

The moduli space of Ricci-flat metrics

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History of the subject

Ricci-flat manifolds and parallel spinors have a rich history, e.g.

- ▶ the work by Lichnerowicz, Koiso and the Besse group

▶ Details

- ▶ Bochner, Calabi and Yau \rightsquigarrow Calabi-Yau manifolds
- ▶ stability results by McK Wang extended by X. Dai, X. Wang and G. Wei
- ▶ recent progress by D. Joyce, M. Haskins, R. Goto, J. Nordström, D. Crowley, S. Goette, K. Kröncke

and many many others

(sorry for not mentioning them!)

My own relation to the subject

- ▶ 2009-2012: H. Weiß and F. Witt.
Studied a geometric weakly parabolic flow on
7-dimensional manifolds
Fixed points = G_2 metrics
- ▶ 2012-2014: B. Ammann, H. Weiß and F. Witt.
Generalisation to arbitrary dimensions: “spinor flow”
Fixed points = metrics with parallel spinors
- ▶ 2015-2017: Further work on the spinor flow by J. Wittmann
(Regensburg) and L. Schiemanowski (Kiel)
- ▶ 2015: B. Ammann, K. Kröncke, H. Weiß and F. Witt
Moduli space of Ricci-flat metrics with parallel spinors
- ▶ 2016: D. Wraith. Space of metrics with $\text{scal} \geq 0$
- ▶ June 2017: B. Ammann, K. Kröncke, Th. Leistner, A.
Lischewski, O. Müller
Application to a constraint equation in Lorentzian manifolds
- ▶ (in progress): B. Ammann, K. Kröncke, H. Weiß:
the structure bundle

Introduction

Goal:

Study the space of all Ricci-flat metrics on a given compact manifold M .

Notation:

$\mathcal{M}(M) := \{\text{Riemannian metrics } g \text{ on } M \text{ with volume } 1\}$

$\mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \text{Ric}^g = 0\}$

$\text{Diff}(M) := \{\text{Diffeomorphisms } M \rightarrow M\}$

$\text{Diff}_{\text{Id}}(M)$ is the identity component of $\text{Diff}(M)$.

$\text{Diff}(M)$ acts on $\mathcal{M}(M)$ and on $\mathcal{M}_0(M)$ via pullback

Main interest:

Moduli space $\mathcal{M}_0(M)/\text{Diff}(M)$,

Premoduli space $\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)$



Example 1: Bieberbach manifolds

Flat manifolds $M = \mathbb{R}^n / \Gamma$.

$\Gamma \subset \mathbb{R}^n \rtimes O(n)$, discrete, cocompact, acts without fixed points.
Every Ricci-flat metric on M is flat.

The connected components of $\mathcal{M}_0(M) / \text{Diff}_{\text{id}}(M)$ are smooth manifolds of dimension ≥ 1 .

Example 2: Calabi-Yau manifolds

Calabi-Yau manifolds = Ricci-flat Kähler manifolds.

Their moduli spaces were studied by Tian and Todorov.

E.g. “the” compact 4-manifold K3:

$$\text{K3} := \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{C}P^3.$$

carries a CY metric.

Main subject of the talk

We will distinguish two classes of metrics with $\text{Ric} = 0$.

Definition

A metric g on a compact manifold M is called *structured* if

- ▶ the universal covering \tilde{M} is spin and
- ▶ if \tilde{M} carries a non-zero parallel spinor.

$$\mathcal{M}_{\parallel}(M) := \{g \in \mathcal{M}(M) \mid g \text{ structured}\} \subset \mathcal{M}_0(M)$$

$\mathcal{M}_{\parallel}(M)$ is open and closed in $\mathcal{M}_0(M)$.

Open Problem

Is every Ricci-flat metric structured?

$$\mathcal{M}_{\parallel}(M) \subsetneq \mathcal{M}_0(M)?$$

Theorem A (A.-Kröncke-Weiß-Witt)

Let M be a closed manifold whose universal covering is spin.

