Disconnecting the G_2 moduli space

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Joint work in progress with
Diarmuid Crowley and Sebastian Goette

C-N, *New invariants of G_2-structures*, Geom. Topol. 19 (2015)
C-G-N, *An analytic invariant of G_2-manifolds*, arXiv:1505.02734

These slides available at
`http://people.bath.ac.uk/jlpn20/disconnect.pdf`

The G_2 moduli space

Let M be a smooth closed 7-manifold admitting metrics with holonomy G_2. The moduli space

$$\mathcal{M} := \{\text{Holonomy } G_2 \text{ metrics on } M\}/\mathsf{Diff}(M)$$

is an orbifold, locally homeomorphic to finite quotients of $H^3_{dR}(M)$.
So far little is known about the *global* properties of $\mathcal{M}$.

Main results:

Exhibit examples of closed G_2-manifolds with $\mathcal{M}$ disconnected, both

- by studying homotopies of G_2-structures, and
- where the G_2-structures are indistinguishable using homotopy theory

Outline:

1. Background and examples
2. Invariants of G_2-structures
3. Constructions
4. Computation

1. Background and examples
The group G_2

$G_2 := \mathrm{Aut}\ \mathbb{O}$, $\quad \mathbb{O} =$ octonions, normed division algebra of real dimension 8.

G_2 acts on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, preserving metric, orientation, cross product

$$a \times b := \mathrm{Im}(ab), \text{ and}$$
$$\varphi_0(a, b, c) := \langle a \times b, c \rangle.$$

In terms of basis $e^1, \dots, e^7 \in (\mathbb{R}^7)^*$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Peculiar algebra facts:

- G_2 is not just contained in stabiliser of φ_0 in $GL(7, \mathbb{R})$, but equality holds.
- The $GL(7, \mathbb{R})$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$.

G_2-structures and holonomy

G_2 is an exceptional case in Berger's list of Riemannian holonomy groups.

A metric with holonomy G_2 is always Ricci-flat.

Parallel tensor fields on Riemannian manifold $M \leftrightarrow$ invariants of $Hol(M)$.

A 3-form $\varphi \in \Omega^3(M^7)$ such that $(T_xM, \varphi) \cong (\mathbb{R}^7, \varphi_0)$ for all $x \in M$ defines a G_2-structure. (This is an *open* condition on φ)

Because $G_2 \subset SO(7)$, this induces a metric and orientation.

$Hol(M) \subseteq G_2 \Leftrightarrow$ metric induced by some G_2-structure φ such that $\nabla\varphi = 0$. Then call φ *torsion-free*. This is equivalent to the first-order non-linear PDE

$$d\varphi = d^*\varphi = 0.$$

Bryant (1985): Local examples

Bryant-Salamon (1987): Complete examples

Joyce (1994): Examples on closed manifolds

Two perspectives on G_2-structures

$$\begin{tikzcd} & Spin(7) \arrow[d] \\ G_2 \arrow[ur, hook] \arrow[r, hook] & SO(7) \end{tikzcd}$$

The spin representation Δ of $Spin(7)$ is real of rank 8.
$Spin(7)$ acts transitively on $S^7 \subset \Delta$ with stabiliser G_2.

G_2	$=$	stabiliser in $GL(7, \mathbb{R})$ of $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$	$=$	stabiliser in $Spin(7)$ of a unit spinor s_0
G_2-structure on M^7	$\leftrightarrow$	positive $\varphi \in \Omega^3(M)$	$\leftrightarrow$	metric g + spin structure + unit spinor field s
Holonomy $\subseteq G_2$	$\Leftrightarrow$	$d\varphi = d^*\varphi = 0$	$\Leftrightarrow$	$\nabla s = 0$
Useful for		differential geometry		homotopy theory

Homotopies of G_2-structures

Let M be a closed 7-dimensional spin manifold.

Given a metric g, the spinor bundle SM is a real vector bundle of rank 8. Two G_2-structures inducing the same metric and spin structure are homotopic if the corresponding unit spinors can be connected by a path of non-vanishing spinors.

