Seminarvortrag: Amenability and Property T for groups and von Neumann algebras

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1 Definitions via representations

1.1 Definition. Relative Property T via representations for a locally compact group G with closed subgroup H:

If every (strongly continuous) unitary representation with almost G-invariant vectors (i.e. a sequence x_n of unit vectors such that $|\pi(g)x_n - x_n| \xrightarrow{n\to\infty} 0$ for all $g \in G$ uniformly on compact subset of G) has a non-zero strictly H-invariant vector.

1.2 Lemma. *In this case, the H-invariant vector x can be chosen close to x_n for some large n.*

On the other hand, it suffices to find a finite dimensional H-invariant subspace.

1.3 Definition. G is amenable if and only if the left regular representation has almost invariant vectors.

1.4 Definition. G has the Haagerup property (is a-T-menable) if and only if there is some strongly continous unitary representaion with almost invariant vectors, and with matrix coefficients $< \pi(g)v, w > \xrightarrow{g\to\infty} 0$.

1.5 Example. The left regular representation λ_G has G-invariant vectors if and only if G is compact, and has almost invariant vectors if and only if G is amenable.

1.6 Lemma. (G, H) *relative property T with H non-compact implies G has not the Haagerup property, which implies that G is not amenable.*

1.7 Example.
- Abelian, solvable groups are amenable.
- $Sl_n(K)$ for $n \geq 3$, $Sp_{2n}(K)$ for $n \geq 2$ and K a local field have property T, more generally the K-points of connected algebraic almost simply groups over K with $rk_K(G) \geq 2$. (Kazhdan, Vaserstein,...).

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- $SO(n,1)$ and $SU(n,1)$ do not have property T, but $Sp(n,1)$ and $F_{4(-20)}$ has (Kazhdan).
- $(Z^2 \rtimes Sl_2(Z), Z^2)$ and $((S^2)^* \rtimes Sl_2(K), (S^2)^*(K))$ have relative property T.
- Free groups neither have property T nor are amenable, but are a-T-menable. Same for $SO(n,1)$, $SU(n,1)$.

For von Neumann algebras: replace unitary representations by bimodules (Connes' correspondences).

1.8 Definition. (L,τ) a von Neumann algebra with faithful normal tracial state, $Q \subset L$ a von Neumann subalgebra have relative property (T) if and only if every L-L bimodule with almost central almost tracial vectors admits a sequence of almost tracial Q-central unit vectors.

Here a bimodule is a Hilbert space H with left-right normal unital actions. $(x_n \in H)$ is almost central, if $|ax_n - x_n a| \xrightarrow{0} \forall a \in L$. (x_n) is almost tracial, if $\|\langle x_n, \cdot x_n\rangle - \tau\| \xrightarrow{0}$ and $\|\langle x_n, x_n \cdot\rangle - \tau\| \xrightarrow{0}$. $x \in H$ is central, if $ax = xa$ for all $a \in L$.

1.9 Definition. (L,τ) is amenable if and only if the Hilbert bimodule $L^2(L,\tau)\otimes L^2(L,\tau)$ with action $axb := (a\otimes 1)x(1\otimes b)$ contains almost central almost tracial vectors.

1.10 Theorem. *Any von Neumann algebra L generated by the representation of a (discrete?) amenable group is amenable.*

Moreover, if Γ is discrete then $L(\Gamma)$ amenable implies Γ amenable.

Proof. Use perhaps $x_n \otimes x_n$, and reasoning similar to the one of the next theorem? □

1.11 Theorem. *Let $H \leq G$ be countable groups. Then: (G,H) has relative property T if and only if $(L(G), L(H))$ has relative property T.*

Proof. "$\Longrightarrow$": Let V be a bimodule with almost central almost tracial vectors (x_n). Define G-action on V by $gx := u_g x u_g^*$. Because of relative property T (strong version) exists to $\epsilon > 0$ an $n \in \mathbb{N}$ and an H-invariant vector x with $|x - x_n| < \epsilon$, $\|\langle x_n, \cdot x_n\rangle - \tau\| < \epsilon$, $\|\langle x_n, x_n\cdot\rangle - \tau\| < \epsilon$. These x produce an $L(H)$-central almost tracial sequence.

