The topology of positive scalar curvature
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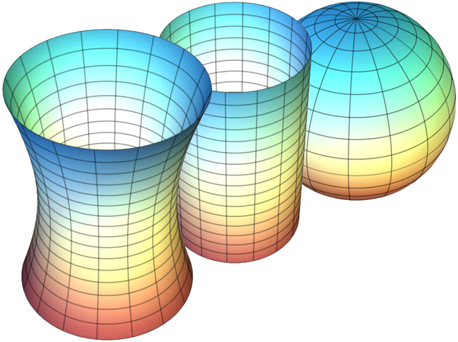
Scalar curvature

My question: Given a smooth compact manifold M, how does the space $\mathrm{Riem}^+(M)$ of Riemannian metrics of positive scalar curvature look like? Is it empty? What are its homotopy groups?



Positive scalar curvature and Gauß-Bonnet

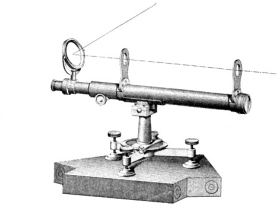
Theorem (Gauß-Bonnet)

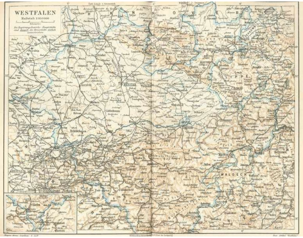
If F is a 2-dimensional compact Riemannian manifold without boundary,

$$\int_F \mathrm{scal}(x)\, d\,\mathrm{vol}(x) = 4\pi\chi(F).$$

Corollary

$\mathrm{scal} > 0$ *on F implies* $\chi(F) > 0$, *i.e.* $F = S^2, RP^2$.



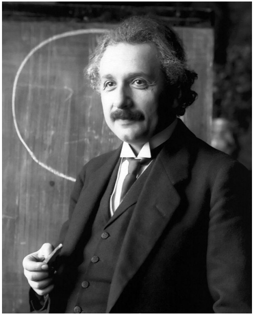


Scalar curvature in dimension m

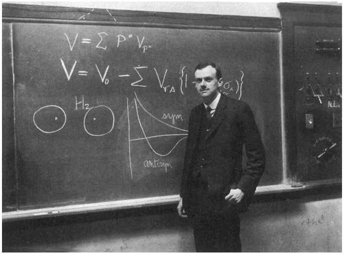
$\mathrm{scal}(x)$ is the integral of the scalar curvatures of all 2-dim surfaces through the point x. It satisfies

$$\frac{\mathrm{vol}(B_r(x) \subset M)}{\mathrm{vol}(B_r(0) \subset \mathbb{R}^m)} = 1 - \frac{\mathrm{scal}(x)}{c_m} r^2 + O(r^4) \qquad \text{small } r.$$

It features in Einstein's general relativity.



Basic Dirac operators



Dirac: Differential operator D as square root of matrix Laplacian (using Pauli matrices).

Schrödinger: generalization to curved space-time (local calculation) satisfies

$$D^2 = \underbrace{\nabla^*\nabla}_{\geq 0} + \frac{1}{4}\,\mathrm{scal}\,.$$

Global Dirac operator

Given an *spin structure* (a strengthened version of orientation) and a Riemannian metric, one gets:

1. the spinor bundle S over M. Sections of this bundle are spinors
2. the Dirac operator D acting on spinors: a first order differential operator which is *elliptic*.

In the following, for easy of exposition we concentrate on even dimensional manifolds and only use complex C^*-algebras.

K-theory of C^*-algebras

Definition

A C^*-algebra A is a norm-closed $*$-subalgebra of the algebra of bounded operators on a Hilbert space.

We have for a (stable) C^*-algebra A:

- $K_1(A)$ are homotopy classes of invertible elements of A.
- $K_0(A)$ are homotopy classes of projections in A.
- 6-term long exact K-theory sequence for ideal $I \subset A$:

$$\to K_0(A/I) \xrightarrow{\delta} K_1(I) \to K_1(A) \to K_1(A/I) \xrightarrow{\delta} K_0(I) \to \\ \to K_0(A) \to K_0(A/I) \to$$

Framework of index

- Using functional calculus, the Dirac operator gives the bounded operator $\chi(D)$, contained in the C^**-algebra* A of bounded operators on $L^2(S)$.

  Here $\chi\colon \mathbb{R} \to [-1,1]$ is any odd functions with $\chi(x) \xrightarrow{x\to\infty} 1$ (contractible choice).
- $\dim M$ even: $S = S^+ \oplus S^-$, $\chi(D) = \begin{pmatrix} 0 & \chi(D)^- \\ \chi(D)^+ & 0 \end{pmatrix}$.
- If M is *compact*, ellipticity implies that

$$\chi^2(D) - 1 \in I, \text{ the ideal of compact operators on } L^2(S),$$

  so also $U\chi(D)^+$ is invertible in A/I (with any unitary $U\colon L^2(S^-) \to L^2(S^+)$: contractible choice).
- It therefore defines a "fundamental class" $[D] \in K_1(A/I)$.