All metrics on M are homotopic, so if we fix the spin structure

$$\left\{\begin{matrix}\text{Homotopy classes of} \\ G_2\text{-structures on } M\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Homotopy classes of non-} \\ \text{vanishing sections of } SM\end{matrix}\right\} \leftrightarrow \mathbb{Z}$$

by counting (with signs) the zeros of an interpolating section of a rank 8 bundle on $M \times [0, 1]$.

$\mathrm{Diff}(M)$ can act by non-trivial translations.

Each component of the G_2 moduli space $\mathcal{M}$ maps to a fixed class of G_2-structures modulo homotopies *and* diffeomorphisms.

Classification of 2-connected manifolds

Let M be a closed smooth 7-manifold with $\pi_1(M) = \pi_2(M) = 0$ and $H^4(M)$ torsion-free. Remaining algebraic topology captured by $b_3(M)$.

Let $d(M) :=$ greatest integer dividing $\frac{1}{2}p_1(M) \in H^4(M)$ ($d(M) := 0$ if $p_1(M) = 0$).

Theorem (Wilkens, 1972)

Such M are classified up to homeomorphism by $(b_3(M), d(M)) \in \mathbb{N} \times 2\mathbb{N}$. The number of inequivalent smooth structures on the topological manifold underlying M is

$$\mathrm{GCD}\big(28,\ \mathrm{Numerator}\big(\tfrac{d(M)}{4}\big)\big).$$

Theorem (C-N)

The number of G_2-structures up to homotopy+diffeomorphism on such M is

$$24\,\mathrm{Numerator}\left(\frac{d(M)}{112}\right).$$

A 2-connected example

Example (C-G-N)

Let M be the unique smooth closed 2-connected 7-manifold with $H^4(M) = \mathbb{Z}^{97}$ and $d = 2$.
There are G_2 metrics g_1, g_2, g_3 on M such that

A the G_2-structures φ_1, φ_2 associated to g_1 and g_2 are not equivalent under homotopies and diffeomorphisms; thus g_1 and g_2 are in different components of the G_2 moduli space $\mathcal{M}$

B the G_2-structures φ_1 and φ_3 *are* homotopic, but nevertheless g_1 and g_3 lie in different components of $\mathcal{M}$.

So for this manifold, the moduli space $\mathcal{M}$ has at least 3 connected components.

Ingredients

Invariants

A The G_2-structures are distinguished by a homotopy and diffeomorphism invariant $\nu(\varphi) \in \mathbb{Z}/48\mathbb{Z}$.

B An analytic refinement $\widehat{\nu}(\varphi) \in \mathbb{Z}$ of $\nu(\varphi)$ is invariant under diffeomorphisms and under deformations through torsion-free G_2-structures (but not under arbitrary homotopies), and can distinguish components of $\mathcal{M}$ even when the G_2-structures are homotopic.

Construction

The "twisted connected sum construction" of Kovalev and Corti-Haskins-N-Pacini produces large numbers of 2-connected G_2-manifolds for which the invariants can be evaluated.

A more complicated version produces some 2-connected examples where $\widehat{\nu}$ takes a range of values.

2. Invariants of G_2-structures
The homotopy invariant

Let X closed spin 8-manifold, and $n(X)$ the signed count of zeros of a transverse positive spinor field ($\Leftrightarrow$ Euler class of rank 8 bundle S^+X). Atiyah-Singer index theorem + $Spin(8)$ characteristic class computation

$$\rightsquigarrow \quad -48 \operatorname{ind} D_X^+ = \chi(X) - 3\sigma(X) - 2n(X). \qquad (*)$$

Let W be a compact spin 8-manifold with boundary M, s a transverse positive spinor field on W, and φ the G_2-structure on M induced by $s_{|M}$. Let $n(W, \varphi)$ be the signed count of zeros of s. $(*)$ implies that

$$\nu(\varphi) := \chi(W) - 3\sigma(W) - 2n(W, \varphi) \mod 48$$

is independent of choice of coboundary W.

On a fixed M, ν takes the 24 values allowed by $\nu(\varphi) = \sum_{i=0}^{3} b_i(M) \mod 2$.

