

Macroscopic scalar curvature and volume

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(Microscopic) Scalar Curvature

(M, g) Riemannian manifold of dim n

$$\text{vol } B(p, \tau) = \underbrace{\omega_n}_{\substack{\text{volume of} \\ \text{Euclidean } \tau\text{-ball} \\ \text{in } \mathbb{R}^n}} \cdot \tau^n \left(1 - \frac{\text{scal}|_p}{6(n+2)} \cdot \tau^2 + O(\tau^3) \right) \quad \text{as } \tau \rightarrow 0$$

\implies lower bound on $\text{scal} \triangleq$ upper volume bound for tiny balls

Macroscopic Scalar Curvature

Let $S \in \mathbb{R}$.

Every model space $\mathbb{H}^n, \mathbb{E}^n, \mathbb{S}^n$ has a scaling that has scalar curvature S .

$V_S(r) :=$ volume of r -ball in that space.

Def: (M, g) has macroscopic scalar curvature S of scale r at the point p if

in universal covering \tilde{M} \rightarrow $\text{vol } B_{\tilde{g}}(\tilde{p}, r) = V_S(r)$

Guth's Volume Theorem

(M, hyp) n -dim. closed hyperbolic manifold
 g another metric on M .

If $V_{(M, g)}(1) \leq V_{\mathbb{H}^n}(1)$, *maximal volume of 1-ball*

then

$$\text{vol}(M, g) \geq \text{const}(n) \cdot \text{vol}(M, \text{hyp}).$$

Reformulating Guth's theorem

We fix the scale to be 1 throughout.

Let (M, hyp) be a closed hyp. n -manifold.

Let g be another metric on M .

macroscopic scalar curvature of (M, g) \geq macroscopic scalar curvature of (M, hyp) everywhere



$$\text{vol}(M, g) \geq \text{const}(n) \cdot \text{vol}(M, \text{hyp}).$$

Context of Guth's theorem

Schoen conjecture

If (M, hyp) is an n -dim. closed hyp. mfd and g is another metric on M with $\text{scal}_g \geq \text{scal}_{\text{hyp}}$, then $\text{vol}(M, g) \geq \text{vol}(M, \text{hyp})$

Theorem (Besson-Courtois-Gallot)

(M, hyp) and g as above.

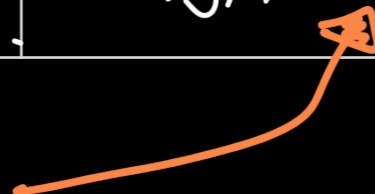
If $\text{vol}(M, g) < \text{vol}(M, \text{hyp})$, then $\exists R_0 \gg 1$

such that $V_{(\tilde{M}, \tilde{g})} (R) > V_{\mathbb{H}^n} (R)$ for $R \geq R_0$.

Different scales

Thm:	Schoen	Guth	BCG
Scale:	small	middle (fixed = 1)	large
Conclusion:	upper bound of volume of small balls ⇓ $\text{vol}(M, g) \geq \text{vol}(M, \text{hyp})$	upper bound of volume of 1-balls ⇓ $\text{vol}(M, g) \geq \text{const} \frac{1}{n} \cdot \text{vol}(\dots)$	upper bound of volume of some large... ⇓ $\text{vol}(M, g) \geq \text{vol}(\dots)$

limitation of
method



Generalizing Guth to arbitrary manifolds

- want to generalize Guth's result from hyperbolic to arbitrary Riemannian manifolds.
- What plays the role of the hyperbolic volume?

Theorem (Gromov-Thurston)

If M^n is a closed hyperbolic manifold, then

$$\text{vol}(M, \text{hyp}) = \frac{\|M\|_{\Delta}}{V_n}$$

$\|M\|_{\Delta}$
simplicial
volume of M

Simplicial Volume

$$\|M\| := \inf \left\{ \sum |c_\sigma| \mid \sum c_\sigma \sigma \text{ singular cycle in } C_n(M; \mathbb{R}) \text{ representing fundamental class} \right\}$$

"Homotopy version of Riemannian volume"

- $\|M\| > 0$ if M admits neg. curved Riem. metric
- $\|M\| = 0$ if $\pi_1(M)$ solvable.

- $\|\bar{M}\| = d \cdot \|M\|$ for a d -sheeted cover $\begin{array}{c} \bar{M} \\ \downarrow \\ M \end{array}$

Main result

Thm (Braun-S.)

Let (M, g) be a closed Riemannian manifold. $\exists \epsilon$

$V_{(M, g)}(1) \leq V_{\| \cdot \|}^n(1)$, then $\text{vol}(M, g) \geq \text{const}(n) \cdot \|M\|$.

macroscopic
scalar curvature
version of

Gromov's "main inequality"

(M, g) closed Riem. mfd with $\text{Ricci}(g) \geq -1$.

Then $\text{vol}(M, g) \geq \text{const}(n) \cdot \|M\|$.


Conjecture (Gromov)

can replace lower Ricci curvature bound by lower scalar curvature bound.

Residually finite groups

- Gull's proof method can be easily generalized to manifolds with residually finite fundamental gr.
- Reason: need large systole in proof. Pass to finite cover and use multiplicativity of $\text{vol}(M)$, $\|M\|$ for covers.