Apply the boundary map $\delta \colon K_1(A/I) \to K_0(I)$ of the long exact K-theory sequence $0 \to I \to A \to A/I \to 0$ to obtain

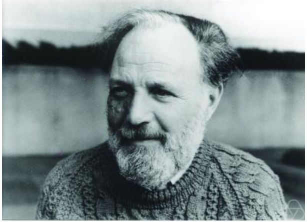
$$\mathrm{ind}(D) := \delta([D]) \in K_0(I) = \mathbb{Z}.$$

We have the celebrated

Theorem (Atiyah-Singer index theorem)

$$\mathrm{ind}(D) = \hat{\mathrm{A}}(M)$$

Here $\hat{\mathrm{A}}(M)$ is a differential -topological invariant, given in terms of the Pontryagin classes of TM, which can be efficiently computed.
It does not depend on the metric (the Dirac operator does).



Schrödinger-Lichnerowicz formula and consequences

Definition

Schrödinger's local calculation relates the Dirac operator to scalar curvature: $D^2 = \nabla^*\nabla + \operatorname{scal}/4 \geq \operatorname{scal}/4$. It implies:
if $\operatorname{scal} > c > 0$ everywhere, $\operatorname{spec}(D) \cap (-\sqrt{c}/2, \sqrt{c}/2) = \emptyset$.
Choose then $\chi = \pm 1$ on $\operatorname{spec}(D)$, therefore $\operatorname{ch}^2(D) = 1$ and $U\chi(D)^+$ is invertible in A, representing a structure class

$$\rho(D_g) \in K_1(A),$$

mapping to $[D] \in K_1(A/I)$.

Potentially, $\rho(D_g)$ contains information about the positive scalar curvature metric g.

Schrödinger-Lichnerowicz formula and consequences II

Theorem

- *If M has positive scalar curvature, then* $\mathrm{ind}(D) = 0 \implies \hat{\mathrm{A}}(M) = 0$*:*
- $\hat{\mathrm{A}}(M) \neq 0$ *is an obstruction to positive scalar curvature!*

Example: Kummer surface. **Non-examples**: $\mathbb{C}P^{2n}$, T^n.

Proof.

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_1(A/I) & \xrightarrow{\delta} & K_0(I) \\ \rho(D_g) & \mapsto & [D] & \mapsto & \mathrm{ind}(D) = 0 \end{array}$$

using exactness of the K-theory sequence. $\square$

Generalized index situation

General goal: find sophisticated algebras $I \subset A$ to arrive at similar index situations. Criteria:

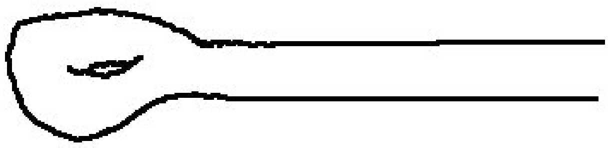
- index construction must be possible (operator in A, invertible modulo an ideal I)
- calculation tools for $K_*(A)$, $K_*(I)$ and the index
- positive scalar curvature must imply vanishing of index (and give structure class $\rho \in K_*(A)$)

Useful/crucial is the context of C^**-algebras*, where *positivity implies invertibility*.

Non-compact manifolds

What can we do if M is not compact?
Why care in the first place?
This is of relevance even when studying compact manifolds:

- extra information can be obtained by studying the covering spaces with their group of deck transformation symmetries (e.g. $\mathbb{R}^n \to T^n$ with deck transformation action by $\mathbb{Z}^n$).
- attaching an infinite half-cylinder to the boundary of a compact manifold with boundary assigns a manifold without boundary, but which is non-compact.



Many important cases where M is non-compact (general relativity).

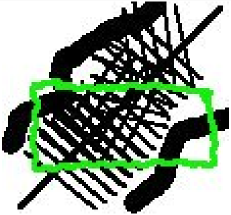
Large scale/Coarse C^*-algebras

Definition (Roe)

M Riemannian spin manifold (not necessarily compact). The coarse algebra/Roe algebra $C^*(M)$ is the closure of the algebra of bounded operators T on $L^2(S)$ satisfying

- T has finite propagation: there is R_T such that the support of $T(s)$ is contained in the R_T-neighborhood of the support of s for each s.
- local compactness: if $\phi \in C_0(M)$ has compact support, the composition of T with multiplication by ϕ (on either side) is a compact operator.