"$leftarrow$": If $\tau\colon G \to U(V)$ has normed almost invariant vectors (x_n), we need to find a finite dimensional H-invariant subspace. Set $K := l^2(G)\otimes V$ with $L(G)$-bimodule structure given by $u_g(\delta_h \otimes x)u_v = \delta_{ghv} \otimes \pi(g)x$. Then $\delta_e \otimes x_n$ are almost central and almost tracial. Therefore we have $\mu \neq 0$ $L(H)$-invariant. Now $K \cong l^2(G,V)$, and $\mu(hgh^{-1}) = \pi(h)\mu(g)$ for all $h \in H$, $g \in G$. If $\mu(g) \neq 0$, L^2 implies that $|\{|\, hgh^{-1} \mid h \in H\} < \infty$. The corresponding linear span of $\mu(hgh^{-1})_{h\in H}$ is a finite dimensional H-invariant subspace of V. □

1.12 Theorem. *(Popa): If $L^\infty(X,\mu) \rtimes G \supset P$ are von Neumann algebras with relative property T, then one can essentially conjugate P to $L(G)$ with a unitary.*

1.13 Theorem. *(Popa) The functor from discrete countable ICC groups with property T (and inclusions as homomorphisms) to von Neumann algebras: $G \mapsto L(\oplus_{g\in G}\mathbb{Z} \rtimes G)$ is injective upon passage to isomorhpism classes.*

1.14 Remark. This is a partial result towards the conjecture of Connes, which states the same for the functor $G \mapsto L(G)$.

1.15 Proposition. *If L is a II_1-factor (really, only a faithful trace is needed) and $Q \subset L$ has relative property T, and $0 \neq p \in Q$ is a projection, then $pQp \subset pLp$ also has relative property T.*

Proof. Because L is a factor there are partial isometries $v_1, \dots, v_k$ with $v_1 = p$, $v_i^* v_i \leq p$ and $\sum v_i v_i^* = 1$.

Assume that K_1 is a pLp bimodules with almost tracial almost invariant vectors x_n. Set $K := LpK_1pL$ with $\langle x\xi y^*, a\eta b^* \rangle := \langle \xi, (x^*a)\eta(b^*y) \rangle$. Upon normalization, $\sum_i v_i x_n v_i^*$ gives almost central almost traical vectors for the L-L bimodule. Therefore, we get a sequence (μ_n) of almost tracial Q-central vectors. Upon normalization, $p\mu_n = \mu_n p$ is pQp-central almost tracial. $\square$

2 Definitions via cohomology

2.1 Definition. Let $\pi\colon G \to U(H_\pi)$ be a strongly continuous representation of a locally compact group. Set

(1) $Z^1(G, \pi) := \{b\colon G \xrightarrow{C} H_\pi \mid g(gh) = \pi(g)b(h) + b(g)\}$

(2) $B^1(G, \pi) = \{b \in Z^1 \mid \exists x \in H_\pi : b(g) = \pi(g)x - x\}$

(3) $H^1(G, \pi) = Z^1/B^1$

(4) $\overline{H^1}(G, \pi) = Z^1/\overline{B}^1$, where we use the topology of uniform convergence on compact sets.

2.2 Remark. There is an interpretation in terms of affine actions: $b\colon G \to H_\pi$ defines $\alpha(g)\colon H_\pi \to H_\pi$ with $\alpha(g)v = \pi(g)v + b(g)$. This is an affine action with linear part π if and only if $b \in Z^1(G, \pi)$. Moreover, $b \in B^1(G, \pi)$ if and only if α has a fixed point (if and only if α is conjugate to π by a translation). The center lemma implies that this is the case if and only if b is bounded.

We observe: $H^1(G, \pi) = 0$ if and only if every affine action (with linear part π) has a fixed point, and $\overline{H^1}(G, \pi) = 0$ if and only if every such action has almost fixed points.

