Vortrag Abstract

Cardona 2017

A vanishing theorem for tautological characteristic classes of aspherical manifolds

F. Hebestreit (Bonn)

joint with M. Land, W. Lück &
O. Randal-Williams

Conventions

Let M be a closed, connected, oriented d-dimensional manifold with

$$\pi_1(M) = \Gamma.$$

$\pi : E \longrightarrow B$ a smooth, oriented M-fibre bundle, e.g. a submersion of closed manifolds.

Thm (–)

Let M be aspherical and satisfy

1) the identity block Borel conj. IBB

2) the central part of Burghelea's conj. CPB

and let $c \in H^k(BSO(d), \mathbb{Q})$ with $k \neq d$.

Then

1) Whenever $\pi : E \to B$ has trivial monodromy,
$$0 = \kappa_c(\pi) \in H^{k-d}(B, \mathbb{Q})$$

2) $$0 = \kappa_c(M) \in H^{k-d}(B\mathrm{Diff}_0(M), \mathbb{Q})$$

3) the scanning map
$$B\mathrm{Diff}_0(M) \to B\mathrm{Diff}^+(M) \to \Omega^\infty MTSO(d)$$
is rationally null homotopic

Definitions

1) tautological classes: $c \in H^{k}(BSO(d))$

$$\kappa_c(\pi) := \pi_! \underbrace{c(T_v \pi)}_{\in H^k(E)} \in H^{k-d}(B)$$

"integration along fibres"

vert. tangent bundle, $\ker D\pi$

$$\kappa(M) : H^{k}(BSO(d)) \longrightarrow H^{k-d}(BDiff^{+}(M))$$

2) (IBB)-conj: The map

$$\widetilde{Top}_0(M) \longrightarrow \mathcal{G}_0(M)$$

is an equivalence.

block homeo's

homotopy equiv's

3) (CPB)-conj: $\forall g \in C(\Gamma)$

$$cd^{tc}_{\mathbb{Q}}(\Gamma / \langle g \rangle) < \infty$$

centre

cohomological dimension

Remarks

1) If $c(\Gamma)=0$ the result is due to Bustamante-Farrell-Jiang, 2015

2) For $d\geq 4$ the (IBB)-conj for M is implied by the Farrell-Jones-conj for Γ (Raimicki).

Yes!

3) For $d\leq 3$ conclusion is known (Earle-Eells, Ebert)

Remarks continued

4) Obtain the conclusion for

- Γ hyperbolic or CAT(0) $\leftarrow$ M non positively curved
- Γ arithmetic $\leftarrow$ $M \cong \Gamma \backslash G / K$
- Γ solvable, elementary amenable, linear over $\mathbb{Q}$, ...
- extensions of the above $\leftarrow$ fibre bundles
- ...

$\rightsquigarrow$ "almost all aspherical manifolds"

Remarks continued

5) Non-vanishing results:

Thm (Tillmann, Madsen-Weiss, Galatius-RW)

Let $W_g = (S^n \times S^n)^{\# g}$, $n \neq 2$

$$\begin{tikzcd}
\mathrm{Sym}\left(H^{*>2n}(BSO(2n)\langle n+1\rangle, \mathbb{Q})\,[2n]\right) \otimes H^*(BSO[n], \mathbb{Q}) \arrow[d, "\kappa \cdot \text{“}\kappa\text{''}"] \\
H^*(B\mathrm{Diff}^+(W_g), \mathbb{Q})
\end{tikzcd}$$

is an isomorphism below degree $\frac{g-3}{2}$.

More generally: Galatius - RW

d even, $\Gamma = 0$

$\rightsquigarrow$ $H^*(BDiff^+(M))$ can be described in degrees below $\frac{g(M)-3}{2}$ using the scanning map

$$\alpha : BDiff^+(M) \longrightarrow \Omega^\infty MT\theta_n$$

- $\Gamma = 0$ can be relaxed (Friedrich)
- d odd is a bit of a mystery (Ebert, H-Perlmutter)

Note $g(M)=0$ if M aspherical

and $d\geq 4$!