If M is 2-connected with $H^4(M)$ torsion-free and d a divisor of 112, then ν distinguishes all classes.

Analytic invariant of G_2-structures

Given a metric on a closed spin M^7, define

$D =$ Dirac operator
$B : \Omega^{ev} \to \Omega^{ev} =$ odd signature operator, $(-1)^k(*d - d*)$ on Ω^{2k}
$h(D) = \dim \ker(D) \in \mathbb{Z}$
$\eta(D) := \eta(D, 0) \in \mathbb{R}$, defined by analytic continuation from

$$\eta(D, s) := \sum_{\lambda \in \mathrm{Spec} D \setminus \{0\}} (\mathrm{sign}\lambda)|\lambda|^{-s} \quad \text{for } \mathrm{Re}\, s \gg 0.$$

For a G_2-structure φ on M, define $MQ(\varphi) \in \mathbb{R}$ in terms of Mathai-Quillen current.

Definition

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$
$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{R}$$

Analytic invariant as refinement

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$

Reversing orientation changes the sign of $\widehat{\nu}_0$.

All terms are continuous in φ, except that the first jumps by 24 when an eigenvalue of D changes between zero and non-zero.

$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{R}$$

$\widehat{\nu}$ is continuous in φ except for jumps by 48.

Theorem (C-G-N)

Let φ be G_2-structure on a closed M^7. Then

$$\nu(\varphi) = \widehat{\nu}(\varphi) \mod 48.$$

(In particular $\widehat{\nu}, \widehat{\nu}_0 \in \mathbb{Z}$.)

Analytic invariant as refinement

$$\widehat{\nu}(\varphi) := -24(\eta + h)(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$
$$\nu(\varphi) := \chi(W) - 3\sigma(W) - 2n(W, \varphi) \in \mathbb{Z}/48\mathbb{Z}.$$

Proof.
For $\partial W = M$ with metric that is product on collar of M

$$\begin{array}{rcccl} \sigma(W) & = & \int_W L(\nabla) & - & \eta(B) \\ \mathrm{ind}\, D_W^+ & = & \int_W \widehat{A}(\nabla) & - & \frac{1}{2}(\eta + h)(D) \\ n(W, \varphi) & = & \int_W e_+(\nabla) & - & MQ(\varphi) \\ & & \text{Chern-Weil term} & & \text{boundary correction} \end{array}$$

Linear combination of Chern-Weil terms gives $\int_W e(\nabla) = \chi(W)$ (essentially by characteristic class formula $(*)$ used to show that ν is well-defined), so

$$\widehat{\nu}(\varphi) = \chi(W) - 3\sigma(W) - 2n(W, \varphi) + 48\, \mathrm{ind}\, D_W^+ \in \mathbb{Z}\,. \qquad \square$$

Analytic invariant of torsion-free G_2-structures

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{Z}$$
$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{Z}$$

For torsion-free φ

- $MQ(\varphi) = 0$
- $h(D) = 1 + b_1(M)$ (so 1 when $Hol = G_2$)
- $\eta(D)$ does not jump

Therefore $\widehat{\nu}_0$ and $\widehat{\nu}$ are constant on connected components of $\mathcal{M}$, and can distinguish components even when the associated G_2-structures are homotopic.

Even if we are only interested in ν, it may be easier to evaluate the intrinsic formula for $\widehat{\nu}$ than to find a spin coboundary to compute ν.

Similarities with *e.g.* the use of Donnelly's analytic refinement of the Eells–Kuiper invariant by Kreck–Stolz and Goette–Kitchloo–Shankar.

3. Constructions
G_2 and $SU(3)$

The action of $SU(3)$ on $\mathbb{C}^3 \cong \mathbb{R}^6$ preserves

$$\omega_0 := \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \in \Lambda^2(\mathbb{R}^6)^*$$
$$\Omega_0 := dz^1 \wedge dz^2 \wedge dz^3 \in \Lambda^3(\mathbb{R}^6)^* \otimes \mathbb{C}$$

On $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$,

$$e^1 \wedge \omega_0 + \mathrm{Re}\,\Omega_0 \;\cong\; e^1 \wedge (e^{23} + e^{45} + e^{67}) + e^{246} - e^{257} - e^{347} - e^{356} \;=\; \varphi_0,$$

the 3-form preserved by G_2.
The stabiliser in G_2 of a non-zero vector is $SU(3)$.