- $\Gamma = \pi_1(M)$ is residually

$$\Gamma = \Gamma_0 > \Gamma_1 > \dots$$


residual tower of finite covers

s.t.h. $\begin{cases} \Gamma_i \triangleleft \Gamma & \text{finite index} \\ \bigcap \Gamma_i = \{1\} \end{cases}$

$$\begin{array}{c} M_i = \Gamma_i \backslash \tilde{M} \\ \downarrow \\ M = \Gamma \backslash \tilde{M} \end{array}$$

A surprising feature of the proof

- In general, we still have a replacement for the solenoidal space:

$$\varprojlim (M \leftarrow M_1 \leftarrow M_2 \leftarrow \dots) =: M_{\text{sol}}.$$

- $M_{\text{sol}} = \varprojlim (\Gamma \backslash \tilde{M} \leftarrow \Gamma_1 \backslash \tilde{M} \leftarrow \Gamma_2 \backslash \tilde{M} \leftarrow \dots)$

$$\cong \varprojlim (\Gamma / \Gamma_i \times_{\Gamma} \tilde{M})$$

$$\cong \left(\varprojlim \Gamma / \Gamma_i \right) \times_{\Gamma} \tilde{M}$$

- free Γ -space
- compact, totally disconnected (Cantor set)
- has Γ -invariant measure

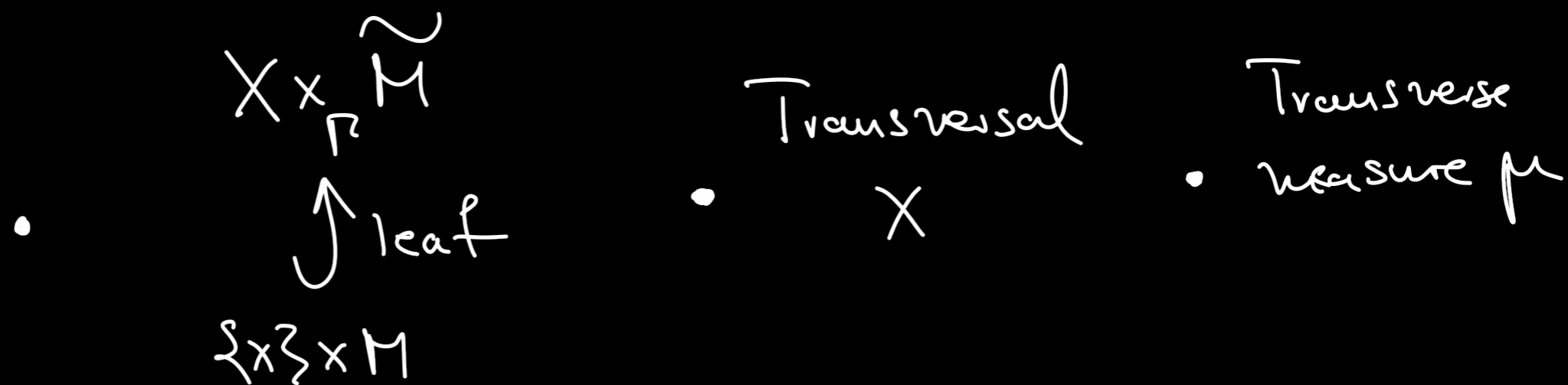
Then from top. dynamics (Elek): Any countable group admits such an action on the Cantor set.

The foliated space

$\Gamma = \pi_1(M) \curvearrowright X$ Cantor set, μ Γ -invariant prob. measure on X

We transfer Gutw's methods to the foliated

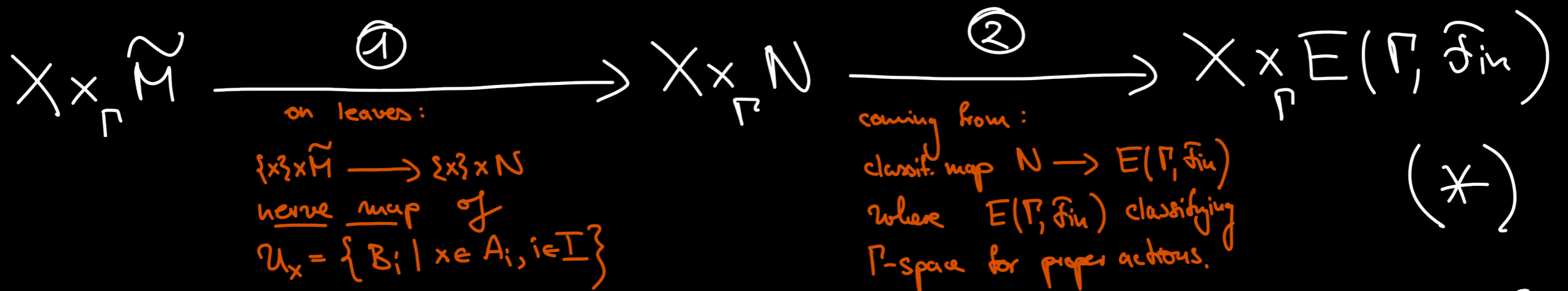
setting:



... and use it to prove the statement about M !

More on the proof

$\Gamma = \pi_1(M) \curvearrowright X$ as in Elek's thm



- Construct Γ -equivariant covering $X \times \tilde{M}$ by $\begin{cases} A_i \subset X \\ B_i \subset \tilde{M} \end{cases}$ clopen \rightarrow $\mathcal{U} = \{A_i \times B_i \mid i \in \mathbb{I}\}$ of $B(\Gamma, r)$ good if
 - Reasonable growth: $|B(p, 100r)| \leq 10^{4(n+3)} |B(p, 100r^{-1})|$
 - Volume bound: $|B(p, r)| \leq 10^{2n+6} v(1) R^{n+3}$
 - $r \leq 1/100$
- Show that we get an isometric map in measure

homology: $\mathcal{H}_n(M) \longrightarrow \mathcal{H}_n(X \times_{\Gamma} \tilde{M}) \xrightarrow{\textcircled{1} \cdot \textcircled{2}} \mathcal{H}_n(X \times_{\Gamma} E(\Gamma, \mathcal{F}_in))$

- Show that $\textcircled{1}$ is leafwise Lipschitz on large volume.