Singular foliations, examples and constructions

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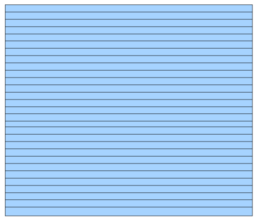
1. Regular foliations
1.1 Definitions

Partition into connected submanifolds local picture:



In other words: there is an open cover of M by foliation charts of the form $U \times T$ where $U \subset \mathbb{R}^p$ and $T \subset \mathbb{R}^q$. T is the *transverse direction* and U is the leafwise direction. Change of charts: $f(u,t) = (g(u,t), h(t))$.

Vectors tangent to the leaves: subbundle F of the tangent bundle.
Integrable subbundle: X, Y vector fields tangent to F, Lie bracket $[X, Y]$ is tangent to F.

Conversely

Frobenius theorem

Every integrable subbundle of the tangent bundle corresponds to a foliation.

Examples of singular foliations

1. $\mathbb{R}$ foliated with 3 leaves. $(-\infty, 0)$, $\{0\}$ and $(0, +\infty)$.
2. $\mathbb{R}^2$, ($\mathbb{R}^n$ for $n \geq 2$) foliated with 2 leaves. $\{(0,0)\}$ and $\mathbb{R}^2 \setminus \{(0,0)\}$.
3. $\mathbb{R}^3$ ($\mathbb{R}^n$ for $n \geq 2$) with co-centric spheres: $\{x \in \mathbb{R}^3;\ \|x\|_2 = r\}$ ($r \in \mathbb{R}_+$).

Remark. Will not allow any partition into manifolds: foliations have to be defined by vector fields.

Not allowed:
$\mathbb{R}^2$ partitioned into $\mathbb{R} \times \{0\}$ and $\{x\} \times (-\infty, 0)$ and $\{x\} \times (0, +\infty)$

Definition of a singular foliations

Theorem (Stefan Sussman)

Let $\mathcal{F} \subset C_c^\infty(M; TM)$ be a submodule such that

1. $\mathcal{F}$ is locally finitely generated
2. if $X, Y \in \mathcal{F}$, then $[X, Y] \in \mathcal{F}$ integrability condition.

Then $\mathcal{F}$ gives rise to a nice partition into leaves.

Definition (Stefan Sussman)

A singular foliation is a partition into leaves associated with such a submodule $\mathcal{F} \subset C_c^\infty(M; TM)$.

Different submodules can give the same partition into leaves: $x\partial_x$ or $x^2\partial_x$...

Our definition of a singular foliation

Definition (Androulidakis-S)

A singular foliation is a locally finitely generated, integrable submodule $\mathcal{F} \subset C_c^{\infty}(M; TM)$.

For us $x\partial_x$ and $x^2\partial_x$ define different foliations.

This because we wish to do index theory along the foliation. We need to know:

1. which are the order 1 differential operators allowed;
2. what is the ellipticity involved.

Very singular foliations

- Constant rank: regular foliation *i.e.* integrable subbundle of the tangent bundle.
- Projective submodule: algebroid with anchor "almost injective" ($\natural_x : \mathfrak{A}_x \to T_x M$ injective on a dense set): nice singular foliation. Debord: integrates to a Lie groupoid.
- just finitely generated: very singular foliation.

Theorem (Androulidakis-S, 2009-2011)

Construction of

- a holonomy groupoid which is a very singular Lie groupoid (longitudinally smooth Debord)
- full and reduced $C^*(M, \mathcal{F})$

Moreover, elliptic operators are regular unbounded multipliers, index takes values in K-theory of $C^*(M, \mathcal{F})$...

Why these constructions?

Why the C*-algebra?

- Contains the resolvants of elliptic differential operators.
- Puts together representations on $L^2(M)$ and on $L^2(leaf)$ of say longitudinal laplacian.
- Essential self-adjointness results, equality of spectra, possible shapes of the spectrum...

Why the groupoid?

