Blowup and deformation groupoids constructions related to index problem

D. & Skandalis - Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus arXiv:1705.09588

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Groupoids in index theory

Smooth compact manifold M : $G = M \times M \rightrightarrows M$ the pair groupoid.
The tangent groupoid of A. Connes :

$$\mathcal{G}_M^t = TM \times \{0\} \cup M \times M \times]0,1] \rightrightarrows M \times [0,1]$$

It defines : $0 \to C^*(\mathcal{G}_M^t|_{M\times]0,1]}) \to C^*(\mathcal{G}_M^t) \overset{e_0}{\to} C^*(\mathcal{G}_M^t|_{M\times\{0\}}) \to 0$
$\simeq \mathcal{K} \otimes C_0(]0,1])$ $\qquad = C^*(TM)$

$[e_0] \in KK(C^*(\mathcal{G}_M^t), C^*(TM))$ is invertible.
Let $e_1 : C^*(\mathcal{G}_M^t) \to C^*(\mathcal{G}_M^t|_{M\times\{1\}}) = C^*(M\times M) \simeq \mathcal{K}$.

The index element

$$\mathrm{Ind}_{M\times M} := [e_0]^{-1} \otimes [e_1] \in KK(C^*(TM), \mathcal{K}) \simeq K^0(C^*(TM)) \ .$$

The algebra $\Psi^*(G) = \Psi^*(M \times M)$ identifies with the C^*-algebra of order 0 pseudodifferential operators on M and

$$0 \longrightarrow \underset{\simeq \mathcal{K}}{C^*(M \times M)} \longrightarrow \Psi^*(M \times M) \longrightarrow C(\mathbb{S}^*TM) \longrightarrow 0$$

which gives a connecting element $\widetilde{Ind}_{M\times M} \in KK^1(C(\mathbb{S}^*TM), \mathcal{K})$.
Let i be the inclusion of $\mathbb{S}^*TM \times \mathbb{R}^*_+$ as the open subset $T^*M \setminus M$ of T^*M then

$$\widetilde{Ind}_{M\times M} = Ind_{M\times M} \otimes [i]$$

Proposition [Connes]

The morphism $\cdot \otimes Ind_{M\times M} : K^0(T^*M) \simeq KK(\mathbb{C}, C^*(TM)) \longrightarrow \mathbb{Z}$ *is* the analytic index map of A-S.

General Lie groupoid $G \rightrightarrows M$ [Monthubert-Pierrot, Nistor-Weinstein-Xu]
The adiabatic groupoid : $\mathcal{G}_M^t = \mathfrak{A}G \times \{0\} \cup G \times]0,1] \rightrightarrows M \times [0,1]$
gives $\mathrm{Ind}_G := [e_0]^{-1} \otimes [e_1] \in KK(C^*(\mathfrak{A}G), C^*(G))$.

Pseudodifferential exact sequence :

$$0 \longrightarrow C^*(G) \longrightarrow \Psi^*(G) \longrightarrow C(\mathbb{S}^*\mathfrak{A}G) \longrightarrow 0$$

which defines $\widetilde{Ind}_G \in KK^1(C(\mathbb{S}^*\mathfrak{A}G), C^*(G))$ with $\widetilde{Ind}_G = Ind_G \otimes [i]$ where i is the inclusion of $\mathbb{S}^*\mathfrak{A}G \times \mathbb{R}_+^*$ as the open subset $\mathfrak{A}^*G \setminus M$ of $\mathfrak{A}^*G$.

Manifold with boundary - $V \subset M$ a hypersurface [Melrose & coauthors]

- 0-calculus, (pseudodifferential) operators vanishing on V : replace $M \times M$ by $G_0 \rightrightarrows M$ equal to the pair groupoid on $M \setminus V$ outside V and isomorphic to $\mathcal{G}_V^t \rtimes \mathbb{R}_+^*$ around V.
- b-calculus, (pseudodifferential) operators vanishing on the normal direction of V : replace $M \times M$ by $G_b \rightrightarrows M$ equal to $M \setminus V \times M \setminus V$ outside V and isomorphic to $V \times V \times \mathbb{R} \rtimes \mathbb{R}_+^*$ around V.