Schwarz (distributional) integral kernel:



Large scale/Coarse C^*-algebras II

- $D^*(M)$ is defined similar to $C^*(M)$, but replacing local compactness by the weaker condition of *pseudolocality*:

$$\phi T \psi \text{ compact whenever } \operatorname{supp}(\phi) \cap \operatorname{supp}(\psi) = \emptyset.$$

Schwarz kernel: 

- $C^*(M) \subset D^*(M)$ is an ideal
- functoriality for Lipschitz maps

Analysis of Dirac operator

Theorem (Roe)

$$\chi(D) \in D^*(M), \qquad \chi(D)^2 - 1 \in C^*(M).$$

- By Fourier inversion

$$\chi(D) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\chi}(\xi) \exp(i\xi D) \, d\xi.$$

  Here, $\exp(i\xi D)$ is the *wave operator*, it has propagation $|\xi|$.
- Consequence: if $\hat{\chi}$ has compact support, $\chi(D)$ has finite propagation.
- By (local) elliptic regularity, $\chi(D)$ is pseudolocal (compact only outside the diagonal as $\hat{\chi}$ is singular at 0)
- and $\chi(D)^2 - 1$ locally compact.

Large scale/Coarse index theory

Exactly as above, the Dirac operator on a spin manifold M defines

$$\textbf{fundamental class} \qquad [D] \in D^*(M)/C^*(M),$$

$$\textbf{coarse index} \qquad \operatorname{ind}_c(D) \in K_*(C^*(M))$$

Theorem

If scal $> c > 0$, *the Dirac operator defines a*

$$\textbf{structure class} \qquad \rho(D_g) \in D^*(M),$$

so $\operatorname{ind}_c(D) = 0 \in K_*(C^*(M))$ *in this case.*

Computability

There are good tools to compute $K_*(C^*(M))$, $K_*(D^*(M))$, e.g.

- coarse *Mayer-Vietoris sequence* to put the information together by breaking up M in simpler pieces
- vanishing for suitable kinds of coarse contractibility, in particular if $M = Y \times [0, \infty)$.
- ...

Example consequence:

Theorem

$$K_0(C^*(\mathbb{R}^{2n})) = \mathbb{Z}; \qquad K_1(C^*(\mathbb{R}^{2n+1})) = \mathbb{Z}.$$

Large scale/Coarse index application

- Let P be a connected compact spin manifold without boundary with $\hat{A}(P) \neq 0$. Let $P \to M \to T^n$ be a fiber bundle. Does M admit a metric of positive scalar curvature? Example: $M = T^n$ if $P = \{*\}$.
- We can pass to the covering $P \to \tilde{M} \xrightarrow{p} \mathbb{R}^n$. Using functoriality, we can map the coarse index of $\tilde{M}$ to $p_*(\mathrm{ind}_c(D)) \in K_n(C^*(\mathbb{R}^n)) = \mathbb{Z}$.

Theorem (Partitioned manifold index theorem (Roe, Higson, Siegel, S.-Zadeh))

$$p_*(\mathrm{ind}_c(D)) = \hat{A}(P).$$

Corollary

$\tilde{M}$ and therefore M does not admit a metric of positive scalar curvature.

Codimension 2 obstruction (Gromov-Lawson)

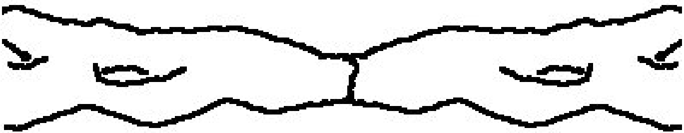
Theorem (Hanke-S.)

M compact spin, $N \subset M$ codimension 2 submanifold with tubular neighborhood $N \times D^2 \subset M$. $\pi_1(N) \to \pi_1(M)$ is injective, $\pi_2(M) = 0$. $\mathrm{ind}(D_N) \neq 0 \in K_*(C^*\pi_1(N)) \implies M$ does not have positive scalar curvature.

Example (Gromov-Lawson, new proof)

No 3-manifolds with $\pi_2 = 0$ and with infinite π_1 admits scal > 0: pick for N a circle which is non-trivial in $\pi_1(M)$.

1. pass to a suitable covering $\bar{M}$, take out $N \times D^2$ and double along $N \times S^1$
2. apply **improved vanishing** (where we glue: no psc) and partitioned manifold index for C^*-coefficients instead of $\mathbb{C}$.