- (1) The premoduli space is $\mathcal{M}_{\parallel}(M)/\text{Diff}_{\text{Id}}(M)$ is a finite-dimensional smooth manifold.
- (2) On $\mathcal{M}_{\parallel}(M)$, the map $g \mapsto \text{Hol}(M, g)$ is locally constant up to conjugation.
- (3) On $\mathcal{M}_{\parallel}(M)$, the map $g \mapsto \dim \Gamma_{\parallel}(\Sigma_g M)$ is locally constant.

If M is simply connected and if g_0 is a metric of irreducible holonomy, then the theorem was already known for metrics close to g_0 .

Sketch of proof: After passing to a finite covering $\widehat{M} \rightarrow M$, we can decompose

$$\widehat{M} = M_1 \times \cdots \times M_r \times T^k$$

into irreducible factors and a torus. Show that deformations “preserve” this structure. Use the known results for the irreducible factors.



Cheeger-Gromoll splitting theorem

The Cheeger-Gromoll splitting theorem implies:

Theorem

If (M, g) is a compact Riemannian manifold with $\text{Ric}^M \geq 0$.

Then the universal covering \tilde{M} is isometric to

$$N \times \mathbb{R}^k$$

where N is a simply-connected compact manifold with $\text{Ric}^N \geq 0$.

Note that k is a topological invariant, namely the growth rate of $\pi_1(M)$.

Killing vector fields

Isometries create singularities in the moduli space.
But luckily we have few isometries:

Lemma

Let K be a Killing vector field on a Riemannian manifold, then

$$\nabla^* \nabla K = \text{Ric}(K)$$

► Proof

Corollary

If (M, g) is a compact Riemannian manifold with $\text{Ric} \leq 0$, then every Killing vector field is parallel.

Proof.

$$\int_M \|\nabla K\|^2 = \int_M \langle \text{Ric}(K), K \rangle \leq 0.$$



Corollary

If (M, g) is a compact Riemannian manifold with $\text{Ric} = 0$.

Then \tilde{M} is isometric to

$$N \times \mathbb{R}^k$$

where N is a simply-connected compact manifold $\text{Ric}^N = 0$,
and $\text{Isom}(N)$ **is finite**.

Consequences

- ▶ It will imply that the singularities of the moduli space $\mathcal{M}_0(M)/\text{Diff}(M)$ are not worse than orbifold singularities. The orbifold singularity at (M, g_0) is modeled by \mathbb{R}^n/Γ where $\Gamma := \text{Isom}(M, g_0)/\text{Isom}_{\text{Id}}(M, g_0)$ is finite.
- ▶ To get rid of orbifold singularities, we pass to the **pre**moduli space $\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)$. This will turn out to be a smooth manifold.
- ▶ Every (closed) structured Ricci-flat manifold is finitely covered by a manifold with a parallel spinor.

Holonomy

Definition

The *holonomy group*

$\text{Hol}(M, g) := \{P_\gamma \mid \gamma \text{ path from } p \text{ to } p \text{ in } M\} \subset \text{GL}(T_p M)$.

The *restricted holonomy group*

$\text{Hol}_0(M, g) := \{P_\gamma \mid \gamma \text{ contract. path from } p \text{ to } p\} = \text{Hol}(\tilde{M}, g)$.

(M, g) is *irreducible* if $\text{Hol}_0(M, g) \rightarrow \text{GL}(T_p M)$ is an irred. repr.

Theorem (de Rham splitting theorem)

Let (\tilde{M}, \tilde{g}) be a complete simply-connected Riemannian manifold. Then as a Riemannian product

$$\tilde{M} = M_1 \times \cdots \times M_r \times \mathbb{R}^k$$

where each M_i is a complete non-flat irreducible Riemannian manifold.

For $\text{Ric} \geq 0$ on a univ. covering \tilde{M} of a closed M this refines the Cheeger-Gromoll splitting.