- Allows to construct the C*-algebra!
- $f(D)$ lives on this groupoid if D elliptic self-adjoint order 1 differential operator, f Schwartz function on $\mathbb{R}$, such that $\hat{f}$ compact support.

The holonomy groupoid

The elements of the holonomy groupoid are germs of (local) diffeomorphisms φ in the subgroup $\exp(\mathcal{F})$ of $\mathit{Diff}(M)$ generated by exponentials of elements of $\mathcal{F}$, divided by an equivalence relation.

- If just one leaf ($\mathcal{F} = C_c(M; TM)$), then $G = M \times M$ and we only retain the pair $(\varphi(x), x)$.
- If regular foliation, we just retain the (germ of the) action of φ on the transversal.

Definition

(x, φ) is the trivial element if $\varphi \in \exp(I_x \mathcal{F})$ where I_x is the ideal of smooth functions vanishing at x.

Topology?

Bi-submersions

In the case of a regular foliation:

- $U \times T$ and $U' \times T'$ foliated charts.
- $h : T \to T'$ a holonomy.
- A chart of the holonomy groupoid: $U' \times U \times T$.
- Source and range maps $s(u', u, t) = (u, t)$ and $r(u', u, t) = (u', h(t))$.

Cannot hope for a manifold with a local diffeomorphism to G. We look for manifolds with a submersion to G. We will call them bi-submersions.

A bi-submersion is a manifold U, with submersions $U \overset{r,s}{\rightrightarrows} M$, with a submersion-family from the fibers of s to the leaves of the foliation.

Formally

Definition

A bi-submersion (U, r, s) where $U \overset{r,s}{\rightrightarrows} M$ submersions such that

$$r^*(\mathcal{F}) = s^*(\mathcal{F}) = C_c^\infty(U; \ker dr) + C_c^\infty(U; \ker ds).$$

Bi-submersions and the holonomy groupoid

- Inverse of a bi-submersion (U, r, s): (U, s, r).
- Composition of bi-submersions: fibered product.
- Existence of bi-submersions: small exponentiation of $\mathcal{F}$ around a point.

Equivalence relation. (U, r, s) at $u \in U$ equivalent to (U', r', s') at $u' \in U'$ if $\exists f : U \to U'$ over r and s with $f(u) = u'$.

Holonomy groupoid: Equivalence quotient. The holonomy groupoid is locally a quotient of a smooth manifold...

Examples

1. $\mathbb{R}$ split in three leaves. Various $\mathcal{F}_n$ generated by $x^n\partial_x$. Different groupoids (but similar C^*-algebras). Of course, one can legitimately choose $\mathcal{F}_1$. Groupoid $\mathbb{R} \rtimes \mathbb{R}_+^*$.
2. $\mathbb{R}^2$ foliated with two leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.

   We put $\mathcal{F} =$ vector fields that vanish at $\{0\}$.

   Generated by
   - ∂_x, ∂_y far from 0 and
   - $x\partial_x,\ y\partial_x,\ x\partial_y,\ y\partial_y$ at 0.

   Given by the action of $GL(2,\mathbb{R})$ on $\mathbb{R}^2$.

   $Hol(M,\mathcal{F})$ is a quotient of $\mathbb{R}^2 \rtimes GL(2,\mathbb{R})$:

$$Hol(M,\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}) \coprod \{0\} \rtimes GL(2,\mathbb{R})$$

Examples (2)

Remark. Could have taken

- vector fields that vanish at higher (k^{th} ?) order at 0. Dimension $2(k+1)$...
- the action of $SL(2,\mathbb{R})$ or of $\mathbb{C}^*$.

Different foliations.

More generally...
Let $G \subset GL_n(\mathbb{R})$ be a connected subgroup. The transformation groupoid $\mathbb{R}^n \rtimes G$ defines a foliation. Holonomy groupoid is a quotient of $\mathbb{R}^n \rtimes G$.

If the action on $\mathbb{R}^n \setminus \{0\}$ transitive, the holonomy groupoid is

$$(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}) \coprod \{0\} \rtimes G.$$

If the action of G on $\mathbb{R}^n \setminus \{0\}$ is not transitive...

