Macroscopic scalar curvature and volume

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Cortona, July 2019

(Microscopic) Scalar Curvature

(M,g) Riemannian manifold of dim n

$$\mathrm{vol}\, B(p,r) = \underbrace{\omega_n \cdot r^n}_{\substack{\text{volume of} \\ \text{Euclidean } r\text{-ball} \\ \text{in } \mathbb{R}^n}} \left(1 - \frac{\mathrm{scal}_g(p)}{6(n+2)} \cdot r^2 + O(r^3)\right)$$

as $r \to 0$

$\rightsquigarrow$ lower bound on scal $\triangleq$ upper volume bound for tiny balls

Macroscopic Scalar Curvature

Let $S \in \mathbb{R}$.

Every model space $\mathbb{H}^n, \mathbb{E}^n, \mathbb{S}^n$ has a scaling that has scalar curvature S.

$V_S(r) :=$ volume of r-ball in that space.

<u>Def</u>: (M,g) has <u>macroscopic scalar curvature S of scale r</u> at the point p if

in universal covering $\tilde{M}$

$$\mathrm{vol}\, B_{\tilde{g}}(\tilde{p}, r) = V_S(r)$$

Guth's Volume Theorem

(M, hyp) n-dim. closed hyperbolic mfd

g another metric on M.

If $V_{(\tilde{M},\tilde{g})}(1) \leq V_{\mathbb{H}^n}(1)$, maximal volume of 1-ball

then

$$\mathrm{vol}(M,g) \geq \mathrm{const}(n) \cdot \mathrm{vol}(M, hyp).$$

Reformulating Guth's theorem

We fix the scale to be 1 throughout.

Let (M, hyp) be a closed hyp. n-mfd.
Let g be another metric on M.

macroscopic scalar curvature of (M,g) $\geq$ macroscopic scalar curvature of (M, hyp) everywhere

$$\Downarrow$$

$$\mathrm{vol}(M,g) \geq \mathrm{const}(n) \cdot \mathrm{vol}(M, hyp).$$

Context of Guth's theorem

Schoen conjecture

If (M, hyp) is an n-dim. closed hyp. mfd and g is another metric on M with $scal_g \geq scal_{hyp}$, then $vol(M,g) \geq vol(M, hyp)$

Theorem (Besson-Courtois-Gallot)

(M, hyp) and g as above.

If $vol(M,g) < vol(M, hyp)$, then $\exists R_0 \gg 1$ such that $V_{(\tilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R)$ for $R \geq R_0$.

Different scales

Thm:	Schoen	Guth	BCG
Scale:	small	middle (fixed = 1)	large
Conclusion:	upper bound of volume of small balls $\Downarrow$ $vol(M,g) \geqslant vol(M,hyp)$	upper bound of volume of 1-balls $\Downarrow$ $vol(M,g) \geqslant const_n \cdot vol(..)$	upper bound of volume of some large... $\Downarrow$ $vol(M,g) \geqslant vol(..)$

limitation of method

Generalizing Guth to arbitrary manifolds

- want to generalize Guth's result from hyperbolic to arbitrary Riemannian manifolds.
- What plays the role of the hyperbolic volume?

<u>Theorem</u> (Gromov-Thurston)
If M^n is a closed hyperbolic mfd, then

$$\mathrm{vol}(M, hyp) = \frac{\|M\|}{V_n}$$

$\|M\|$ simplicial volume of M

Simplicial Volume

$$\|M\| := \inf \left\{ \sum |c_\sigma| \;\middle|\; \sum c_\sigma \cdot \sigma \text{ singular cycle in } C_n(M;\mathbb{R}) \text{ representing fundamental class} \right\}$$

"Homotopy version of Riemannian volume"

- $\|M\| > 0$ if M admits neg. curved Riem. metric
- $\|M\| = 0$ if $\pi_1(M)$ solvable.
- $\|\overline{M}\| = d \cdot \|M\|$ for a d-sheeted cover $\begin{array}{c}\overline{M}\\ \downarrow \\ M\end{array}$

Main result

Thm (Braun-S.)