What about more general situations ...

Can we mix the above situations ?

Framework : $G \rightrightarrows M$ a Lie groupoid, $V \subset M$ a submanifold, $\Gamma \rightrightarrows V$ a subgroupoid of G and operators that "slow down" near V in the normal direction and "propagate" along Γ inside V.

Today, in this talk :

- Present the general groupoid constructions involved in such situations.
- Compute, compare the corresponding index elements and connecting maps arising.

The Deformation to the Normal Cone construction

Let V be a closed submanifold of a smooth manifold M with normal bundle N_V^M. The deformation to the normal cone is

$$DNC(M,V) = M \times \mathbb{R}^* \cup N_V^M \times \{0\}$$

It is endowed with a smooth structure thanks to the choice of an exponential map $\theta : U' \subset N_V^M \to U \subset M$ by requiring the map

$$\Theta : (x,X,t) \mapsto \begin{cases} (\theta(x,tX),t) \ for \ t \neq 0 \\ (x,X,0) \ for \ t = 0 \end{cases}$$

to be a diffeomorphism from the open neighborhood $W' = \{(x,X,t) \in N_V^M \times \mathbb{R} \mid (x,tX) \in U'\}$ of $N_V^M \times \{0\}$ in $N_V^M \times \mathbb{R}$ on its image.

We define similarly

$$DNC_+(M,V) = M \times \mathbb{R}_+^* \cup N_V^M \times \{0\}$$

Alternative description of DNC

Let V be a closed submanifold of a smooth manifold M with normal bundle N_V^M :

$$DNC(M,V) = M \times \mathbb{R}^* \cup N_V^M \times \{0\}$$

The smooth structure is generated by requiring the following maps to be smooth

For any $x \in M$, $\lambda \in \mathbb{R}^*$, $y \in V$, $X \in T_yM/T_yV$:

- $p : DNC(M,V) \to M \times \mathbb{R}$: $\quad p(x,\lambda) = (x,\lambda)\,,\;\; p(y,X,0) = (y,0)$;
- given $f : M \to \mathbb{R}$, smooth with $f_{|V} = 0$, $\tilde{f} : DNC(M,V) \to \mathbb{R}, \quad \tilde{f}(x,\lambda) = \dfrac{f(x)}{\lambda}, \; \tilde{f}(y,X,0) = (df)_y(X)$

Functoriality of DNC

Consider a commutative diagram of smooth maps

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "f_V"'] & M \arrow[d, "f_M"] \\ V' \arrow[r, hook] & M' \end{tikzcd}$$

Where the horizontal arrows are inclusions of submanifolds. Let

$$\begin{cases} DNC(f)(x,\lambda) = (f_M(x),\lambda) & \text{for } x \in M,\ \lambda \in \mathbb{R}_* \\ DNC(f)(x,X,0) = (f_V(x),(df_M)_x(X),0) & \text{for } x \in V,\ X \in T_xM/T_xV \end{cases}$$

We get a smooth map $DNC(f) : DNC(M,V) \to DNC(M',V')$.

Deformation groupoid

Let Γ be a closed Lie subgroupoid of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$.
Functoriality implies :

$$DNC(G,\Gamma) \rightrightarrows DNC(G^{(0)},\Gamma^{(0)})$$

is naturally a Lie groupoid; its source and range maps are $DNC(s)$ and $DNC(r)$; $DNC(G,\Gamma)^{(2)}$ identifies with $DNC(G^{(2)},\Gamma^{(2)})$ and its product with $DNC(m)$ where $m : G_i^{(2)} \to G_i$ is the product.