If X is a Calabi-Yau 3-fold (6-manifold with $\mathrm{Hol}(X) = SU(3)$) then $\mathrm{Hol}(S^1 \times X) = SU(3) \subset G_2$, so $S^1 \times X$ has a torsion-free G_2-structure.
But we are more interested in manifolds with full holonomy G_2.

Proposition (Joyce)

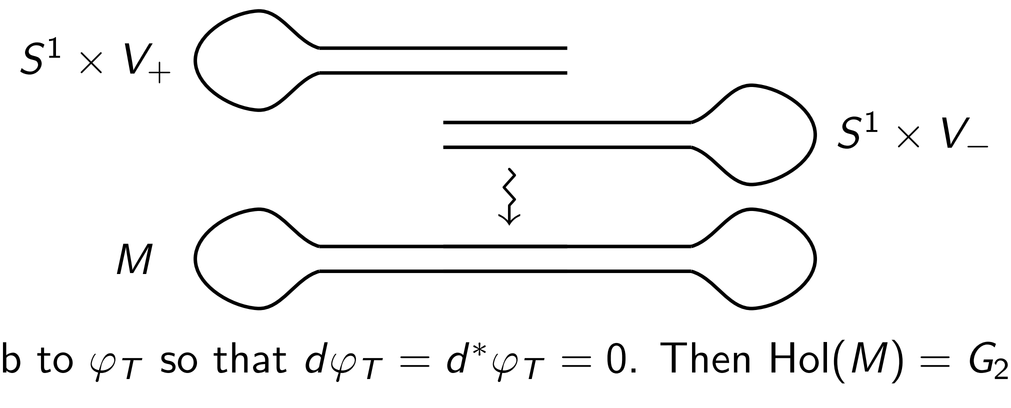
If M^7 is closed and $Hol(M) \subseteq G_2$ then

$$Hol(M) = G_2 \Leftrightarrow \pi_1(M) \textit{ finite}$$

Twisted connected sums

Donaldson, Kovalev, Corti-Haskins-N-Pacini

- Construct simply-connected, complete Calabi-Yau 3-folds V, with "asymptotically cylindrical end" $\mathbb{R} \times S^1 \times K3$.
- $\mathrm{Hol}(S^1 \times V) = \mathrm{SU}(3) \subset G_2$, so $S^1 \times V$ has torsion-free G_2-structure
- Find pairs of such $V_\pm$, with a diffeomorphism F of the cylindrical ends of $S^1 \times V_+$ and $S^1 \times V_-$ ensuring
  - Gluing G_2-structures on the halves with "neck length" $T \gg 0$ defines φ_T on M with $\nabla\varphi_T$ exponentially small in T.
  - $M = S^1 \times V_+ \cup_F S^1 \times V_-$ is simply-connected (F is "twisted")



- Perturb to φ_T so that $d\varphi_T = d^*\varphi_T = 0$. Then $\mathrm{Hol}(M) = G_2$.

Matching

The ACyl end of $S^1 \times V_\pm$ is $\mathbb{R} \times S^1 \times S^1 \times K3_+ \cong \mathbb{R} \times T^2_\pm \times K3_\pm$.
Glue the cylindrical ends using a product isometry

$$F \;:=\; (-1) \times m \times r : \;\mathbb{R} \times T^2_+ \times K3_+ \;\to\; \mathbb{R} \times T^2_- \times K3_-,$$

where $m : T^2_+ \to T^2_-$ is the reflection $S^1 \times S^1 \to S^1 \times S^1$, $(u, v) \mapsto (v, u)$.
m swaps "internal" and "external" circles $\Rightarrow \pi_1 M = 0$ by van Kampen.

