \quad \sigma_{r+1} = u^{\alpha_1}$$

Corollary. *HP in two-dimensional case*

Proof of (local) HP in three dimensions

In general (dimension n), we can begin with
$\sigma_1 = u_1^\alpha$ (e.g., at a point of log rank 0) and write

$$\sigma_j = g_j(u) + u^\delta T_j(u, v), \quad j \geq 2,$$

where u_1^α divides σ_j, and $g_j(u)$ comprises all monomials
u^γ of σ_j, with γ linearly dependent on α_1 over $\mathbb{Q}$

Say $T_j = \sum_{\epsilon \in \mathbb{N}^{n-s}} c_{j\epsilon}(u)\, v^\epsilon$

Let d denote smallest ϵ such that $c_{j\epsilon}$ is a unit, for some j
(maybe $d = \infty$)

Then d is a local invariant of the Fitting ideal F_{n-2}

We can reduce to the case d finite, by resolution of singularities of the ideal generated by the coefficients $c_{j\epsilon}$

At an n-point (i.e., $u = (u_1, \dots, u_n)$) this means we get

$$u^\delta T_j(u) = u^{\alpha_2} \cdot \text{unit}, \quad \text{for some } j, \text{ say } j = 2,$$

and can absorb the unit to put σ_2 in HP form

In three dimensions $(n = 3)$: we have

3-points:	coordinates	(u_1, u_2, u_3)
2-points:		(u_1, u_2, w)
1-points:		(u, v, w)

I. We reduce to the following normal forms:

at a 3-point: $u^\delta T_2 = u^{\alpha_2}$, α_2 independent of α_1 (HP)

at a 2-point: $T_j = a_{jd}\, w^d + \sum_{i=0}^{d-1} a_{ji}\, u^{\gamma_{ji}}\, w^i$

at a 1-point: $T_j = a_{jd} w^d + \sum_{i=0}^{d-1} a_{ji}\, u^{q_{ji}}\, v^{s_{ji}}\, w^i$

where

a_{jd} unit and $a_{j,d-1} \equiv 0 \quad (j = 2 \text{ or } 3)$,

a_{ji} unit or 0, $\quad i \le d-1$,

there is a term of order ≤ 1 in (v, w),

we can assume $a_{20} = 1$, and $s_{20} = 1$ (1-point case)

by HP in dimension 2,

all $s_{ji} = 0$ or 1

II. We principalize the ideal generated by

$$w^d, \quad u^{\gamma_{ji}}\, w^i \quad (\text{or } \; u^{q_{ji}}\, w^i)$$

to achieve order reduction over every point $\square$

Example

u^2
$u^3\,(w^3 + (v^2 + u\,x^2)\,w + u^2\,y)$
$u^6\,w$
$u^6\,v$
$u^6\,x$

Thank you for your attention!