Remarks

- No transversality assumption !
- N_Γ^G is a $\mathcal{VB}$-groupoid over $N_{\Gamma^{(0)}}^{G^{(0)}}$ denoted $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$.

$$DNC(G,\Gamma) = G \times \mathbb{R}^* \cup \mathcal{N}_\Gamma^G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R}^* \cup N_{\Gamma^{(0)}}^{G^{(0)}} \times \{0\}$$

Examples

1. The adiabatic groupoid is the restriction of $DNC(G, G^{(0)})$ over $G^{(0)} \times [0, 1]$.
2. If V is a saturated submanifold of $G^{(0)}$ for G, $DNC(G, G_V^V)$ is the normal groupoid of the immersion $G_V^V \hookrightarrow G$ which gives the shriek map [M. Hilsum, G. Skandalis].
3. $\pi : E \to M$ a vector bundle; consider $\Delta E \subset E \underset{M}{\times} E \subset E \times E$:

$$\mathcal{T} = DNC\big(DNC(E \times E, E \underset{M}{\times} E), \Delta E \times \{0\}\big) \rightrightarrows E \times \mathbb{R} \times \mathbb{R}$$

Let $\mathcal{T}^{\square} = \mathcal{T}|_{E\times[0,1]\times[0,1]}$ and $\mathcal{T}hom = \mathcal{T}|_{E\times\{0\}\times[0,1]}$.

Examples

3. $\pi : E \to M$ a vector bundle; consider $\Delta E \subset E \underset{M}{\times} E \subset E \times E$:

$$\mathcal{T} = DNC\big(DNC(E \times E, E \underset{M}{\times} E), \Delta E \times \{0\}\big) \rightrightarrows E \times \mathbb{R} \times \mathbb{R}$$

Let $\mathcal{T}^{\square} = \mathcal{T}|_{E\times[0,1]\times[0,1]}$ and $\mathcal{T}hom = \mathcal{T}|_{E\times\{0\}\times[0,1]}$.

$$\begin{tikzcd}
E \underset{M}{\times} TM \underset{M}{\times} E \arrow[r, dash] \arrow[rr, dotted, bend left, "Ind_a^M"] \arrow[d, dash, "\mathcal{T}hom"] & \underset{\simeq(\mathcal{G}^t_M)^{\pi}_{\pi}}{DNC(E \times E, E \underset{M}{\times} E)|_{E\times[0,1]}} \arrow[r, dash] \arrow[d, phantom, "\mathcal{T}^{\square}"] & E \times E \arrow[d, dash, "\mathcal{G}^t_E"] \\
TE \arrow[r, dash] \arrow[u, dotted, bend left, "Thom^{-1}"] & TE \times [0,1] \arrow[r, dash] & TE \arrow[u, dotted, bend right, "Ind_a^E"']
\end{tikzcd}$$

Gives $Ind_a^M = Ind_t^M$ [D.-Lescure-Nistor].

The Blowup construction

The scaling action of $\mathbb{R}^*$ on $M \times \mathbb{R}^*$ extends to the gauge action on $DNC(M,V) = M \times \mathbb{R}^* \cup N_V^M \times \{0\}$:

$$\begin{array}{ccc} DNC(M,V) \times \mathbb{R}^* & \longrightarrow & DNC(M,V) \\ (z,t,\lambda) & \mapsto & (z, \lambda t)\ for\ t \neq 0 \\ (x, X, 0, \lambda) & \mapsto & (x, \frac{1}{\lambda} X, 0)\ for\ t = 0 \end{array}$$

The manifold $V \times \mathbb{R}$ embeds in $DNC(M,V)$:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & V \arrow[d] \\ V \arrow[r, hook] & M \end{tikzcd}$$

The gauge action is free and proper on the open subset $DNC(M,V) \setminus V \times \mathbb{R}$ of $DNC(M,V)$. We let :

$$Blup(M,V) = \big(DNC(M,V) \setminus V \times \mathbb{R}\big)/\mathbb{R}^* = M \setminus V \cup \mathbb{P}(N_V^M) \ \text{ and}$$

$$SBlup(M,V) = \big(DNC_+(M,V) \setminus V \times \mathbb{R}_+\big)/\mathbb{R}_+^* = M \setminus V \cup \mathbb{S}(N_V^M)\ .$$

Functoriality of *Blup*

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "f_V"'] & M \arrow[d, "f_M"] \\ V' \arrow[r, hook] & M' \end{tikzcd}$$

gives $DNC(f) : DNC(M,V) \to DNC(M',V')$

which is equivariant under the gauge action : it passes to the quotient $Blup$ as soon as it is defined.