Berger's holonomy list

Theorem

If (M, g) is an irreducible Riemannian manifold, $n = \dim M$, then (M, g) is locally symmetric or $\text{Hol}_0(M, g)$ is one of the following:

- (1) $\text{Hol}_0(M, g) = \text{SO}(n)$ (generic)
- (2) $\text{Hol}_0(M, g) = \text{U}(n/2)$ (Kähler)
- (3) $\text{Hol}_0(M, g) = \text{SU}(n/2)$ (Ricci-flat Kähler)
- (4) $\text{Hol}_0(M, g) = \text{Sp}(n/4)$ (hyper-Kähler)
- (5) $\text{Hol}_0(M, g) = \text{Sp}(n/4) \cdot \text{Sp}(1)$ (quaternionic-Kähler)
- (6) $\text{Hol}_0(M, g) = \text{G}_2$ and $n = 7$
- (7) $\text{Hol}_0(M, g) = \text{Spin}(7)$ and $n = 8$

There are compact examples in each case.

In cases (3), (4), (6) and (7) we have $\text{Ric} = 0$.

If M is compact with $\text{Ric} = 0$, then we are in case (1), (3), (4), (6) or (7).

Parallel spinors and holonomy

- ▶ If (M, g) carries a parallel spinor, then (\tilde{M}, \tilde{g}) carries a parallel spinor (pullback).
- ▶ If (\tilde{M}, \tilde{g}) carries a parallel spinor, then g is a **infinitesimally stable** Ricci-flat metric.
- ▶ (\tilde{M}, \tilde{g}) carries a parallel spinor if and only if $\text{Hol}_0(M, g)$ is a product of
 - (1) $\{1\} = \text{SO}(1)$
 - (3) $\text{SU}(k)$ (Ricci-flat Kähler)
 - (4) $\text{Sp}(k)$ (hyper-Kähler)
 - (6) G_2
 - (7) $\text{Spin}(7)$

Deformations of Ricci-flat metrics

Let (M, g) be a compact Ricci-flat Riemannian manifold.

$$T_g(\mathcal{M}(M)) = \{\text{symmetric } (0, 2)\text{-tensors, } \int \text{tr}_g h = 0\}$$

Einstein-Hilbert functional $\mathcal{E}(g) = \int_M \text{scal}^g \, dv^g$.

$$\text{Hess}_g \mathcal{E} : T_g(\mathcal{M}(M)) \times T_g(\mathcal{M}(M)) \rightarrow \mathbb{R}$$

The *formal tangent space* of $\mathcal{M}_0(M)$ is

$$T_g^{\text{form}}(\mathcal{M}_0(M)) := \ker(\text{Hess}_g \mathcal{E}).$$

Corollary (.../AKWW 2015)

If g is a structured Ricci-flat metric, then $\mathcal{M}_0(M)$ is a manifold in a neighborhood of g and

$$T_g^{\text{form}}(\mathcal{M}_0(M)) = T_g(\mathcal{M}_0(M)).$$



Tangent space of the premoduli space

$$h \in T_g(\mathcal{M}(M)) = \{\text{symmetric } (0, 2)\text{-tensors, } \int \text{tr}_g h = 0\}$$

h is orthogonal to the conformal class $[g] \Leftrightarrow \text{tr}_g h = 0$

h is orthogonal to the $\text{Diff}(M)$ -orbit of $g \Leftrightarrow \text{div}_g h = 0$

$\text{TT} := \{h \text{ is a symm. } (0, 2)\text{-tensor with } \text{tr}_g h = 0 \text{ and } \text{div}_g h = 0\}$.

The *formal tangent space of the premoduli space* is

$$T_g^{\text{form}}(\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)) := \ker(\text{Hess}_g \mathcal{E}) \cap \text{TT}.$$

Corollary (.../AKWW 2015)

If g is a structured Ricci-flat metric, then $\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)$ is a manifold in a neighborhood of $g \text{Diff}_{\text{Id}}(M)$ and

$$T_g^{\text{form}}(\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)) = T_g(\mathcal{M}_0(M)/\text{Diff}_{\text{Id}}(M)).$$



Infinitesimal stability of Ricci-flat metrics

$\text{TT} := \{h \text{ is a symm. } (0, 2)\text{-tensor with } \text{tr}_g h = 0 \text{ and } \text{div}_g h = 0\}$.