Matching problem: Find pairs V_+ and V_- such that there is an isometry $r : K3_+ \to K3_-$ making F an isomorphism of the ACyl G_2-structures.

Kovalev (2003):	Use Fano 3-folds to produce examples of pairs V_+, V_- with solution to the matching problem.
Corti-Haskins-N-Pacini (2014):	Millions of examples from weak Fano 3-folds. Topological type determined in many cases. Many gluings give same smooth manifold.

Invariants of twisted connected sums

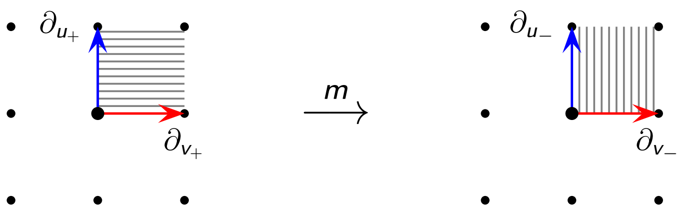
Theorem (C-N)

Any twisted connected sum has $\nu = 24 \in \mathbb{Z}/48\mathbb{Z}$.

Theorem (C-G-N)

Any twisted connected sum has $\widehat{\nu} = -24 \in \mathbb{Z}$.

Analytic computation reveals the result to be related to a geometric feature: $m : T^2_+ \to T^2_-$ aligns "external" circle tangents ∂_v at right angle.



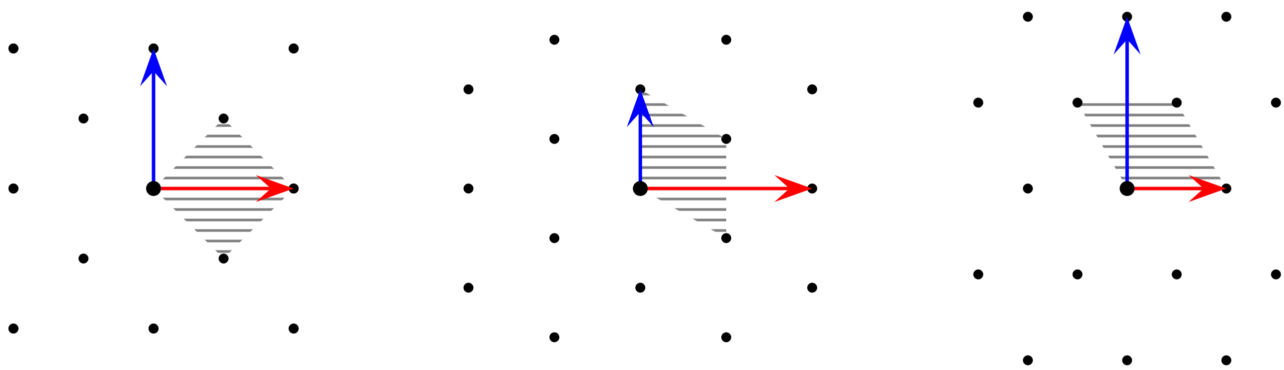
Inevitable, because m is an isometry of rectangular tori, and is not allowed to align the external circles: otherwise M would have an S^1 factor.

Tori with symmetries

Warm-up question:

Let $a\colon S^1 \to S^1$ be the antipodal map $z \mapsto -z$.
Let $T^2 := S^1 \times S^1 \,/\, a \times a$ where the S^1 factors have circumference 1 and x.
For how many different x does T^2 have rotation symmetries other than ± 1?