However:

Surfaces (and products thereof) show: Vanishing theorem does **not** hold in $H^*(B\mathrm{Diff}^+(M))$

$\rightsquigarrow$ "almost vanish"

Remarks continued

6) $k = d$

$\rightsquigarrow$ $\kappa_c(\pi) = \langle c(TM), \Gamma \rangle \in H^0(B)$

can be "arbitrary", since

Thm (Ontaneda)

Every class $\in \Omega_*$ can be represented by a negatively curved manifold M.

Note: this implies $C(\Gamma) = \emptyset$!

$\rightsquigarrow$ "almost all classes"

7) Our results are actually stronger!

i.e. there is

- an integral version
- a version for diffeomorphisms homotopic to the identity
- a version for topological/piecewise linear (block) bundles
- a version including char. numbers

Recall

1) tautological classes: $c \in H^{\xi}(BSO(d))$

$$\kappa_c(\pi) = \pi_! \, c(T_v \pi) \in H^{\xi - d}(B)$$

$$\kappa(M) : H^{\xi}(BSO(d)) \longrightarrow H^{\xi-d}(B\mathrm{Diff}^+(M))$$

exists integrally!

2) (IBB)-conj: The map

$$\widetilde{\mathrm{Top}}_0(M) \longrightarrow \mathcal{G}_0(M)$$

is an equivalence.

3) (CPB)-conj: $\forall g \in C(\Gamma)$

$$\mathrm{cd}^{tc}_{\mathbb{Q}}\left(\Gamma/\langle g \rangle\right) < \infty$$

Sketch of proof

Step 1 Some tautological classes can be defined for topological block bundles, in particular all rational ones can.

$\rightsquigarrow$ suffices to show

$$0 = \kappa_c(M) \in H^{k-d}(B\widetilde{\mathrm{Top}}_0(M), \mathbb{Q})$$

more precisely: We construct a lift

$$\begin{tikzcd} H^*(BSTOP) \otimes \mathbb{Z}[e] \arrow[r, "\kappa"] \arrow[d] & H^*(B\widetilde{\mathrm{Top}}^+(M)) \arrow[d] \\ H^*(BSO(d)) \arrow[r, "\kappa"] & H^*(B\mathrm{Diff}^+(M)) \end{tikzcd}$$

generalising work of Ebert-RW.

Sketch of proof continued

Now (IBB)-conj implies

$$\widetilde{BTop_0}(M) \xrightarrow{\simeq} B\mathcal{G}_0(M) = B^2 C(\Gamma)$$

$$\pi_i(\ \ ) = \begin{cases} C(\Gamma) & i=2 \\ 0 & \text{else} \end{cases}$$

$\rightsquigarrow$ If $C(\Gamma) = 0$

$$\widetilde{BTop_0}(M) \simeq *$$

$\rightsquigarrow$ recover Bustamante-Farrell-Jiang immediately

Sketch of proof continued

Now consider the universal block bundle with trivial monodromy

$$p : \widetilde{E}(M) \longrightarrow B\widetilde{\mathrm{Top}}_0(M)$$

$$\rightsquigarrow \quad \begin{tikzcd}
M \arrow[r, phantom, "\simeq"] \arrow[d, hook] & B\Gamma \arrow[d, dashed] \\
\widetilde{E}(M) \arrow[r, phantom, "\simeq"] \arrow[d, "p"] & B(\Gamma/C(\Gamma)) \arrow[d] \\
B\widetilde{\mathrm{Top}}_0(M) \arrow[r, phantom, "\simeq"] & B^2C(\Gamma)
\end{tikzcd}$$

classifies the central extension

$$0 \longrightarrow C(\Gamma) \longrightarrow \Gamma \longrightarrow \Gamma/C(\Gamma) \longrightarrow 0$$

Step 2: (CPB)-conj implies

$$C(\Gamma) \neq 0 \implies p_! \simeq 0$$

(IBB) + (CPB)

Corollary

M aspherical as above with

$$C(\Gamma) \neq 0.$$

Then all Pontryagin numbers and the Euler characteristic of M vanish

due to Gottlieb

Question

M aspherical with

$$C(\Gamma) \neq 0.$$

Is M null cobordant?

Cappell - Weinberger - Yan: $C(\Gamma)$ need not be induced by a principal torus action!