Counterintuitive approximations

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The approximation theorem

Approximation

low regularity flexibility	$\leftrightarrow$	high regularity rigidity

Usually:
Approximate mathematical objects by ones with better regularity
(mollifiers, finite-dimensional approximations, ...)

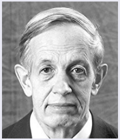
Here:
Approximate maps by ones with less regularity
plus additional (unexpected) properties

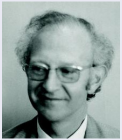
Prototypical example: Nash-Kuiper embedding

Theorem (Nash 1954, Kuiper 1955)

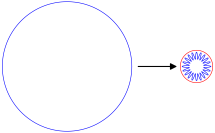
Let M^n be a compact Riemannian manifold and $f : M \hookrightarrow \mathbb{R}^{n+1}$ a short C^∞-embedding.
Then for any $\varepsilon > 0$ $\exists$ embedding $g : M \hookrightarrow \mathbb{R}^{n+1}$ s.t.

- $|f - g| < \varepsilon$;
- g is C^1;
- g is isometric!





In particular: M^n embedds isometrically into small ball.
Trivial for $n = 1$:



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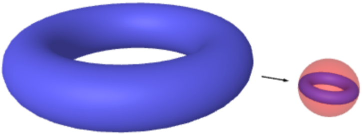
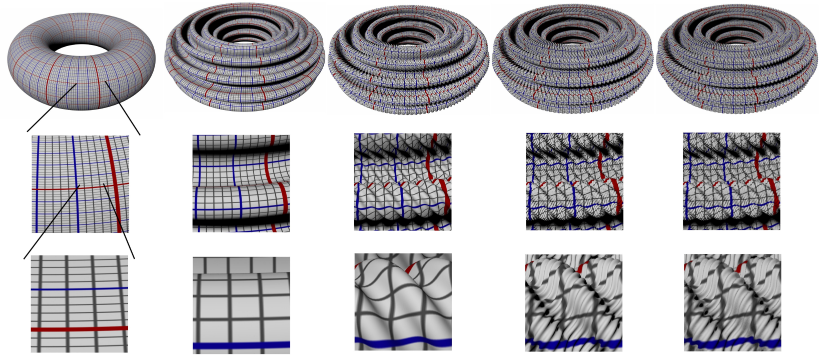
False for $n \geq 2$? If C^1 is replaced by C^2:



True for C^1:



Nash-Kuiper embedding

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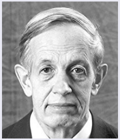
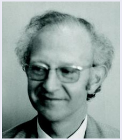




- False if C^1 replaced by C^2.
- Cao, Székelyhidi (2019): True if C^1 replaced by $C^{1,\alpha}$ with $\alpha < \frac{1}{n^2+n+1}$ (and $\alpha < \frac{1}{5}$ for $n = 2$).
- Borisov (1959): False if $n = 2$ and C^1 replaced by $C^{1,\alpha}$ with $\alpha > 2/3$.

Our setting

- V a smooth manifold;
- $\pi : X \to V$ a smooth vector bundle;
- $k \in \mathbb{N}$ a positive integer;
- f a C^k-section on V;
- Γ a subsheaf of the sheaf of C^k-sections of X;
- $\pi_k : J^k X \to V$ k-jet bundle of X;
- $j^k f(p)$ k-jet of f at $p \in V$;

Commutative diagram:

$$\begin{tikzcd} J^k X \arrow[rr, "\pi_{k,k-1}"] \arrow[dr, "\pi_k"'] & & J^{k-1} X \arrow[dl, "\pi_{k-1}"] \\ & V & \end{tikzcd}$$

Define

$$J^k\Gamma := \{j^k\gamma(p) \mid \gamma \text{ local section of } \Gamma \text{, defined near } p, p \in V\} \subset J^k X$$

The approximation theorem

Theorem 1 (B.-Hanke 2019)

Assume $\forall\, p \in V$ $\exists$ open neighborhood W of $j^{k-1}f(p)$ in $J^{k-1}X$ and a continuous map $\sigma_W : W \to J^k X$ s.t.

- $\pi_{k,k-1} \circ \sigma_W = \mathrm{id}_W$;
- $\sigma_W(\omega) \in J^k\Gamma$ for each $\omega \in W$.

Then $\exists$ section $\hat{f}$ of $X \to V$ s.t.:

- $\hat{f}$ is C^{k-1}-close to f;
- $\hat{f} \in C^{k-1,1}_{\mathrm{loc}}(V, X)$;
- $\hat{f}|_{\mathscr{U}} \in \Gamma(\mathscr{U})$ for some open and dense $\mathscr{U} \subset V$.

False if $C^{k-1,1}_{\mathrm{loc}}$ is replaced by C^k.

▸ Theorem 2

Examples

Example 1: Lipschitz functions

- $V = \mathbb{R}$;
- $\pi : X \to V$ trivial line bundle;
- $k = 1$;
- Γ sheaf of locally constant C^∞-functions on $\mathbb{R}$;
- f C^1-function on $\mathbb{R}$;

Put $W := J^0 X = X = \mathbb{R} \times \mathbb{R}$ and $\sigma_W(p, \xi) = j^1(t \mapsto \xi)(p)$.