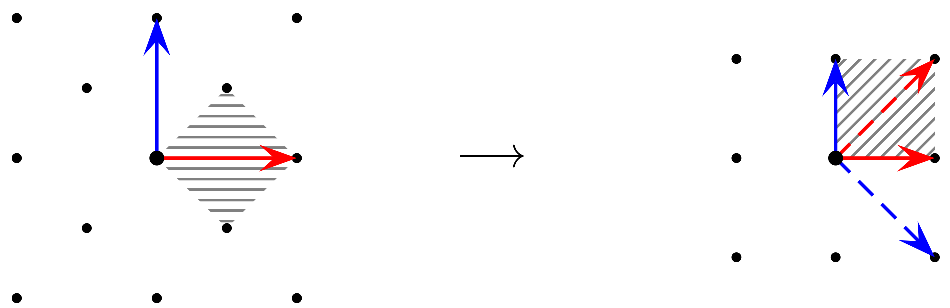
$$x = 1, \sqrt{3}, \text{ or } \frac{1}{\sqrt{3}}$$



Isometries between tori

Consider a pair of tori that are either rectangular (metric product $S^1 \times S^1$) or quotient of a rectangular one by an involution ($S^1 \times S^1 / a \times a$).
For isometries between such tori, at what angles θ can the sides of the rectangles be aligned?

Can achieve $\theta = \dfrac{\pi}{4}$ with an involution on one side.



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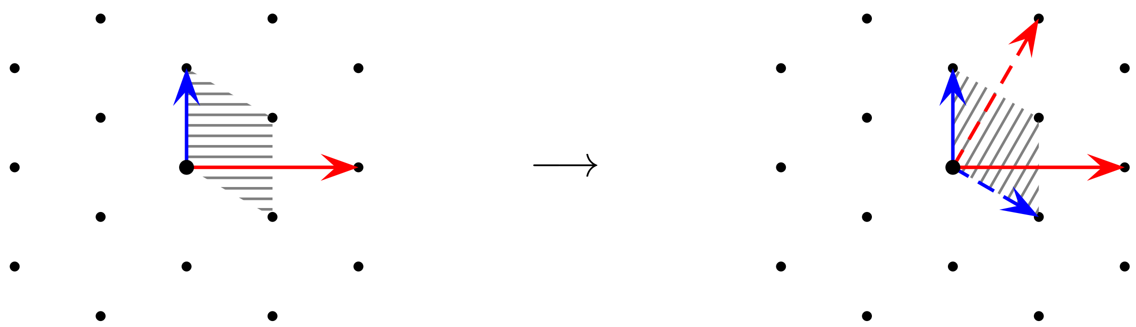
For isometries between such tori, at what angles θ can the sides of the rectangles be aligned?

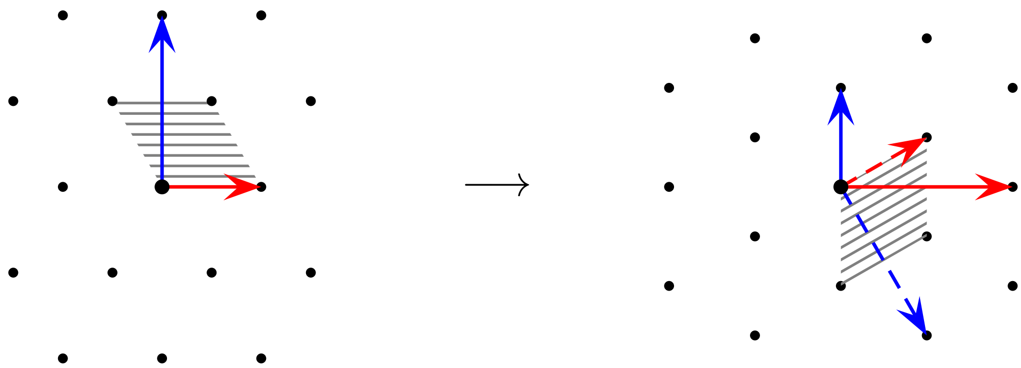
With involutions on both sides, one can achieve $\theta = \frac{\pi}{3}$.



Isometries between tori

Consider a pair of tori that are either rectangular (metric product $S^1 \times S^1$) or quotient of a rectangular one by an involution ($S^1 \times S^1 / a \times a$).
For isometries between such tori, at what angles θ can the sides of the rectangles be aligned?

With involutions on both sides, one can achieve $\theta = \dfrac{\pi}{6}$.