2.3 Lemma. *If G is locally compact σ-compact, and π has almost invariant vectors (but no invariant vectors), then $H^1(G, \bigoplus_{k=1}^{\infty} \pi) \neq 0$.*

Proof. Write $G = \bigcup L_n$ for compact subsets L_n. We find normed x_n with $\max_{g \in L_n} |gx_n - x_n| \leq 2^{-n}$. Set $b(g) := \bigoplus n(gx_n - x_n)$; then b converges uniformly on all compact subsets. But b is unbounded by the following reasoning: Otherwise: if C is the closed convex hull of $\pi(G)x_n$ (for n large fixed), then C is G-invariant, $|gx_n - x_n| < 1$ implies $Re(\langle gx_n, x_n \rangle) \geq 1/2$ for all $g \in G$, so $0 \notin C$. The vector of minimal length in C is then G-invariant, contradicting our assumption on π. $\square$

2.4 Theorem. *(Delorme-Guichardet): If G is locally compact and σ-compact then we have the equivalence: G has property T $\iff$ $H^1(G, \pi) = 0$ for every unitary representation π $\iff$ every affine isometric action on a Hilbert space has a fixed point.*

2.5 Corollary. *Every action of a σ-compact group G with property T on a tree fixed an edge or a vertex*

Proof. Otherwise, construct an affine isometric action which is metrically proper, i.e. doesn't have almost fixed points. □

2.6 Theorem. *(Shalom) If G is locally compact 2nd countable and compactly generated, we have the further equivalence: G has property $T \iff \overline{H^1}(G,\pi) = 0 \forall\pi \iff \overline{H^1}(G,\pi) = 0$ for all irreducible $\pi \iff H^1(G,\pi) = 0$ for all irreducible π.*

2.7 Proposition. *Equivalent to: G has the Haagerup property is: every continuous isometric action of G on an affine Hilbert space H is metrically proper (i.e. if $B \subset H$ is bounded, then $\{g \in G \mid gB \cap B \neq \emptyset\} \subset G$ is precompact).*

3 Some properties

3.1 Theorem. *$f\colon G_1 \to G_2$ continuous with dense image.*

- *If G_1 has property T, so has G_2*
- *If G_1 has property T and G_2 is a-T-menable, then $\operatorname{im}(f)$ is precompact.*
- *$G/\overline{[G,G]}$ is compact, G is unimodular*

Proof. For the first statement, use the restrictions $G_1 \to G_2 \xrightarrow{\pi} U(H_\pi)$ and denseness. The other two follow from the first. □

3.2 Theorem. *If G locally compact has property T, it is compactly generated.*

Proof. Write $G = \bigcup H$ over the set C all compactly generated open subgroups. If $H \in C$, then G/H is discrete.

The representation $\pi = \bigoplus_{H \in C} \lambda_{G/H}$ has almost invariant vectors (since every compact subset L is contained in some $H_L \in C$, then δ_{H_L} is L-fixed). Property T yields $0 \neq x = \oplus x_H$. In particular, for some H, $x_H \in l^2(G/H)$ is G-fixed. Because of the l^2-condition, G/H is finite, since H was compactly generated, therefore so is G. □

3.3 Proposition. *Let $1 \to N \to G \to G/N \to 1$ be an exact sequence of locally compact groups, N closed in G.*

If $N, G/N$ have property T, so has G. If G has property T, so have G/N and N.

Same hold for amenability.

Proof. Use induction and restriction of representations, not trivial. □

3.4 Proposition. *If $C \leq G$ is closed and central, G/C has property T and $G/\overline{[G,G]}$ is compact, then G has property T.*

3.5 Proposition. *G is amenable if and only if the canonical map $C^*_{max}G \to C^*_{red}G$ is an isomorphism.*

4 Amenability and means

4.1 Theorem. *G locally compact is amenable if and only if there is a finitely additive G-invariant mean (non-negative, $\mu(1) = 1$) on $L^\infty(G)$.*

4.2 Example. The Haar measure produces this if G is compact; otherwise one needs axiom of choice, even for $G = \mathbb{Z}$.

4.3 Theorem. *G is amenable if and only if any continuous affine action of G on a non-empty compact convex subset X of a locally convex topological vector space has a fixed point.*

Proof. Part: the set of means in the dual of L^∞ is such a set, an invariant mean is a fixed point for this action. □

5 Amenability and von Neumann algebras

5.1 Theorem. *$L \subset B(H)$ is amenable if and only if there is a linear projection $P\colon B(H) \to L$ of norm 1. (Such a P is automatically an L-bimodule map). (This property is called "injective", and is injectivity in an appropriate abelian category of von Neumann algebras).*

This is equivalent to: L is hyperfinite (i.e. generated by an increasing sequence of finite dimensional subalgebras).