Let $U_f(M,V) = DNC(M,V) \setminus DNC(f)^{-1}(V' \times \mathbb{R})$ and define

$$Blup_f(M,V) = U_f/\mathbb{R}^* \subset Blup(M,V)$$

Then $DNC(f)$ passes to the quotient :

$$Blup(f) : Blup_f(M,V) \to Blup(M',V')$$

Analogous constructions hold for $SBlup$.

Blowup groupoid

Let Γ be a closed Lie subgroupoid of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$. Define

$$\widetilde{DNC}(G,\Gamma) = U_r(G,\Gamma) \cap U_s(G,\Gamma)$$

elements whose image by $DNC(s)$ and $DNC(r)$ are not in $\Gamma^{(0)} \times \mathbb{R}$. Functoriality implies :

$$Blup_{r,s}(G,\Gamma) = \widetilde{DNC}(G,\Gamma)/\mathbb{R}^* \rightrightarrows Blup(G^{(0)},\Gamma^{(0)})$$

is naturally a Lie groupoid; its source and range maps are $Blup(s)$ and $Blup(r)$ and its product is $Blup(m)$.

Analogous constructions hold for $SBlup$.

$$SBlup_{r,s}(G,\Gamma) = \widetilde{DNC_+}(G,\Gamma)/\mathbb{R}_+^* \rightrightarrows SBlup(G^{(0)},\Gamma^{(0)})$$

In summary

We start with :

$$\begin{array}{ccc} \Gamma & \hookrightarrow & G \\ \downdownarrows & & \downdownarrows \\ \Gamma^{(0)} & \hookrightarrow & G^{(0)} \end{array}$$

Let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$ to $N_{\Gamma^{(0)}}^{G^{(0)}} \setminus \Gamma^{(0)}$ and $\mathring{G}$ the restriction of G to $G^{(0)} \setminus \Gamma^{(0)}$.

$\mathring{\mathcal{N}}_\Gamma^G / \mathbb{R}^*$ inherits a structure of Lie groupoid : $\mathcal{P}\mathcal{N}_\Gamma^G \rightrightarrows \mathbb{P}N_{\Gamma^{(0)}}^{G^{(0)}}$.

$\mathring{\mathcal{N}}_\Gamma^G / \mathbb{R}_+^*$ inherits a structure of Lie groupoid : $\mathcal{S}\mathcal{N}_\Gamma^G \rightrightarrows \mathbb{S}N_{\Gamma^{(0)}}^{G^{(0)}}$.

$$Blup_{r,s}(G,\Gamma) = \mathring{G} \cup \mathcal{P}\mathcal{N}_\Gamma^G \rightrightarrows G^{(0)} \setminus \Gamma^{(0)} \cup \mathbb{P}N_{\Gamma^{(0)}}^{G^{(0)}} .$$

$$SBlup_{r,s}(G,\Gamma) = \mathring{G} \cup \mathcal{S}\mathcal{N}_\Gamma^G \rightrightarrows G^{(0)} \setminus \Gamma^{(0)} \cup \mathbb{S}N_{\Gamma^{(0)}}^{G^{(0)}} .$$

These Lie groupoids come with natural local compactification :

$$Blup_{r,s}(G,\Gamma) \subset Blup(G,\Gamma) \ \text{ and } \ SBlup_{r,s}(G,\Gamma) \subset SBlup(G,\Gamma)$$

Remark about corners

Let $\Gamma \rightrightarrows V$ be a closed Lie subgroupoid of $G \rightrightarrows M$ where $V \subset M$ is a hypersurface, $M = M_- \cup M_+$ with $M_- \cap M_+ = V$.

Facts

1. $SBlup(M, V) = M \setminus V \cup \mathcal{S}N_V^M \simeq M_- \sqcup M_+$,
2. The restriction of $SBlup_{r,s}(G, \Gamma)$ to M_+ is a longitudinally smooth groupoid over the manifold with boundary M_+.

Consequence: The construction holds for groupoid over manifold with corners.