Extra-twisted connected sums

Suppose V is an ACyl Calabi-Yau with an involution τ, that acts on the asymptotic cross-section $S^1 \times K3$ by $a \times \mathrm{Id}_{K3}$.
Then $S^1 \times V \,/\, a \times \tau$ is an ACyl G_2-manifold with cross-section

$$(S^1 \times S^1 \,/\, a \times a) \times K3 = T^2 \times K3.$$

Let $M_\pm$ be a pair of ACyl G_2-manifolds of this form, or of the form $S^1 \times V$.
Let $m : T^2_+ \to T^2_-$ be a reflection. Depending on the circumferences of the circles, m can align the external circle directions at angle $\theta = \frac{\pi}{3}, \frac{\pi}{4}$ or $\frac{\pi}{6}$.

θ-matching problem: Find pairs V_+ and V_- with involution, and with an isometry $r : K3_+ \to K3_-$ such that $(-1) \times m \times r$ is an isomorphism of the limits of the ACyl G_2-structures of M_+ and M_-.

Can obtain some ACyl Calabi-Yau manifolds with involution, and solutions to the matching problem, from branched double covers of Fano 3-folds.

Examples from extra-twisted connected sums

For each $\theta \neq \frac{\pi}{2}$, a range of values of $\widehat{\nu}$ can be realised by θ-TCSs.

Claim A

For a certain $\frac{\pi}{4}$-TCS M_2, compute that $\pi_2 M_2 = 0$, $H^4(M_2) \cong \mathbb{Z}^{97}$, $d(M_2) = 2$, and $\nu(\varphi_2) = 36 \in \mathbb{Z}/48\mathbb{Z}$.

Among the millions of 2-connected ordinary TCS, find one that also has $H^4(M_1) = \mathbb{Z}^{97}$ and $d(M_1) = 2$. By the classification of 2-connected 7-manifolds, it is diffeomorphic to M_2.

However, the ordinary TCS has $\nu(\varphi_1) = 24 \in \mathbb{Z}/48\mathbb{Z}$, so the G_2-structures φ_1 and φ_2 are not homotopic (not even after changing the diffeomorphism that identifies M_1 and M_2). Hence the constructed G_2-metrics g_1 and g_2 lie in different components of the G_2 moduli space on M.

Examples from extra-twisted connected sums

Claim B

For a certain $\frac{\pi}{6}$-TCS M_3, compute that $\pi_2 M_3 = 0$, $H^4(M_3) \cong \mathbb{Z}^{97}$, $d(M_3) = 2$, and $\widehat{\nu}(\varphi_3) = -72 \in \mathbb{Z}$.

M_3 is thus diffeomorphic to M_1 above.

On this manifold, there are precisely $24\,\mathrm{Numerator}(\frac{d}{112}) = 24$ classes of G_2-structures modulo homotopy and diffeomorphism, all distinguished by ν.

Since $\nu(\varphi_1) = \nu(\varphi_3) = 24 \in \mathbb{Z}/48\mathbb{Z}$, the diffeomorphism $M_1 \cong M_3$ can therefore be chosen so that the torsion-free G_2-structures are homotopic.

However, the ordinary TCS has $\widehat{\nu}(\varphi_1) = -24$, so the two torsion-free G_2-structures lie in different components of the G_2 moduli space.

$\frac{\pi}{3}$-TCSs have 3-torsion in $H^4(M)$, making it harder to apply classification results to find different examples realising the same smooth manifold.

4. Computation
Limits of the eta invariants

$M_\pm := S^1 \times V_\pm$ or $S^1 \times V_\pm/a \times \tau$, with asymptotic limit $\mathbb{R} \times T^2_\pm \times K3$.
$m : T^2_+ \to T^2_-$ reflection, aligning external circle factors at angle $\theta \in (0, \frac{\pi}{2}]$.
Construct family of torsion-free G_2-structures φ_T on M the result of gluing M_+ to M_- by $(-1) \times m \times r$ with "neck length" T.

Theorem
Let $\rho := \pi - 2\theta$. Then $\eta(D) \to \frac{\rho}{\pi}$ as $T \to \infty$.