Change of scalars

- Throughout, we can replace the complex numbers by any C^*-algebra A; get algebras $C^*(M;A), D^*(M;A)$ (Mishchenko, Fomenko, Higson, Pedersen, Roe,...).
- In particular: $C^*\pi_1(M)$, a C^*-closure of the group ring $\mathbb{C}\pi_1(M)$.
- The whole story then relates to the *Baum-Connes conjecture*.
- Throughout, we can use the Dirac operator twisted with a flat bundle of Hilbert A-modules, e.g. the Mishchenko bundle. All constructions and results carry over (in our setup without too much extra work).
- For compact M, we get Rosenberg index

$$\mathrm{ind}(D) \in K_{\dim M}(C^*\pi_1 M)$$

refining $\mathrm{ind}(D) \in \mathbb{Z}$ we started with.

Gromov-Lawson-Rosenberg conjecture

- Former conjecture (Gromov-Lawson-Rosenberg): if M compact spin, $\dim M \geq 5$:
  M positive scalar curvature $\iff \quad 0 = \mathrm{ind}(D) \in K_{\dim M}(C^*_{\mathbb{R}}\pi_1 M)$.
- 5-dimensional Counterexample (S.) with $\pi_1(M) = \mathbb{Z}^3 \times (\mathbb{Z}/3\mathbb{Z})^2$
- Question: how much exactly does $\mathrm{ind}(D) \in K_*(C^*\pi_1 M)$ see about positive scalar curvature?

Improved vanishing

Theorem (Roe, Hanke-Pape-S.)

If M contains a geodesic ray $R \subset M$ and scal $> c > 0$ *outside an r-neighborhood of R for some $r > 0$, then already*

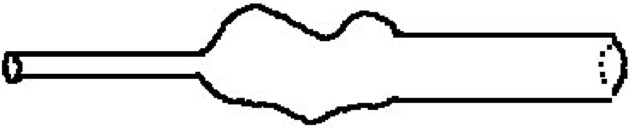
$$\mathrm{ind}_c(D) = 0 \in K_*(C^*(M)).$$

For the proof, consider the ideal $C^*(R \subset M)$ in $D^*(M)$ of operators in $C^*(M)$ *supported near* R.

- local analysis shows that $\chi(D)$ **is invertible module** $C^*(R \subset M)$
- $$\begin{array}{ccc} K_1(D^*(M)/C^*(R \subset M)) & \longrightarrow & K_0(C^*(R \subset M)) {\color{red}= 0} \\ \downarrow & & \downarrow \\ K_1(D^*(M)/C^*(M)) & \longrightarrow & K_0(C^*(M)) \end{array}$$
- apply naturality.

Relative index of metrics

- Goal: we want to compare two families of Riemannian metrics $(g_0^p)_{p\in S^k}$, $(g_1^p)_{p\in S^k}$ with scal > 0 on M, representing two elements in $\pi_k(\mathrm{Riem}(M))$.
- We choose an interpolating family of metrics $(g_t^p)_{p\in S^d}$ on $M \times [0,1]$ —leading to a metric g on $M \times [0,1]$.
- Adding the half cylinders left and right provides a non-compact Riemannian manifold M_∞ without boundary with positive scalar curvature near the two ends.
- Partial vanishing gives an Atiyah-Patodi-Singer index in $K_*(C^*(M \subset M_\infty; C^0(S^k, C^*\pi_1 M))) \cong K_{*-k}(C^*\pi_1 M) \oplus K_*(C^*\pi_1 M)$.



Space of metrics of positive scalar curvature

Theorem (Hanke-S.-Steimle)

Given $k \in \mathbb{N}$, *as long as* $\dim(M)$ *is big enough and* $\dim(M) + k \equiv 1 \pmod 4$, *if* $g_0 \in \mathrm{Riem}^+(M) \neq \emptyset$ *then there are elements of infinite order in* $\pi_k(\mathrm{Riem}^+(M), g_0)$ *detected by this method.*
If M *is a sphere, these classes remain of infinite order in* π_d *of the (observer) moduli space of metrics of positive scalar curvature.*

- Construction is based on Gromov-Lawson surgery construction of psc metrics in families (Walsh)
- and the construction (via surgery theory and smoothing theory) of interesting bundles over S^k whose total space has non-vanishing $\hat{A}$-genus (Hanke-S.-Steimle).

Questions.

- What of this remains true for manifolds without spin structure (where even the universal cover is non-spin)?
- Does the Rosenberg index $\mathrm{ind}(D) \in K_*(C^*\pi_1 M)$ capture all information about positive scalar curvature obtainable via index theory (weak Gromov-Lawson-Rosenberg conjecture)? In particular, how does it relate to the codimension 2 obstruction?
- What about the "modified n-torus with fundamental group $(\mathbb{Z}/3\mathbb{Z})^n$ obtained by doing surgery on T^n to adjust π_1?
- Fully compute the homotopy type of $\mathrm{Riem}^+(M)$ when non-empty.

In any case: THANK YOU for your attention.