Let (M^n, g) be a closed Riemannian manifold. If

$V_{(M,g)}(1) \le V_{\mathbb{H}^n}(1)$, then $\mathrm{vol}(M,g) \geq \mathrm{const}(n) \cdot \|M\|$.

macroscopic scalar curvature version of

Gromov's "main inequality"

(M,g) closed Riem. mfd with $\mathrm{Ricci}(g) \geq -1$.

Then $\mathrm{vol}(M,g) \geq \mathrm{const}(n) \cdot \|M\|$.

Conjecture (Gromov)

can replace lower Ricci curvature bound by lower scalar curvature bound.

Residually finite groups

- Guth's proof method can be easily generalized to manifolds with residually finite fundamental gr.

- Reason: need large systole in proof. Pass to finite cover and use multiplicativity of vol(M), ||M|| for covers.

- $\Gamma = \pi_1(M)$ is residually

  $$\Gamma = \Gamma_0 > \Gamma_1 > \cdots \quad \text{s.th.} \quad \begin{cases} \Gamma_i \triangleleft \Gamma & \text{finite index} \\ \bigcap \Gamma_i = \{1\} \end{cases}$$

  $\Updownarrow$

  residual tower of finite covers

  $$\begin{array}{c} M_i = \Gamma_i \backslash \widetilde{M} \\ \downarrow \\ M = \Gamma \backslash \widetilde{M} \end{array}$$

A surprising feature of the proof

- In general, we still have a replacement for the solenoidal space:

$$\varprojlim \left(M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \cdots \right) =: M_{sol} .$$

- $$M_{sol} = \varprojlim \left(\Gamma \backslash \widetilde{M} \leftarrow \Gamma_1 \backslash \widetilde{M} \leftarrow \Gamma_2 \backslash \widetilde{M} \leftarrow \cdots \right)$$
$$\cong \varprojlim \left(\Gamma / \Gamma_i \times_\Gamma \widetilde{M} \right)$$
$$\cong \left(\varprojlim \Gamma / \Gamma_i \right) \times_\Gamma \widetilde{M}$$

- free Γ-space
- compact, totally disconnected (Cantor set)
- has Γ-invariant measure

Thm from top. dynamics (Elek): Any countable group admits such an action on the Cantor set.

The foliated space

$\Gamma = \pi_1(M) \curvearrowright X$ Cantor set, μ Γ-invariant prob. measure on X

We transfer Guth's methods to the foliated setting:

- $X \times_\Gamma \widetilde{M}$ $\uparrow$ leaf $\{x\} \times M$
- Transversal X
- Transverse measure μ

.... and use it to prove the statement about M!

More on the proof

$\Gamma = \pi_1(M) \curvearrowright X$ as in Elek's thm

$$X \times_\Gamma \widetilde{M} \xrightarrow{\quad ① \quad} X \times_\Gamma N \xrightarrow{\quad ② \quad} X \times_\Gamma E(\Gamma, \mathcal{F}in) \qquad (*)$$

on leaves:
$\{x\} \times \widetilde{M} \longrightarrow \{x\} \times N$
nerve map of
$\mathcal{U}_x = \{ B_i \mid x \in A_i, i \in I \}$

coming from:
classif. map $N \longrightarrow E(\Gamma, \mathcal{F}in)$
where $E(\Gamma, \mathcal{F}in)$ classifying Γ-space for proper actions.

- Construct Γ-equivariant covering $\mathcal{U} = \{ A_i \times B_i \mid i \in I \}$ of $X \times \widetilde{M}$ by
$$\begin{cases} A_i \subset X & \text{clopen} \\ B_i \subset \widetilde{M} & \text{good ball} \end{cases}$$

$B(p,r)$ good if
1) Reasonable growth:
$|B(p,100r)| \leq 10^{4(n+3)} |B(p,100^{-1}r)|$
2) Volume bound: $|B(p,r)| \leq 10^{2n+6} V(1) R^{n+3}$
3) $r \leq 1/100$

- Show that we get an isometric map in measure homology:
$$\mathcal{H}_n(M) \longrightarrow \mathcal{H}_n(X \times_\Gamma \widetilde{M}) \xrightarrow{① \circ ②} \mathcal{H}_n(X \times_\Gamma E(\Gamma, \mathcal{F}in))$$
- Show that ① is leafwise Lipschitz on large volume.