For $h, k \in \text{TT}$ we have

$$(\text{Hess}_g \mathcal{E})(h, k) = -\frac{1}{2} \int_M \langle h, \Delta_E k \rangle \text{dvol}^g$$

where $\Delta_E = \nabla^* \nabla - 2\mathring{R}$ is the Einstein operator.

A Ricci-flat metric g is called *infinitesimally stable*

$$:\Leftrightarrow (\text{Hess}_g \mathcal{E})|_{\text{TT} \times \text{TT}} \leq 0$$

$$\Leftrightarrow \Delta_E \text{ is positive semi-definite}$$

Theorem (McKenzie Wang, Ind. Univ. Math. J. 1991)

Structured Ricci-flat metrics are infinitesimally stable.

More precisely: McK Wang assumes a parallel spinor on M . Using what we explained before, the theorem then directly follows as claimed.

Theorem (X. Dai, X. Wang, G. Wei, Inv. Math. 2005)

Structured Ricci-flat metrics g_0 with irreducible holonomy on a simply connected closed manifold are locally stable, i.e. there is no metric of positive scalar curvature in a neighborhood of g_0 .

▶ Historical details



Kröncke's product formula

Theorem (Kröncke)

If (M^m, g) and (N^n, h) are two infinitesimally stable Ricci-flat manifolds, then $(M \times N, g + h)$ is also infinitesimally stable. Furthermore, on TT we have

$$\ker(\Delta_E^{M \times N}) = \mathbb{R}(n \cdot g - m \cdot h) \oplus (\Gamma_{\parallel}(TM) \odot \Gamma_{\parallel}(TN)) \\ \oplus \ker(\Delta_E^M) \oplus \ker(\Delta_E^N).$$

Corollary (Kröncke/AKWW)

The Cheeger-Gromoll/de Rham splitting is preserved under deformations.

Other Ingredient: The Montgomery-Zippin theorem allows to pass from simply-connected manifolds to arbitrary manifolds.

Corollary (Kröncke/AKWW)

Local stability is true for arbitrary closed manifolds.

Corollary

M is a closed manifold such that $\mathcal{M}_{\parallel}(M)$ has at least k connected components, then $\mathcal{M}_{\parallel}(M \times M)$ has at least k^2 connected components.

Application: Metrics of non-negative scalar curvature

Let M be a closed spin manifold.

$$\text{Scal}_{>0}(M) := \{g \in \mathcal{M}(M) \mid \text{scal}^g > 0\}$$

$$\text{Scal}_{\geq 0}(M) := \{g \in \mathcal{M}(M) \mid \text{scal}^g \geq 0\}$$

Theorem (D. Wraith, July 2016 (arxiv))

If g_t is a path in $\text{Scal}_{\geq 0}(M)$ with $g_0 \in \text{Scal}_{>0}(M)$, then $\ker D_{g(t)}^V = 0$ for all t and all Dirac operators twisted with flat bundles V .

Proof: Assume $\ker D_{g_1}^V \ni \varphi \neq 0$. Then $\nabla^{g_1} \varphi = 0$ and $\text{scal}^{g_1} = 0$. Apply Ricci flow to the path g_t . Let $\tau \in [0, 1]$ be minimal such that $\text{scal}^{g_t} = 0$ for $t \in [\tau, 1]$. Then $\text{Ric}^{g_t} = 0$ for $t \in [\tau, 1]$ and $\text{scal}^{g_{\tau-\epsilon}} > 0$. Thus g_τ has a parallel spinor. Thus cannot be deformed to $\text{scal} > 0$. \downarrow

Consequences

- ▶ $\text{Scal}_{\geq 0}(M) = \overline{\text{Scal}_{> 0}(M)} \sqcup \mathcal{M}_{\parallel}(M) \sqcup \{\text{Unstruct. Ricci flat}\}$
- ▶ On $\overline{\text{Scal}_{> 0}(M)} \subset \text{Scal}_{\geq 0}(M)$ the kernel of the Dirac operator vanishes.

“Corollary”

Every **known** non-trivial element in $\pi_k(\text{Scal}_{> 0}(M))$ is also non-trivial in $\pi_k(\text{Scal}_{\geq 0}(M))$.

