# Twisted spin$^c$-bordism & twisted K-homology

(On joint work with Baum, Khorami & Schick)

## §1 Geometry & K-homology: untwisted

$K(X)$, geometrically defined via vector-bundle over space $X$.

$\rightsquigarrow$ generalized cohomology theory

$K^n(X) = [X_+, \mathbb{K}_n]$, $(\mathbb{K}_n)$ spectrum

$\rightsquigarrow$ generalized homology theory

$K_n(X) = \operatorname{colim}_{k\to\infty} [S^{k+n}, X_+ \wedge \mathbb{K}_k]$

Poincaré duality for K-theory

$M$ closed, spin$^c$ $\rightsquigarrow$ $[M]_K \in K_n(M)$

$$K^*(M) \xrightarrow{\cong} K_{n-*}(M)$$
$$x \longmapsto [M]_K \cap x$$

$M$, $\dim M = 2k$, by Bott periodicity

$$K_0(M) \longrightarrow \mathrm{Hom}(K^0(M), \mathbb{Z})$$
$$[M] \longmapsto (E \text{ vbl} \longmapsto \mathrm{ind}(\not{D}_M \otimes E))$$

Atiyah's idea : $K_0(M) =$ set of equivalence classes of some sort of operators like $\not{D}$

⇝ Def'n of $\mathcal{Ell}(X)$ via $C(X)$-modules

⇝ Def'n of KK-theory by Kasparov.

More geometric definitions:

- Baum-Douglas theory $K_*^{BD}(X)$
- Stolz' definition via a quotient of spin$^c$-bordism by suitable bordism classes of bundles (only carried out for KO-theory, so far)

Def. (Baum-Douglas theory)

$K_*^{BD}(X)$ = set of equivalence classes $(M, E, f)$

- $M$ a closed spin$^c$-mfd
- $f: M \to X$ continuous map
- $E$ vector bundle over $M$

The equivalence relation is generated by

- spin$^c$-bordism
- direct sum of vector bundles:

  $(M, E_1 \oplus E_2, f) \sim (M \amalg M, E_1 \amalg E_2, f \amalg f)$
- spherical modification

  (which takes care of Bott periodicity)

Spherical modification:

Given $(M, E, f)$, $W \to M$ spin$^c$-vbl, $\dim W = 2k$

Define: $F_W$ = dual of $S_V^{ev}$, where $S_V$ is the reduced vertical spinor bundle over $S(W \oplus 1) \xrightarrow{\pi} M$.

Then $(M, E, f) \sim (S(W\oplus 1), f \circ \pi, F_W \otimes \pi^* E)$

Thm (Baum-Douglas, 1982, Baum-Higson-Schick, 2007)

$$K_*^{BD}(X) \cong K_*(X)$$

Using the Atiyah orientation

$$\alpha : \Omega_*^{spin^c} \longrightarrow K_*$$

one obtains a canonical map

$$(*) \qquad \Omega_*^{spin^c}(X) \otimes_{\Omega_*^{spin^c}} K_* \longrightarrow K_*(X)$$

Thm (Hovey - Hopkins 1992):

The homomorphism $(*)$ is an isomorphism.

We thus have a triangle of isomorphisms

$$\begin{tikzcd}
\Omega_*^{spin^c}(X) \otimes_{\Omega_*^{spin^c}} K_* \arrow[rr, "\cong"] \arrow[dr, dash, "\cong"'] & & K_*^{BD}(X) \arrow[dl, dash, "\cong"] \\
& K_*(X) &
\end{tikzcd}$$

<u>Note</u>: The horizontal arrow is induced by:

$$(M^n, f) \otimes (E \to S^k) \longmapsto (M^n \times S^k, pr_2^* E, f \circ pr_1)$$

Thus it is somewhat surprising that this map is surjective.

## §2 Twisted analogues

Assume, $M$ is not spin$^c$:

- have no Poincaré duality for $K_*$
- have no canonical Dirac operator on $M$.

$\rightsquigarrow$ General way out: twistings

Recall: definition of spin$^c$-structure

$U(1) \to Spin^c \to SO \to \cdots$

$$\begin{tikzcd}
\rightsquigarrow BU(1) \arrow[r] & BSpin^c \arrow[r] & BSO \arrow[r, "W_3"] & BBU(1) \simeq K(\mathbb{Z},3) \\
 & & M \arrow[ul, dashed, "F"] \arrow[u, "\nu"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Put $B = BBU(1)$. Let $X \xrightarrow{\tau} B$ be a map.