Let $R_\pm : H^2(K3;\mathbb{R}) \to H^2(K3;\mathbb{R})$ be reflection in $\mathrm{Im}(H^2(V_\pm) \to H^2(K3))$

Theorem
Define a unitary map $U : H^2(K3;\mathbb{C}) \to H^2(K3;\mathbb{C})$ by $e^{\pm i\rho}R_+R_-$ on $H^{2,\pm}(K3;\mathbb{C})$. Then

$$\eta(B) \to \frac{1}{\pi} \sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda$$

as $T \to \infty$, where the branch of arg *takes values in $(-\pi, \pi)$.*

Evaluating $\widehat{\nu}$

$U := e^{\pm i\rho} R_+ R_-$ on $H^{2,\pm}(K3;\mathbb{C})$. The theorems imply

$$\widehat{\nu}_0 = -24\eta(D) + 3\eta(B) = -24\frac{\rho}{\pi} + \frac{3}{\pi} \sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda.$$

If $\theta = \frac{\pi}{2}$ then $\rho = \pi - 2\theta = 0$, and U is the real orthogonal map $R_+ R_-$. Hence eigenvalues are ± 1 or occur in conjugate pairs, so $\sum \arg \lambda = 0$, and

$$\widehat{\nu}_0 = 0.$$

In general

$$\sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda \;=\; \sum \pm\rho \;+\; \sum_{\substack{\lambda \in \mathrm{Spec}\, R_+ R_- \\ \lambda \neq -1}} \arg \lambda \;+\; \pi\, b \;=\; -16\rho + \pi\, b,$$

where $b \in \mathbb{Z}$ counts "half branch jumps" between λ and $e^{\pm i\rho}\lambda$. Then

$$\widehat{\nu}_0 = -72\frac{\rho}{\pi} + 3b.$$

Sketch proof of theorem for $\eta(B)$

Theorem

$$\eta(B) \to \frac{1}{\pi} \sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda$$

as $T \to \infty$, for $U := e^{\pm i\rho} R_+ R_-$ on $H^{2,\pm}(K3; \mathbb{C})$.

The proof relies on

Kirk-Lesch gluing formula:

$$\eta(B) \ \to \ \eta(B_+) \ + \ \eta(B_-) \ + \ \text{Maslov index}$$

as $T \to \infty$, for $B_\pm$ the odd signature operators on manifolds with boundary.

Because $M_\pm$ have an S^1-factor they have an orientation-reversing isometry. Therefore $B_\pm$ has spectral symmetry, so $\eta(B_\pm) = 0$!

Hence it remains only to evaluate the Maslov index.

The Maslov index

Consider $H^3(T^2 \times K3)$ as a complex vector space, with complex structure $*$.

$$\text{Maslov index} := \frac{1}{\pi} \sum_{\substack{\lambda \in \mathrm{Spec}(-\widetilde{R}_+ \widetilde{R}_-) \\ \lambda \neq -1}} \arg \lambda$$

where $\widetilde{R}_\pm$ is reflection of $H^3(T^2 \times K3)$ in the image of $H^3(M_\pm)$.
Thus it suffices to prove that $-\widetilde{R}_+ \widetilde{R}_-$ has the same spectrum as U.

$$H^3(T^2 \times K3) \cong H^1(T^2) \otimes H^2(K3) \cong \mathbb{C} \otimes H^{2,+}(K3) \oplus \overline{\mathbb{C}} \otimes H^{2,-}(K3).$$

$\widetilde{R}_\pm \cong R_{\partial_{u_\pm}} \otimes R_\pm$, where
$R_{\partial_{u_\pm}} : H^1(T^2) \to H^1(T^2)$ is reflection in internal circle direction of $M_\pm$
$R_\pm : H^2(K3) \to H^2(K3)$ is reflection in $\mathrm{Im}(H^2(V_\pm) \to H^2(K3))$ as before.

$-R_{\partial_{u_+}} R_{\partial_{u_-}}$ is rotation by $\rho = \pi - 2\theta$. Therefore on $H^{2,\pm}(K3; \mathbb{C})$

$$-\widetilde{R}_+ \widetilde{R}_- \cong e^{\pm i\rho} R_+ R_-$$