Examples of blowup groupoids

1. Take $G \rightrightarrows G^{(0)}$ a Lie groupoid and $\mathbb{G} = G \times \mathbb{R} \times \mathbb{R} \rightrightarrows G^{(0)} \times \mathbb{R}$.
Recall that $DNC(G, G^{(0)}) = G \times \mathbb{R}^* \cup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R}$.

$$Blup_{r,s}(\mathbb{G}, \mathbb{G}^{(0)} \times \{(0,0)\}) = DNC(G, G^{(0)}) \rtimes \mathbb{R}^* \rightrightarrows G^{(0)} \times \mathbb{R}$$

Gauge adiabatic groupoid [D.-Skandalis]
2. Let $V \subset M$ be a hypersurface,

$$\underbrace{G_b = SBlup_{r,s}(M \times M, V \times V)}_{\text{The b-calculus groupoid}} \subset \underbrace{SBlup(M \times M, V \times V)}_{\text{Melrose's b-space}}$$

$$\underbrace{G_0 = SBlup_{r,s}(M \times M, \Delta(V))}_{\text{The 0-calculus groupoid}} \subset \underbrace{SBlup(M \times M, \Delta(V))}_{\text{Mazzeo-Melrose's 0-space}}$$

Iterate these constructions to go to the study of manifolds with corners [Monthubert]. Or consider a foliation with no holonomy on V [Rochon]. Define the holonomy groupoid of a manifold with iterated fibred corners [D.-Lescure-Rochon].

Exact sequences coming from deformations and blowups

Let $\Gamma \rightrightarrows V$ be a closed Lie subgroupoid of a Lie groupoid $G \rightrightarrows M$, suppose that Γ is amenable and let $\mathring{M} = M \setminus V$. Let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of the groupoid $\mathcal{N}_\Gamma^G \rightrightarrows \mathcal{N}_V^M$ to $\mathring{\mathcal{N}}_V^M = \mathcal{N}_V^M \setminus V$.

$$DNC_+(G,\Gamma) = G \times \mathbb{R}_+^* \cup \mathcal{N}_\Gamma^G \times \{0\} \rightrightarrows M \times \mathbb{R}_+^* \cup \mathcal{N}_V^M$$

$$\widetilde{DNC_+(G},\Gamma) = G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^* \cup \mathring{\mathcal{N}}_\Gamma^G \times \{0\} \rightrightarrows \mathring{M} \times \mathbb{R}_+^* \cup \mathring{\mathcal{N}}_V^M$$

$$SBlup_{r,s}(G,\Gamma) = \widetilde{DNC_+(G},\Gamma)/\mathbb{R}_+^* = G_{\mathring{M}}^{\mathring{M}} \cup \mathcal{S}\mathcal{N}_\Gamma^G \rightrightarrows \mathring{M} \cup \mathbb{S}(\mathcal{N}_V^M)$$

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(\widetilde{DNC_+(G},\Gamma)) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \longrightarrow 0$$

Connecting elements

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(DN\widetilde{C_+(G},\Gamma)) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0 \quad \partial_{\widetilde{DNC_+}}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{SN}_\Gamma^G) \longrightarrow 0 \quad \partial_{SBlup}$$

Connecting elements : $\partial_{DNC_+} \in KK^1(C^*(\mathcal{N}_\Gamma^G), C^*(G \times \mathbb{R}_+^*))$, $\partial_{\widetilde{DNC_+}} \in KK^1(C^*(\mathring{\mathcal{N}}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*))$ and $\partial_{SBlup} \in KK^1(C^*(\mathcal{SN}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}}))$.

Connecting elements

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[d, dash, "\mathring{\beta}"'] & C^*(\widetilde{DNC_+(G,\Gamma)}) \arrow[r] \arrow[d, dash, "\beta"'] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] \arrow[d, dash, "\beta^\partial"'] & 0 & \partial_{\widetilde{DNC_+}} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] & C^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] & C^*(\mathcal{SN}_\Gamma^G) \arrow[r] & 0 & \partial_{SBlup}
\end{tikzcd}$$

The β's being KK-equivalences given by Connes-Thom elements.

