Intersection Spaces, Perverse Sheaves and Type IIB String Theory

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Motivation: Poincaré Duality

- Convention:
  - *singular space* = n-dim. complex projective variety
  - *manifold* = n-dim. complex projective manifold
  - coefficients: $\mathbb{Q}$
- Manifolds have a nice hidden symmetry: *Poincaré Duality*.
- In particular: X manifold $\Longrightarrow b_k(X) = b_{2n-k}(X)$.
- Singular spaces do **not** posses such symmetry.

Intersection homology

- much of the manifold theory (e.g., Poincaré Duality, Kähler package) is recovered in the singular context if one uses instead the *intersection (co)homology groups* $IH_*(X)$ of Goresky-MacPherson.
- these are homology groups of a complex of allowed chains, defined by imposing restrictions on how chains meet the singularities.

Properties of intersection homology

- if $\widetilde{X}$ is a *resolution of singularities* of X, then:

$$IH_*(X) \subset H_*(\widetilde{X})$$

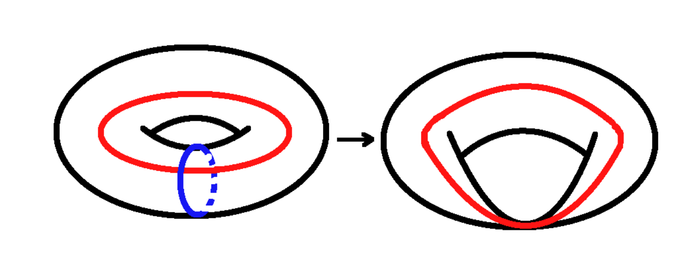
- if $\widetilde{X}$ is a *small resolution*: $IH_*(X) \cong H_*(\widetilde{X})$.
- $IH_*(X)$ carries the *Kähler package*: Hodge structure, weak and hard Lefschetz.
- IH_* is a topological invariant, but is *not* functorial, and has *no* ring structure in general.
- $IH^*(X)$ is realized by a perverse, self-dual, constructible sheaf complex $\mathcal{IC}_X$, i.e.

$$IH^i(X) \cong \mathbb{H}^i(X, \mathcal{IC}_X[-n]).$$

- IH_* (like H_*) is *unstable* under *smoothings of singularities*.

Example

Let $X_s := \{y^2 = x(x-1)(x-s)\} \subset \mathbb{P}^2$, with $|s| < 1$.
For $s \neq 0$, X_s is an elliptic curve (i.e., a 2-torus T^2).
$X = X_0$ has a nodal singularity (X is a pinched torus).



$H_1(X) = \mathbb{Q}$, $IH_1(X) = 0$, but $H_1(X_s) = IH_1(X_s) = \mathbb{Q} \oplus \mathbb{Q}$
so neither $H_*(-)$ nor $IH_*(-)$ is invariant under the smoothing
$X \rightsquigarrow X_s$.

Questions: *Is there a (co)homology theory for singular spaces, so that:*

- *Poincaré Duality holds;*
- *it has a ring structure;*
- *it is more stable under deformations;*
- *it carries a Kähler package;*
- *it is realized by a self-dual perverse sheaf?*

Intersection Spaces

- Banagl (2010) uses homotopy theory to associate to a (certain) singular space X a cellular complex IX, the intersection space of X.
- **Theorem** (Banagl): $\widetilde{H}_*(IX)$ satisfies Poincaré Duality.
- Note: The intersection space cohomology $H^*(IX)$ has an internal *ring structure* defined by the usual cup product in cohomology, so $H^*(IX) \neq IH^*(X)$!!!

Construction of Intersection Spaces

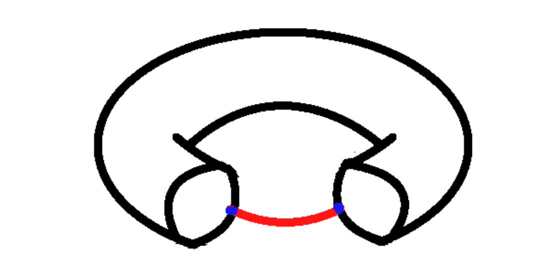
- IX is defined via spatial homology truncation, i.e., by replacing links of singularities of X by their *Moore approximations*.
- Let K be a simply-connected CW complex, and fix $n \in \mathbb{Z}_{\geq 0}$.
- Moore approximation of (K, n) is a CW complex $K_{<n}$ together with a structural map $f : K_{<n} \to K$, so that:
  - $f_* : H_r(K_{<n}) \to H_r(K)$ is an isomorphism if $r < n$
  - $H_r(K_{<n}) = 0$ for all $r \geq n$.
- Moore approximations exist (P. Hilton), and can often be constructed in the non-simply-connected case, e.g., if $\partial_n = 0$, choose $K_{<n} := K^{(n-1)}$ with $f = incl$.