This is furthermore equivalent to: all finite derivations of L with coefficients in a normal dual Banach bimodule over L are inner (a condition on H^1!).

5.2 Theorem. *If an action on a measure space is amenable (or more generally a groupoid is amenable) then the induced (crossed product) von Neumann algebra is amenable.*

5.1 Actions

We consider action of countable G "freely" and ergodically on the standard probability space $([0, 1], d\mu)$.

5.3 Definition. (1) Such actions are *conjugate* if they are transported by group isomorphisms and measure space isomorphism.

(2) They are orbit equivalence, if we have a measure space isomorphism mapping (almost all) orbits bijectively to orbits.

(3) We can also just consider the von Neumann algebra $L^\infty(X, \mu) \rtimes G$.

5.4 Theorem. *(Connes-Feldmann-Weiss, Ornstein-Weiss): if the equivalence relation is amenable, then it is orbit equivalent to the standard action of $\mathbb{Z}$ on $\prod_{g\in\mathbb{Z}}(X, \mu)$.*

Here, the equivalence relation $Q \subset X \times X$ is amenable if for every Banach space E and cocycle $m\colon Q \to Iso(E)$ $(m(x, y)m(y, z) = m(x, z))$ and m-invariant Borel field A_x of compact convex subsets of E^* (m-invariant menas that $(m(x, y)^*)^{-1}A_y = A_x$ almost everywhere), there is an m-invariant section $\phi\colon X \to E^*$ with $\phi(x) \in A_x$, where m-invariant means $(m(x, y)^*)^{-1}\phi(y) = \phi(x)$ almost e.

5.5 Remark. Amenable actions produce amenable equivalence relations.

5.6 Definition. An action (of a discrete group) on X is amenable if there are $b^n\colon X \to prob(G)$ such that $\forall g \in G$ $\lim_{n\to\infty} \sup_{x\in X} \left|gb^n_x - b^n_{gx}\right|_{l^1} = 0$.

If X is compact Hausdorff, we want w^*-continuity of b^n and talk of *topological amenability*, if X is a measure space: need suitable measurability (?).

5.7 Remark. G is amenable if and only if it acts amenably on a point, then every action of G is amenable.

G acts amenably on some compact space, if it acts amenably on its Stone-Cech compactification.

6 Spectral approaches to property T and amenability

6.1 Definition. Assume μ is a probability measure on G and π a unitary action. We define a Laplacian $\pi(\mu) \in B(H_\pi)$ by

$$\langle \pi(\mu)x, y\rangle = \int_G \langle \pi(g)x, y\rangle \, d\mu(g).$$

Then $\|\pi(\mu)\| \leq 1$.

6.2 Proposition. *If G is locally compact, μ is Haar absolutely continuous and the support of μ topologically generates G, then π has almost invariant vectors if and only if $1 \in spec(\pi(\mu))$.*

If the support of $\mu^* * \mu$ topologically generates G, this is equivalent to $\|\pi(\mu)\| = 1$.

6.3 Proposition. *If G is discrete, it is amenable if and only if it admits a Folner exhaustion.*

6.4 Theorem. *If $G = \pi_1(X)$ (discrete) then $H^1(G, \pi) = H^1_{dR}(X, \tilde{X} \times_G H_\pi)$ by Hodge theory. This gives property T if X is Riemannian symmetric irreducible of non-compact type (different from $H^n(R), H^n(C)$), using Bochner formulas.*

In a different direction (without Hodge theory):

6.5 Theorem. *(Zuk): If X is a locally finite simplicial complex (simply connected) and for every vertex $v \in X$, the first (non-zero) eigenvalue of the Laplacian of the link of v is $> 1/2$ ($(\Delta f)(x) = f(x) - 1/deg(x) \sum_{y\sim x} f(y)$), and if G acts properly and cocompactly on X, then G has property T.*

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