Connecting elements

$$
\begin{tikzcd}
0 \arrow[r] & C^*(G \times \mathbb{R}_+^*) \arrow[r] & C^*(DNC_+(G,\Gamma)) \arrow[r] & C^*(\mathcal{N}_\Gamma^G) \arrow[r] & 0 & \partial_{DNC_+} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[u, hook, "\mathring{j}"] & C^*(\widetilde{DNC_+(G,\Gamma)}) \arrow[r] \arrow[u, hook, "j"] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] \arrow[u, hook, "j^\partial"] & 0 & \partial_{\widetilde{DNC_+}} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[u, dash, "\mathring{\beta}"] & C^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] \arrow[u, dash, "\beta"] & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \arrow[r] \arrow[u, dash, "\beta^\partial"] & 0 & \partial_{SBlup}
\end{tikzcd}
$$

The j's coming from inclusion.

Connecting elements

$$
\begin{tikzcd}
0 \arrow[r] & C^*(G \times \mathbb{R}_+^*) \arrow[r] & C^*(DNC_+(G,\Gamma)) \arrow[r] & C^*(\mathcal{N}_\Gamma^G) \arrow[r] & 0 & \partial_{DNC_+} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[u, hook, "\mathring{j}"] & C^*(\widetilde{DNC_+(G,\Gamma)}) \arrow[r] \arrow[u, hook, "j"] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] \arrow[u, hook, "j^\partial"] & 0 & \partial_{\widetilde{DNC_+}} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[u, dash, "\mathring{\beta}"] & C^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] \arrow[u, dash, "\beta"] & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \arrow[r] \arrow[u, dash, "\beta^\partial"] & 0 & \partial_{SBlup}
\end{tikzcd}
$$

Proposition

$\partial_{SBlup} \otimes \mathring{\beta} \otimes [\mathring{j}] = \beta^\partial \otimes [j^\partial] \otimes \partial_{DNC_+} \in KK^1(C^*(\mathcal{S}\mathcal{N}_\Gamma^G), C^*(G))$.

Proposition

If $\mathring{M}$ meets all the G-orbits, $\mathring{j}$ is a Morita equivalence - and therefore ∂_{DNC_+} determines ∂_{SBlup}.

Connecting elements

$$
\begin{tikzcd}
0 \arrow[r] & C^*(G\times\mathbb{R}_+^*) \arrow[r] & C^*(DNC_+(G,\Gamma)) \arrow[r] & C^*(\mathcal{N}_\Gamma^G) \arrow[r] & 0 & \partial_{DNC_+} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{M}\times\mathbb{R}_+^*) \arrow[r] \arrow[u, hook, "\mathring{j}"] & C^*(\widetilde{DNC_+}(G,\Gamma)) \arrow[r] \arrow[u, hook, "j"] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] \arrow[u, hook, "j^\partial"] & 0 & \partial_{\widetilde{DNC_+}} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[u, dash, "\mathring{\beta}"] & C^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] \arrow[u, dash, "\beta"] & C^*(\mathcal{SN}_\Gamma^G) \arrow[r] \arrow[u, dash, "\beta^\partial"] & 0 & \partial_{SBlup}
\end{tikzcd}
$$

Proposition

$\partial_{SBlup}\otimes\mathring{\beta}\otimes[\mathring{j}] = \beta^\partial\otimes[j^\partial]\otimes\partial_{DNC_+}\in KK^1(C^*(\mathcal{SN}_\Gamma^G), C^*(G))$.

Proposition

If $\mathfrak{A}_x\to(N_V^M)_x$ is nonzero for every $x\in V$, then $\mathring{j}, j, j^\partial$ are isomorphisms.