“Proof”

All known proofs showing that elements in $\pi_k(\text{Scal}_{> 0}(M))$ are non-trivial use index theory. As $\ker D$ vanishes not only on $\text{Scal}_{> 0}(M)$ but on $\overline{\text{Scal}_{> 0}(M)}$ the non-triviality extends to $\overline{\text{Scal}_{> 0}(M)}$.

Remark

- ▶ I did **not** claim that $\pi_k(\iota) : \pi_k(\text{Scal}_{> 0}(M)) \rightarrow \pi_k(\text{Scal}_{\geq 0}(M))$ is injective. The kernel of $\pi_k(\iota)$ might be large, but I do not know methods to detect non-trivial elements in $\ker \pi_k(\iota)$.

Application: Lorentzian manifolds with parallel spinors

BBGM parallel transport for spinors

BBGM = Bär-Bourguignon-Gauduchon-Moroianu

Let g_t , $t \in [0, T]$ be a path of Riemannian metrics on M .

Get metric $G := g_t + dt^2$ on $M \times [0, T]$

Definition

For any $\varphi \in \Gamma(\Sigma^{g_0} M)$ we define the *BBGM parallel transport* $\mathcal{P}_{0,T}(\varphi)$ as follows:

$$\Sigma^{g_0} M \hookrightarrow \Sigma^G(M \times [0, T])|_{M \times \{0\}}$$

Extend φ to $\Phi \in \Gamma(\Sigma^G(M \times [0, T]))$ with $\frac{\nabla}{dt}\Phi = 0$.

$$\mathcal{P}_{0,T}(\varphi) := \Phi_{M \times \{T\}} \in \Gamma(\Sigma^{g_T} M).$$

Theorem B (A., Kröncke, Müller 2017)

Assume g_t is a family of Ricci-flat metrics on a closed manifold M , and $\operatorname{div}^{g_t} \frac{d}{dt} g_t = 0$. Let $\varphi \in \Gamma(\Sigma^{g_0} M)$ be parallel. Then $\mathcal{P}_{0,t}(\varphi) \in \Gamma(\Sigma^{g_t} M)$ is parallel for all t .

Using “good” identifications $\Phi \in \Gamma(\Sigma N)$, $N := M \times [0, T]$.

$$\begin{aligned}\frac{\partial}{\partial t} \cdot \Phi &= i\Phi \\ \nabla_{(\partial/\partial s)}^N \Phi &= 0 \\ \nabla_X^N \Phi &= \frac{1}{2} W(X) \cdot \frac{\partial}{\partial s} \cdot \Phi = \frac{i}{2} W(X) \cdot \Phi\end{aligned}$$

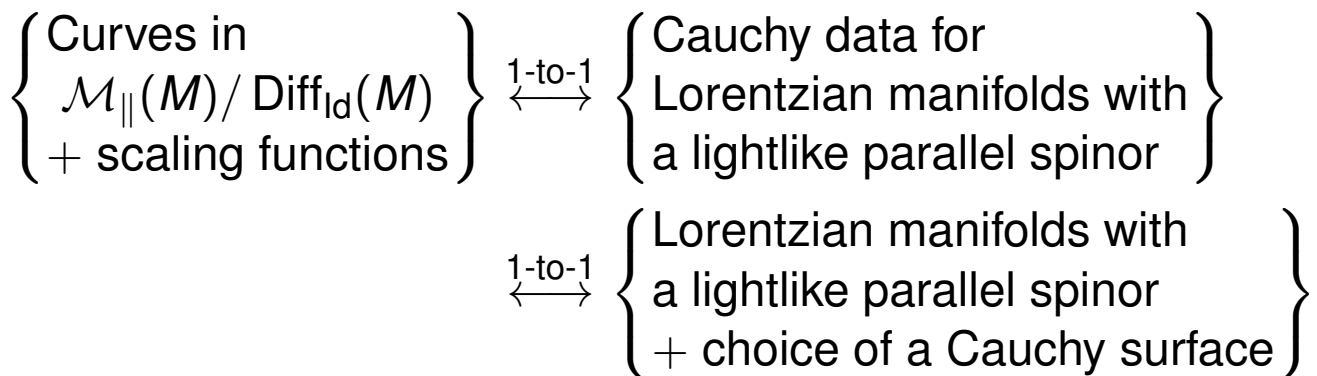
These are solutions for the constraint equations for Lorentzian manifolds with a parallel spinor.