3. Action of $SO(3)$ on $\mathbb{R}^3$. Groupoid

$$SO(3) \times \{0\} \coprod \{(x,y) \in (\mathbb{R}^3 \setminus 0) \times (\mathbb{R}^3 \setminus 0);\ \|x\|_2 = \|y\|_2\}.$$

4. Let $G_n =$ upper triangular matrices with positive diagonal. Invariant subspaces $\{0\} = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_{n-1} \subset E_n = \mathbb{R}^n$.

   For $0 < k \leq n$, let $\Omega_k = \mathbb{R}^n \setminus E_{k-1}$ and $Y_k = E_k \setminus E_{k-1}$. The set Y_k consists of two G orbits. Put $Y_0 = \{0\}$ and $\Omega_0 = \mathbb{R}^n$.

   Let also $p_k : \mathbb{R}^n \to \mathbb{R}^n/E_k = \mathbb{R}^{n-k}$. Let then

$$\mathcal{G}_k = \{(x,g,y) \in \Omega_k \times G_{n-k} \times \Omega_k;\ p_k(x) = gp_k(y)\} \rightrightarrows \mathcal{G}_k.$$

Proposition

The holonomy groupoid of the foliation of $\mathbb{R}^n$ is a union $\coprod_{j=0}^n (\mathcal{G}_j)|_{Y_j}$.

Other examples of the same flavor

Remarks

1. Use any parabolic subgroup instead of the minimal parabolic G_n.
2. Let $P_1, P_2 \subset GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ subgroups, and let $P_1 \times P_2$ act on $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ by left and right multiplication.
   - If $P_1 = P_2 = GL_n$ classes labeled by rank.
   - If $P_1 = P_2 =$ upper triangular matrices, orbits labeled $\mathfrak{S}_n$ (Bruhat decomposition).
3. One may write projective (or spherical) analogues of these examples.

Baum-Connes conjecture?

Why a Baum-Connes conjecture? We constructed a C^*-algebra which is a receptacle for index problems. Is there a guess of what this K-theory should be?

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- Set a general framework where such a conjecture can be formulated: we assume the foliation to be nicely described by a sequence of groupoids.
- Prove it in some cases... Essentially when the groupoids are amenable.

Remark

Even in the regular case, there are counterexamples to the Baum-Connes conjecture (Higson-Lafforgue-S), so we cannot hope our BC conjecture to hold...

Computation of examples

1. The $SO(3)$ action on $\mathbb{R}^3$. $Hol(\mathbb{R}^3, \mathcal{F}) = \mathbb{R}_+^* \times (\mathbb{S}^2 \times \mathbb{S}^2) \coprod SO(3)$.

   Exact sequence:

   $$0 \to C_0(R_+^*) \otimes \mathcal{K} \to C^*(M, \mathcal{F}) \to C^*(SO(3)) \to 0$$

   Also the mapping cone of $C^*(SO(3)) \to \mathcal{K}(L^2(\mathbb{S}^2))$.
   $K_0(C^*(M, \mathcal{F}))$ is the kernel of the K-theory map $\mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}$ ($K_1 = 0$).

2. $G = SL(n, \mathbb{R})$ or $GL(n, \mathbb{R})$ or... acting on $\mathbb{R}^n$ with two orbits 0 and $\mathbb{R}^n \setminus \{0\}$: $0 \to \mathcal{K} \to C^*(M, \mathcal{F}) \to C^*(G) \to 0$ (full C^*-algebras).
   Diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{R}^n \setminus \{0\}) \rtimes G \arrow[r] \arrow[d, "0"] & C_0(\mathbb{R}^n) \rtimes G \arrow[r] \arrow[d] & C^*(G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & C^*(\mathbb{R}^n, \mathcal{F}) \arrow[r] & C^*(G) \arrow[r] & 0
\end{tikzcd}$$

$$K_*(C^*(\mathbb{R}^n, \mathcal{F})) = \mathbb{Z} \oplus K_*(C^*(G)).$$

Thank you for your attention.