Construction of Intersection Spaces, cont'd

- IX is defined by replacing links of singularities of X by their Moore approximations.
- Assume X has only one isolated singularity x, with link K.
- Let $M := X - c^\circ(K)$. Then $\partial M = K$, and $X = M \cup_K c(K)$.
- Let $g : K_{<n} \xrightarrow{f} K = \partial M \hookrightarrow M$, with $n = dim_{\mathbb{C}} X$.
- Set
$$IX := cone(g) := M \cup_g cone(K_{<n})$$
- If X has several isolated singularities, IX is defined by performing this procedure on each of the links, simultaneously.
- $H_*(IX; \mathbb{Q})$ is well-defined, i.e., independent of all choices.

Example

In the case of the nodal curve X, with $n = 1$, the link of the singular point is two circles $K = S^1 \sqcup S^1$, so $K_{<1} = \bullet \sqcup \bullet$.
The intersection space IX of X is



$$H_1(IX) = \mathbb{Q} \oplus \mathbb{Q} = H_1(X_s).$$

Conifold transitions in (10d) String Theory

- In ($10d$) string theory, the 6 (real) dimensions needed for a string to vibrate form a CY space.
- But which Calabi-Yau 3-fold should model the *string*?
- **Conjecture** [Green-Hübsch]
  Topologically distinct Calabi-Yau's are connected to each other by conifold transitions, which induce a phase transition between the corresponding string models.

Conifold transition:

$$X_s \overset{s\to 0}{\rightsquigarrow} S \xleftarrow{\pi} Y,$$

where:

- X_s and Y are topologically distinct smooth Calabi-Yau 3-folds.
- S is obtained from X_s by deforming the complex structure.
- π is a *small* resolution.
- The deformation collapses embedded 3-spheres (*vanishing cycles*) to isolated o.d.p.'s.
- π resolves the singularities of S by replacing each of them with a $\mathbb{CP}^1$.

In type IIA string theory, there are charged 2-branes that wrap around the $\mathbb{CP}^1$ 2-cycles, and which become massless when these 2-cycles are collapsed to points by the resolution map π. The *intersection homology* of the conifold accounts for all of these so it *is the physically correct homology theory for type IIA string theory*.

In type IIB string theory there are charged 3-branes wrapped around the vanishing cycles, and which become massless as these vanishing cycles are collapsed by the deformation of complex structure. Neither $H_*(S)$ nor $IH_*(S)$ account for these massless 3-branes, but $H_*(IS)$ yields the correct count. So the *homology of intersection spaces is the physically correct homology theory in the IIB string theory*.

Relation to mirror symmetry

Given a Calabi-Yau 3-fold X, the *mirror map* associates to it another Calabi-Yau 3-fold X° so that type IIB string theory on $\mathbb{R}^4 \times X$ corresponds to type IIA string theory on $\mathbb{R}^4 \times X^\circ$.
If X and X° are smooth, their Betti numbers are related by precise algebraic identities, e.g., $b_3(X^\circ) = b_2(X) + b_4(X) + 2$, etc.
Conjecture: [Morrison] The mirror of a conifold transition is again a conifold transition, but performed in the reverse order (mirror symmetry should exchange resolutions and deformations).
So, if S and S° are *mirrored conifolds* (in mirrored conifold transitions), the intersection space homology of one space and the intersection homology of the mirror space form a mirror-pair, i.e.,

$$b_3(IS^\circ) = Ib_2(S) + Ib_4(S) + 2,$$

etc., where Ib_i is the i-th intersection homology Betti number.

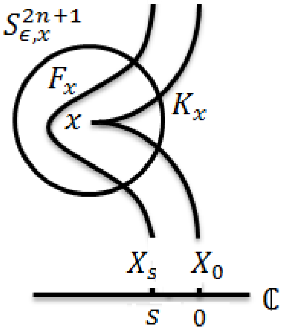
Upshot: The above mirror symmetry considerations suggest that one should be able to compute $H_*(IX)$ for a variety X in terms of the topology of a smoothing family X_s, by "*mirroring*" known results relating the intersection homology groups $IH_*(X)$ of X to the topology of a resolution of singularities $\widetilde{X}$.

Hypersurface singularities

Let f be a homogeneous polynomial s.t.

$$X = \{f(x) = 0\} \subset \mathbb{CP}^{n+1}$$

has only *one isolated singularity* x.
Let K_x, F_x and $T_x : H^n(F_x) \to H^n(F_x)$ be the *link*, *Milnor fiber* and local *monodromy operator* of (X, x).
$F_x \overset{h.e.}{\simeq} \bigvee_{\mu_x} S^n$, μ_x = Milnor number (=number of *vanishing cycles* at x).



Theorem A (Banagl-M.)

Let $X \subset \mathbb{CP}^{n+1}$ be a hypersurface with only one isolated singular point x. Let X_s be a nearby smoothing of X. Then:

$$\dim H^i(IX) = \begin{cases} \dim H^i(X_s) & \text{if } i \neq n, 2n \\ \dim H^i(X_s) - \mathrm{rank}(T_x - 1) & \text{if } i = n \\ 0 & \text{if } i = 2n. \end{cases}$$

Corollary

$H^n(IX) \cong H^n(X_s) \iff T_x$ *is trivial.*

Remark

Recall that $IH^*(-)$ is invariant under small resolutions. We regard the local trivial monodromy condition as "mirroring" that of the existence of small resolutions.