Corollary 1

Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and $\varepsilon > 0$. Then $\exists\ \hat{f} : \mathbb{R} \to \mathbb{R}$ s.t.

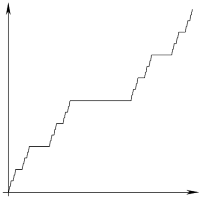
- $|f - \hat{f}| < \varepsilon$;
- $\hat{f}$ is locally Lipschitz on $\mathbb{R}$;
- $\hat{f}$ is smooth and $\hat{f}' = 0$ on an open dense subset of $\mathbb{R}$.

Example 1: Lipschitz functions

Apply to $f(t) = t$ and restrict to $[0, 1]$. Get Lipschitz function

$\hat{f} : [0, 1] \to \mathbb{R}$ s.t. $\hat{f}(0) = 0$, $\hat{f}(1) = 1$, $\hat{f}' = 0$ on open dense subset

Comparison with Cantor function:



	our $\hat{f}$	Cantor function
regularity	$C^{0,1}$	C^α with $\alpha = \ln(2)/\ln(3)$
fund. thm. applies	yes	no
$f' = 0$ on	open dense	open dense of full measure

Open dense subsets need not have full measure

Example

Enumerate $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$. Put

$$\mathcal{U} := \bigcup_{j=1}^{\infty} (q_j - \varepsilon_j, q_j + \varepsilon_j)$$

Then $\mathcal{U} \subset \mathbb{R}$ is open and dense and

$$|\mathcal{U}| \leq 2 \sum_j \varepsilon_j$$

Example 2: Riemannian metrics

- V any manifold of dimension ≥ 2;
- $\pi : X \to V$ bundle of symmetric (0,2)-tensors;
- $k = 2$;
- f a C^2-metric on V;
- Γ sheaf of smooth metrics with sectional curvature $\equiv K$ for given $K \in \mathbb{R}$;

Pick local chart $(U, x^1, \ldots, x^n)$ on V. Put

$$W := \{\omega \in \pi_1^{-1}(U) \mid \pi_{1,0}(\omega) \text{ is positive definite}\}.$$

Express 1-jet $\omega \in \pi_1^{-1}(U)$ as $\omega = \omega_0 + \sum_j \omega_j x^j$.
Associate the metric h_ω given by $h_\omega = \omega_0 + \sum_j \omega_j x^j$. Put

$$\sigma_W(\omega) := j^2\Big(\big((\exp_p^{h_\omega})^{-1}\big)^* g_{T_pV}^{[K]}\Big)(p) \in J^2\Gamma.$$

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Example 2: Riemannian metrics

Corollary 2

Let V be a smooth manifold with a C^2-Riemannian metric g. Then there exists a Riemannian metric $\hat{g}$ on V with the following properties:

- $\hat{g}$ is C^1-close to g;
- $\hat{g}$ has local $C^{1,1}$-Lipschitz regularity;
- the curvature tensor of $\hat{g}$ exists as an L^∞_{loc}-tensor field and $\mathrm{sec}_{\hat{g}} \equiv K$ on an open dense subset of V.

How about Gauss-Bonnet?

Corollary 3

Each differentiable manifold of dimension ≥ 2 has a complete $C^{1,1}$-Riemannian metric with curvature $\equiv 1$ (and others with curvature $\equiv 0$ and $\equiv -1$, resp.) on an open dense subset.

If V is a compact surface then the Gauss-Bonnet theorem

$$\int_V K\, dA = 2\pi\chi(V)$$

holds for these metrics!

Example 3: Embeddings of surfaces

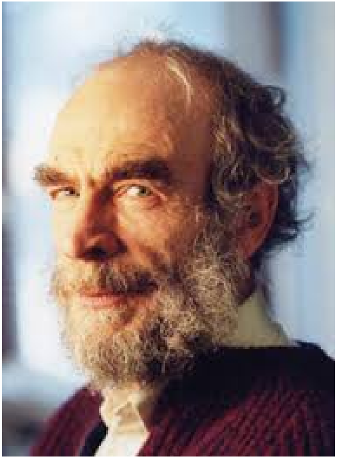
- V analytic 2-dimensional manifold;
- $\pi : X \to V$ trivial $\mathbb{R}^3$-bundle;
- $k = 2$;
- f a C^2-embedding $V \hookrightarrow \mathbb{R}^3$;
- Γ sheaf of analytic embeddings with Gauss curvature $\equiv K$ for given $K \in \mathbb{R}$;

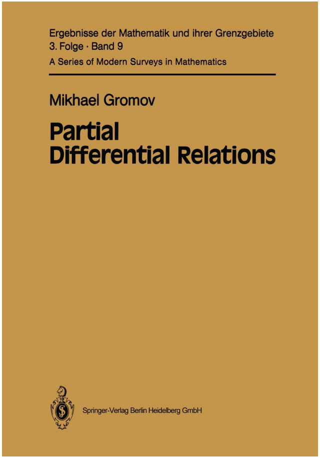
Corollary 4

There exists a $C^{1,1}$-embedding $\hat{f} : V \hookrightarrow \mathbb{R}^3$ C^1-close to f which is analytic on an open dense subset $\mathscr{U} \subset V$ and has constant Gauss curvature K on $\mathscr{U}$ (w.r.t. the induced metric).