<u>Def.</u> A twisted $\mathrm{spin}^c$ structure for an oriented manifold with a map to $X$ is represented by an isomorphism of stable vector bundles

$$\begin{array}{ccccc}
\nu^s & \xrightarrow{\varphi} & pr_1^*\gamma & \longrightarrow & \gamma \\
\downarrow & & \downarrow & & \downarrow \\
M & \longrightarrow & BSO \times_K X & \xrightarrow{pr_1} & BSO
\end{array}$$

Two such isomorphisms provide the same f-twisted structure, if they differ by a $\mathrm{spin}^c$-isomorphism (we omit the details).

Def. (twisted spin$^c$-bordism for $X \xrightarrow{\tau} B$)

$$\Omega_*^{spin^c}(X;\tau) = \text{twisted spin}^c\text{-bordism classes of pairs } (M, f: M \to X) \text{ with a twisted spin}^c\text{-str.}$$

Remark: There are also other ways to introduce twisted spin$^c$-bordism and the notion of a twisted spin$^c$ structure.

There are also several different ways to introduce twisted K-theory groups $K_*(X;\tau)$ for a space with a reference map $X \to B$.

In any case: there is twisted spin$^c$ Atiyah homomorphism (given by an index)

$$\Omega_*^{spin^c}(X;\tau) \longrightarrow K_*(X;\tau)$$

and it factors through a homomorphism

$$\begin{tikzcd}
\Omega_*^{spin^c}(X;\tau) \arrow[rr] \arrow[dr] & & K_*(X;\tau) \\
& \Omega_*^{spin^c}(X;\tau) \otimes_{\Omega_*^{spin^c}} K_* \arrow[ur] &
\end{tikzcd}$$

## Def. (D-brane homology)

Baum - Carey - Wang, 2013

$K_*^{DB}(X;\tau)$ = set of equivalence classes $(M, E, f)$

- closed manifold $M$ with a map $f: M \to X$
- a twisted spin$^c$-str. for $(M, f)$
- a vector bundle $E$ over $M$

The equivalence relation is generated by

- twisted spin$^c$ bordism
- direct sum of vector bundles
- spherical modifications.

<u>Remark</u> : Baum - Carey - Wang conjectured that

$$K_*^{DB}(X,\tau) \cong K_*(X;\tau)$$

<u>Thm</u> ( Baum - Khorami - J. - Schick )

There is a triangle of isomorphisms

$$\begin{tikzcd}
\Omega_*^{spin^c}(X;\tau) \otimes_{\Omega_*^{spin^c}} K_* \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & K_*^{DB}(X;\tau) \arrow[dl, "\cong"] \\
& K_*(X;\tau) &
\end{tikzcd}$$

Note again: a priori one would expect the horizontal homomorphism to be injective but not to be surjective.

Remark on proof:

- In the theorem one compares three generalized homology theories, which are defined on the category of spaces over a fixed space, namely $B = BBU(1)$.
- We prove the theorem, using techniques from algebraic topology. In particular this allows to prove the statements localized at the individual primes in order to get the integral statement.

## § 3 Aspects of the proof of the twisted version of the Hopkins-Hovey theorem at the prime 2

→ need to dive into deep sea of algebraic topology

- Homology theories for spaces over a fixed base space $B$ can be represented through a sequence of spaces over $B$: $E_n \to B$
- Natural transformations of such theories can be represented by sequences of maps over $B$; i.e. for $n \in \mathbb{N}$ have

$$\begin{tikzcd} E_n \arrow[rr] \arrow[dr] & & E'_n \arrow[dl] \\ & B & \end{tikzcd}$$

In order to obtain representing spaces for the natural transformation $\Omega_*^{spin^c}(X;\tau) \xrightarrow{\alpha} K_*(X;\tau)$ it is useful to use a concrete model for $BBU(1)$

Def. Put $B = BPU = BPU(H)$, $PU(H) = U(H)/S^1 \cdot \mathrm{Id}_H$ for a separable Hilbert space $H$.