Consequence (& Lischewski und Leistner):






There is a time- and space-oriented Lorentzian manifold Q of dimension $\dim M + 2$ with

- ▶ $(N, g_t + dt^2)$ is a Cauchy hypersurface for Q ,
- ▶ Q carries a parallel spinor whose associated vector field is lightlike (and parallel).




Expected picture:



Some related literature

-  B. AMMANN, K. KRÖNCKE, H. WEISS, AND F. WITT, *Holonomy rigidity for Ricci-flat metrics*, <http://arxiv.org/abs/1512.07390>
-  A. BESSE, *Einstein manifolds*, Springer, Berlin, 1987.
-  R. GOTO, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler, G_2 and Spin(7) structures*, Internat. J. Math. **15** (2004), no. 3, 211–257.
-  K. KRÖNCKE *On infinitesimal Einstein deformations*, Differ. Geom. Appl. **38** (2015), 41–57.
<http://arxiv.org/abs/1508.00721>
-  J. NORDSTRÖM, *Ricci-flat deformations of metrics with exceptional holonomy*, Bull. Lond. Math. Soc. **45** (2013), 1004–1018. <http://arxiv.org/abs/1008.0663>.

Literature continued

-  M. WANG, *Preserving parallel spinors under metric deformations*, Ind. Math. J. **40** (1991), 815–844.
-  X. DAI, X. WANG, G. WEI, *On the Stability of Riemannian Manifold with Parallel Spinors*, Invent. Math. **161** (2005), 151–176.
-  D. WRAITH, *Non-negative versus positive scalar curvature*, <http://arxiv.org/abs/1607.00657>

Besse group

The authors of the book “Einstein manifolds” by “Arthur Besse” are:

Averous, Bérard-Bergery, Berger, Bourguignon, Derdzinski, DeTurck, Gauduchon, Hitchin, Houillot, Karcher, Kazdan, Koiso, Lafontaine, Pansu, Polombo, Thorpe, Valère

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Proof of the Lemma for Killing fields

Let K be a Killing field on a Riemannian manifold. Then $Y \mapsto \nabla_Y K$ is skew-symmetric, and thus $Y \mapsto \nabla_{X,Y}^2 K$ is also skew-symmetric for fixed X .

$$\begin{aligned} 0 &= g(R(X, Y)K, W) + g(R(Y, W)K, X) + g(R(W, X)K, Y) \\ &= g(\nabla_{X,Y}^2 K, W) - g(\nabla_{Y,X}^2 K, W) + g(R(Y, W)K, X) \\ &\quad + g(\nabla_{W,X}^2 K, Y) - g(\nabla_{X,W}^2 K, Y) \\ &= 2g(\nabla_{X,Y}^2 K, W) + 2g(R(Y, W)K, X) \end{aligned}$$

Thus

$$\nabla_{X,Y}^2 K + R(K, X)Y = 0.$$

Taking the trace we get

$$\nabla^* \nabla K = \text{Ric}(K).$$

Historical comments concern stability

- ▶ Infinitesimal stability: This was proven by McK Wang for manifolds with parallel spinors. The general case of structured Ricci-flat manifolds then follows using the fact that any closed structured Ricci-flat manifold is finitely covered by one with parallel spinor. Dai-Wang-Wei reprove this. Their argument for infinite fundamental group can be simplified using our arguments.
- ▶ For local stability Dai-Wang-Wei require simple-connectedness (an assumption which is not hard to remove). Their proof however also uses irreducibility without stating it. The irreducibility is used at the step to argue that every formal deformation integrates to a deformation. In reducible metrics Dai-Wang-Wei's arguments only yield proofs for formal deformations of product type.