The proof

Gromov’s exercise





Gromov's exercise

Exercises. (a) Let $\mathscr{R} \subset X^{(r)} \to X \to V$ be an *open differential relation, let* $V_0 \subset V$ be an arbitrary submanifold and let $f_0 \colon V \to X$ be a C^r-solution of $\mathscr{R}$ [i.e. $J^r_{f_0}(V) \subset \mathscr{R}$]. Let F denote the space of C^r-solutions $f \colon V \to X$ of $\mathscr{R}$, such that $J^{r-1}_f | V_0 = J^{r-1}_{f_0} | V_0$, and let F_0 be the space of jets $\varphi \colon V_0 \to \mathscr{R}$ of such solutions near V_0. That is $\varphi \in F_0$ if and only if there exists a solution $f' \colon \mathcal{O}\!\!\not{p}\, V_0 \to X$ of $\mathscr{R}$ such that $J^{r-1}_{f'} | V_0 = J^{r-1}_{f_0} | V_0$ and for which $J^r_{f'} | V_0 = \varphi$. Prove the following

Weak Flexibility Lemma. *The map $f \mapsto J^r_f V_0$ is a Serre fibration $F \to F_0$.*

Hint. Use the induction in dim V and codim V_0, starting with $\dim V = 1, \dim V_0 = 0$.

(b) Apply (a) to the differential relations $K(g) > 0$, $K(g) < 0$, $S(g) > 0$, and to a closed *geodesic* $V_0 \subset (V, g_0)$. Thus deform a given Riemannian metric g_0 which satisfies one of the above inequalities to a metric g whose sectional curvature is *constant* near V_0, while satisfying the same curvature inequality as g_0 everywhere on V.

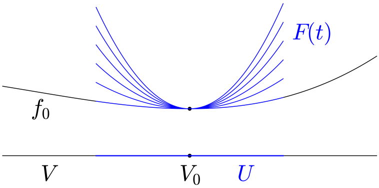
Partial differential relations (PDRs)

- $X \to V$ a vector bundle;
- $k \in \mathbb{N}_0$;
- $J^k X \to V$ the k^{th} jet bundle;
- $\mathscr{R} \subset J^k X$ a subset.

Definition

$\mathscr{R}$ is called a partial differential relation of order k.
A section $u : V \to X$ solves $\mathscr{R}$ if $j^k u(v) \in \mathscr{R}$ for all $v \in V$.

Local flexibility - Setup



- $X \to V$ a vector bundle;
- $\mathscr{R}$ an **open** PDR of order k;
- $V_0 \subset V$ a **closed** subset;
- U an open neighborhood of V_0 in V;
- f_0 a C^k-section on V solving $\mathscr{R}$;
- $F \in C^0([0,1], C^k(U,X))$ s.t. each $F(t)$ solves $\mathscr{R}$ over U.

Moreover, assume

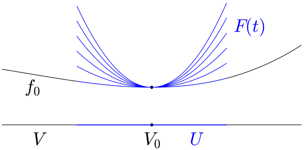
- $f_0|_U = F(0)$;
- $j^{k-1}F(t)|_{V_0} = j^{k-1}f_0|_{V_0}$ for all $t \in [0,1]$.

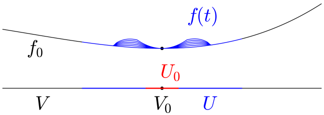
Solution of Gromov's exercise

Theorem 2 (B.-Hanke 2019)

$\exists$ open subset U_0 with $V_0 \subset U_0 \subset U \subset V$ and a continuous $f : [0,1] \to C^k(V, X)$ s.t.

- $f(0) = f_0$;
- $f(t)|_{U_0} = F(t)|_{U_0}$;
- $f(t)|_{V \setminus U} = f_0|_{V \setminus U}$;
- each $f(t)$ solves $\mathscr{R}$.





Theorem 1

Proof of Theorem 1

Pick dense countable subset $\{p_1, p_2, p_3, \ldots\} \subset V$.
Inductively construct f_j and U_j s.t.

- $f_0 = f$ and $U_0 = \emptyset$;
- $U_j \supset U_{j-1}$;
- $p_j \in U_j$;
- $f_j = f_{j-1}$ on U_{j-1};
- $\|f_j - f_{j-1}\|_{C^{k-1}(V)} < 2^{-j} \cdot \varepsilon$;
- $\|f_j - f_{j-1}\|_{C^k(V)} < C + 2^{-j}$;
- $f_j|_{\overline{U_j}} \in \Gamma(\overline{U_j})$.

Then $\hat{f} = \lim_{j\to\infty} f_j$ does the job with $\mathscr{U} = \bigcup_j U_j$.

Thanks for your attention!

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