Intersection space complex: Existence and Properties

Theorem B (Banagl-Budur-M., *ATMP (to appear)*)

(a) There exists a perverse sheaf complex $\mathcal{IS}_X$ on X so that there are (abstract) isomorphisms

$$\mathbb{H}^i(X; \mathcal{IS}_X[-n]) \simeq \begin{cases} H^i(IX) & \text{if } i \neq 2n \\ H^{2n}(X_s) = \mathbb{Q} & \text{if } i = 2n. \end{cases}$$

(b) $\mathbb{H}^i(X; \mathcal{IS}_X)$ carries a natural mixed Hodge structure, $\forall i$.
(c) If the local monodromy T_x at x is semi-simple in the eigenvalue 1, the intersection-space complex $\mathcal{IS}_X$ is self-dual.
(d) If the local monodromy T_x at x is semi-simple in the eigenvalue 1, and the global monodromy T acting on $H^*(X_s)$ is semi-simple in the eigenvalue 1, then $\mathbb{H}^*(X; \mathcal{IS}_X)$ carry pure Hodge structures satisfying the Hard Lefschetz theorem.

Nearby and vanishing cycles.

- Let $X \subset \mathbb{CP}^{n+1}$ be a hypersurface with $Sing(X) = \{x\}$.
- Assume $n > 2$, so $\pi_1(K_x) = 0$.
- Let $\pi : \widetilde{X} \to S \subset \mathbb{C}$ be a family of hypersurfaces with $X = \pi^{-1}(0)$, s.t. $\widetilde{X}$ is smooth and $X_s := \pi^{-1}(s)$ for $s \neq 0$ is a smooth hypersurface in $\mathbb{CP}^{n+1}$.
- Let $\psi_\pi, \varphi_\pi : D^b_c(\widetilde{X}) \to D^b_c(X)$ be the nearby and vanishing cycle functors of π, with monodromy T and resp. $\widetilde{T}$.
- $H^i(X_s) \cong \mathbb{H}^i(X; \psi_\pi \mathbb{Q}_{\widetilde{X}})$, with compatible monodromies.
- If $i_x : \{x\} \hookrightarrow X$ is the inclusion, then:

$$H^i(F_x) \cong H^i(i_x^* \psi_\pi \mathbb{Q}_{\widetilde{X}}) \ , \quad \widetilde{H}^i(F_x) = H^i(i_x^* \varphi_\pi \mathbb{Q}_{\widetilde{X}})$$

  with compatible monodromies.
- $Supp(\varphi_\pi \mathbb{Q}_{\widetilde{X}}) = Sing(X) = \{x\}$.

- There are decompositions:

$$\psi_\pi = \psi_{\pi,1} \oplus \psi_{\pi,\neq 1} \quad \text{and} \quad \varphi_\pi = \varphi_{\pi,1} \oplus \varphi_{\pi,\neq 1}$$

- There are canonical morphisms:

$$\mathit{can} : \psi_\pi \to \varphi_\pi \quad \text{and} \quad \mathit{var} : \varphi_\pi \to \psi_\pi$$

s.t. $\mathit{can} \circ \mathit{var} = \widetilde{T} - 1$, $\mathit{var} \circ \mathit{can} = T - 1$.

Recall:

$$\dim H^i(IX) = \begin{cases} \dim H^i(X_s) & \text{if } i \neq n, 2n \\ \dim H^i(X_s) - \mathrm{rank}(T_x - 1) & \text{if } i = n \\ 0 & \text{if } i = 2n. \end{cases}$$

where:

- $H^i(X_s) \cong \mathbb{H}^i(X; \psi_\pi \mathbb{Q}_{\widetilde{X}})$
- $\widetilde{H}^i(F_x) = H^i(i_x^* \varphi_\pi \mathbb{Q}_{\widetilde{X}}) \cong \mathbb{H}^i(X; \varphi_\pi \mathbb{Q}_{\widetilde{X}})$

with compatible monodromies.

Intersection space complex: Construction

- Let
$$\mathcal{C} := Image(\widetilde{T} - 1) \overset{\iota_\varphi}{\hookrightarrow} \varphi_\pi \mathbb{Q}_{\widetilde{X}}[n] \in Perv(X)$$
- Then $\mathcal{C} \in Perv(X)$, $Supp(\mathcal{C}) = \{x\}$, and
$$\mathbb{H}^i(X; \mathcal{C}) = \begin{cases} 0 & , \text{ if } i \neq 0, \\ Image(T_x - 1) & , \text{ if } i = 0. \end{cases}$$
- Let
$$\iota := var \circ \iota_\varphi : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n].$$
- Define:
$$\mathcal{IS}_X := Coker\left(\iota : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n]\right) \ \in \ Perv(X)$$
- $\mathcal{IS}_X$ underlies a mixed Hodge module.
- If T_x is semi-simple in the eigenvalue 1, then:
$$\mathcal{IS}_X \cong \psi_{\pi,1} \mathbb{Q}_{\widetilde{X}}[n].$$

THANK YOU !!!