Index type connecting elements

$$\begin{tikzcd}
0 \arrow[r] & C^*(G\times\mathbb{R}_+^*) \arrow[r] & \Psi^*(DNC_+(G,\Gamma)) \arrow[r] & \Sigma_{DNC_+} \arrow[r] & 0 & \widetilde{Ind}_{DNC_+} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) \arrow[r] \arrow[u, hook, "\mathring{j}"'] \arrow[d, dash, "\mathring{\beta}"'] & \Psi^*(DN\widetilde{C_+(G},\Gamma)) \arrow[r] \arrow[u, hook, "j"'] \arrow[d, dash, "\beta"'] & \Sigma_{\widetilde{DNC_+}} \arrow[r] \arrow[u, hook, "j^\partial"'] \arrow[d, dash, "\beta^\partial"'] & 0 & \widetilde{Ind}_{\widetilde{DNC_+}} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] & \Psi^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] & \Sigma_{SBlup} \arrow[r] & 0 & \widetilde{Ind}_{SBlup}
\end{tikzcd}$$

Proposition

$$\widetilde{Ind}_{SBlup}\otimes\mathring{\beta}\otimes[\mathring{j}] = \beta^\partial\otimes[j^\partial]\otimes\widetilde{Ind}_{DNC_+} \in KK^1(C^*(\Sigma_{SBlup}), C^*(G)).$$

Proposition

- The β's are K-equivalences.
- If $\mathfrak{A}_x \to (N_V^M)_x$ is nonzero for every $x\in V$, the j's are K-equivalences. $\diamond$

Boutet de Monvel type construction

$G \rightrightarrows M$ is a Lie groupoid, $V \subset M$ a closed transverse submanifold : $G_V^V \rightrightarrows V$ is a Lie groupoid - $\mathring{M} = M \setminus V$ and $\mathring{G} = G_{\mathring{M}}^{\mathring{M}}$.

Following [Debord S. (2014)] : Poisson-trace bimodule $\mathcal{E}_{PT}(G, V)$.

$$\begin{tikzcd}
0 \arrow[r] & C^*(\mathring{G}) \arrow[r] \arrow[d, dash, "\mathcal{E}'_{PT}(G,V)"] & C^*(SBlup_{r,s}(G,V)) \arrow[r] \arrow[d, dash, "\mathcal{E}_{PT}(G,V)"] & C^*(\mathcal{S}(\mathcal{N}_V^G)) \arrow[r] \arrow[d, dash, "\mathcal{E}''_{PT}(G,V)"] & 0 \\
0 \arrow[r] & C^*(G_V^V) \arrow[r] & \Psi^*(G_V^V) \arrow[r] & C(\mathbb{S}^*\mathfrak{A}G_V^V) \arrow[r] & 0
\end{tikzcd}$$

Boutet de Monvel type pdo algebra $\Psi^*_{BM}(G, V)$: algebra of matrices $R = \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix}$ with $\Phi \in \Psi^*(SBlup_{r,s}(G, V))$, $P \in \mathcal{E}_{PT}(G, V)$, $T \in \mathcal{E}_{PT}(G, V)^*$ and $Q \in \Psi^*(G_V^V)$.

Boundary symbol algebra $\Sigma^{\Psi^*}_{bound}(G, V)$: algebra of matrices $\begin{pmatrix} \phi & p \\ t & q \end{pmatrix}$ where $\phi \in \Psi^*(\mathcal{S}N_V^G)$, $q \in C(\mathbb{S}^*\mathfrak{A}G_V^V)$, $p, t^* \in \mathcal{E}''_{PT}$.

Boutet de Monvel type construction

Two symbols:

- *classical symbol* $\sigma_c : \Psi^*_{BM}(G,V) \to C_0(\mathbb{S}^*\mathfrak{A}SBlup_{r,s}(G,V))$, $\sigma_c \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix} = \sigma_0(\Phi)$;
- *boundary symbol* $r_{BM} : \Psi^*_{BM}(G,V) \to \Sigma_{BM}(G,V)$ defined by

$$r_V \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix} = \begin{pmatrix} r_V^{\psi}(\Phi) & r_V^{\propto}(P) \\ r_V^{\backpropto}(T) & \sigma_V(Q) \end{pmatrix}$$

where

- $r_V^{\psi} : \Psi^*(SBlup_{r,s}(G,V)) \to \Psi^*(\mathcal{S}N_V^G)$ is the restriction;
- σ_V: ordinary order 0 principal symbol on the groupoid G_V^V;
- $r_V^{\propto}, r_V^{\backpropto}$: restrictions to the boundary.

Computation of all kinds of connecting maps and index maps...

Thank you for your attention !