The spaces $E_n$ for $\Omega_*^{spin^c}(\;;\;)$, and $K_*(\;;\;)$ and the natural transformation are roughly given by

$$\underset{\overset{\shortparallel}{MSpin^c_{B,n}}}{EPU \times_{PU} PU(H\otimes H_n) \times_{O(n)} S^n} \xrightarrow{\alpha_n} \underset{\overset{\shortparallel}{K_{B,n}}}{EPU \times_{PU} \mathrm{Fred}_{Cl(n)}(H\otimes H_n)}$$

The spaces $MSpin^c_{B,n}$ yield a spectrum $MSpin^c_B$ while the spaces $K_{B,n}$ yield a spectrum $K_B$

Thm (Hebestreit 2013, J.): K-locally there is a homotopy equivalence at 2 of spectra over B

$$MSpin^c_B \xrightarrow{\simeq} \bigoplus_{J \text{ Partition}} \Sigma^{4|J|} k_B$$

where

- $k_B$ is the so-called 0-connected cover of $K_B$
- $\Sigma^n$ denotes formal suspension

(parametrized Anderson-Brown-Peterson-Splitting)

Remark: One can see from the parametrized Anderson Brown-Peterson splitting that a copy of (connected) parametrized K-theory splits off, but this yields just an additive result.

Hopkins and Hovey used the unparametrized version of the Anderson-Brown-Peterson splitting to construct at 2 a sort of $MSpin^c$ resolution of K-theory to prove the Hopkins-Hovey theorem in the unparametrized setting. Their argument is of algebraic nature and can be carried over.

## §4 Sketch of proof of the twisted version of the Hopkins-Hovey theorem away from 2

We build on:

Thm (Khorami 2011): Let $\tau : X \to BPU$, $P = f^* EPU$.

Then there is a isomorphism

$$\boxed{K_*(P) \otimes_{K_*(PU)} K_* \xrightarrow{\cong} K_*(X;\tau)}$$

where $K_*(PU)$ acts on $K_*(P)$ by the multiplication induced by the action $P \times PU \to P$,

and $K_*(PU)$ acts on $K_*$ via the homomorphism

$$K_*(PU) = K_*(BU(1)) \to K_*K \xrightarrow{\mu_*} K_*$$

Using the untwisted Hopkins-Hovey theorem we get

**Cor:**

$$\Omega_*^{spin^c}(P) \otimes_{\Omega_*^{spin^c}(PU)} K_* \overset{!}{\cong} K_*(X;\tau)$$

$$\cong$$

$$\left(\Omega_*^{spin^c}(P) \otimes_{\Omega_*^{spin^c}} K_*\right) \otimes_{\Omega_*^{spin^c}(PU) \otimes_{\Omega_*^{spin^c}} K_*} K_*$$

$$\cong$$

$$K_*(P) \otimes_{K_*(PU)} K_* \xrightarrow{\cong} K_*(X;\tau)$$

Observation: Let $\pi : P = f^* EPU \to X$ be the projection of the PU-bundle over X. Then we have a factorisation:

$$
\begin{tikzcd}
\Omega^{spin^c}_*(P) \otimes_{\Omega^{spin^c}_*(PU)} K_* \arrow[d, "\cong"'] \arrow[r] & K_*(X;\tau) \\
\Omega^{spin^c}_*(P;\tau\circ\pi) \otimes_{\Omega^{spin^c}_*(PU)} K_* \arrow[d, two heads] \arrow[ur, bend right] & \\
\Omega^{spin^c}_*(X;\tau) \otimes_{\Omega^{spin^c}_*} K_* \arrow[uur, bend right] &
\end{tikzcd}
$$

Crucial lemma for the proof away from 2

Let $\Lambda = \mathbb{Z}[\frac{1}{2}]$, then the composition below is split surjective.

$$\Omega_*^{spin^c}(P) \otimes \Lambda \cong \Omega_*^{spin^c}(P, \tau\circ\pi) \otimes \Lambda$$

$$\longrightarrow \Omega_*^{spin^c}(X;\tau) \otimes \Lambda \qquad {}^{*)}$$

Remark: At 2 the homomorphism

$$\Omega_*^{spin^c}(P;\tau\circ\pi) \otimes_{\Omega_*^{spin^c}(PU)} \Omega_*^{spin^c} \longrightarrow \Omega_*^{spin^c}(X;\tau)$$

in general is not a surjection.

*) The argument uses: $BSpin \simeq BSO \times BU(1)$ away from 2

Thank you for your attention!