A parametrized version of the Borsuk Ulam theorem

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Abstract

In this note, we establish special homological properties of the set of solutions of the Borsuk-Ulam theorem. This solves a special case of a conjecture of Simon. This conjecture would be relevant in connection with new existence results for equilibria in certain games.

1 A parametrized Borsuk-Ulam theorem

1.1 Cech homology

Throughout this note, all spaces encountered will be subspaces of (smooth) manifolds. The homology groups we are using will exclusively be *Cech homology* groups. Their properties can be found in [2, Chapters IX, X] and in [1, VIII, 13]. We list the most important properties:

(1) Cech homology is defined for subsets of finite dimensional manifolds.

(2) A homeomorphism $f\colon A \to B$ induces an isomorphism of Cech homology groups, i.e. the groups are independent of the particular embedding. (This is a property which is not particularly relevant to us.)

(3) Cech homology satisfies excision in a very strong form: if $f\colon (X,A) \to (Y,B)$ is a map of compact pairs such that $f|\colon X \setminus A \to Y \setminus B$ is a homeomorphism, then $f_*\colon H_*(X,A) \to H_*(Y,B)$ is a isomorphism of Cech homology groups.

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(4) More generally, the *Vietoris theorem* about maps with acyclic fibers holds: if $f\colon (X,A) \to (Y,B)$ is a map of compact pairs such that the Cech homology groups $H_*(f^{-1}(y))$ are trivial for all $y \in Y$, then $f_*\colon H_*(X,A) \to H_*(Y,B)$ is an isomorphism of Cech homology groups. **(Kommentar: Check, give reference, probably one needs this only for $y \in Y \setminus B$.)**

(5) For ENRs, e.g. for topological manifolds, Cech homology and singular homology are canonically isomorphic.

(6) For *compact* subsets of manifolds, and with coefficients in a field (e.g. $\mathbb{Z}/2$), Cech homology is a homology theory, in particular with a long exact sequence of a pair. This does not hold in general for $\mathbb{Z}$-coefficients, this is one of the reasons why we work with $\mathbb{Z}/2$-coeffients. The second reason is that with coefficients $\mathbb{Z}/2$, every compact manifold has a unique orientation class.

(7) For Cech homology, there is a natural intersection pairing: If (X,A) and (Y,B) are two compact subsets of an m-dimensional manifold W (possibly with non-empty boundary), then there is an intersection pairing

$$H_p(X,A) \otimes H_q(Y,B) \to H_{p+q-m}(X \cap Y, (A \cap Y) \cup (X \cap B))$$

which is natural for inclusions of pairs.

1.2 Transport of the spanning property

Let W be a submanifold of $\mathbb{R}^{m+1}$ of codimension zero (of course with boundary ∂W). Assume that W is either a smooth or a PL submanifold. Note that, even if the embedding is only PL, as a codimension zero submanifold we can find a homeomorphism to a smooth codimension zero submanifold, so that we can use a smooth structure in any case. **(Kommentar: Check this and elaborate on it!)**

Let $X \subset \partial W \times D^m$ be a compact correspondence with values in the interior $ID^m)^\circ$ of D^m (by definition, a correspondence is nothing but such a subset). Note that, upto a radial dilation, every compact correspondence on ∂W with values in $\mathbb{R}^m$ will have values in the interior of D^m. Let $p_W\colon X \to W$ be the projection. Assume that there is a Cech homology class

$$[X] \in H_m(X;\mathbb{Z}/2),$$

such that $(p_{\partial W})_*[X] = [\partial W] \in H^{m-1}(\partial W;\mathbb{Z}2)$, the "fundamental class" $[X]$ maps to the fundamental class of the manifold ∂W (since we are working with coefficients $\mathbb{Z}/2$, this class is unambiguously defined).

1.1 Example. A particular example of such a correspondence is the graph of a continuous function $f\colon \partial W \to (D^m)^\circ$. If $X = \{(p, f(p)) \mid p \in \partial W\}$ then the projection $p_{\partial W}$ is a homeomorphism, and correspondingly the required homology class $[X]$ exists.

1.2 Definition. The *spherical correspondence* Y associated to X is defined as follows: it is the subset $Y \in W \times S^m \times D^m$ with

$$\begin{aligned}(p,v,w) \in Y \qquad \iff \qquad & (\exists \lambda > 0 : p + \lambda v \in \partial W \text{ and } (p+\lambda v, w) \in X) \text{ or}\\ & (p \in \partial M, v \text{ not inward pointing }, (p,w) \in X).\end{aligned}$$

Here, *inward pointing* means strictly inward pointing, not tangent to ∂W or to any face of ∂W (in the case of a PL embedding). We define $\partial Y := p_W^{-1}(\partial W)$, where p_W is the obvious map $Y \to W$, induced by the projection $W \times S^m \times D^m \to W$.

Note that, with this definition, Y is a compact subset of $W \times S^m \times (D^m)^\circ$.

1.3 Theorem. *The spherical correspondence Y associated to X as above has itself a "fundamental class" $[Y, \partial Y]$ in its relative Cech homology*

$$[Y] \in H_{2m+1}(Y, \partial Y; \mathbb{Z}/2),$$

such that $(p_{W\times S^m})_*([Y, \partial Y]) = [(W, \partial W)\times S^m] \in H_{m+1}(W\times S^m, \partial W \times S^m; \mathbb{Z}/2)$.

Proof. We have to construct a number of intermediate correspondences. First, by the Künneth formula, there is a fundamental class

$$[X \times (W, \partial W)] \in H_{2m+1}(X \times W, X \times \partial W; \mathbb{Z}/2)$$

which maps under the projection map to

$$[\partial W \times (W, \partial W)] \in H_{2m+1}(\partial W \times W, \partial W \times \partial W; \mathbb{Z}/2).$$

Define

$$Y_2 \subset W \times S^m \times X \subset W \times S^m \times \partial W \times D^m$$

by

$$(q, v, (p, w)) \in Y_2 \iff (p = q \text{ and } v \text{ not inward pointing}) \\ \text{or } p \neq q, v = \frac{q-p}{|q-p|},$$

and set $\partial Y_2 := p_W^{-1}(\partial W)$. The evident map $(Y_2, \partial Y_2) \to (W \times X, \partial W \times X)$ restricts to a homeomorphism $Y_2 \setminus \partial Y_2 \to (W \setminus \partial W) \times X$ and therefore induces in Cech homology an isomorphism

$$H_*(Y_2, \partial Y_2; \mathbb{Z}/2) \to H_*(W \times X, \partial W \times X; \mathbb{Z}/2).$$

In particular, there is a fundamental class $[Y_2, \partial Y_2] \in H_{2m+1}(Y_2, \partial Y_2; \mathbb{Z}/2)$ which maps under the projection map to $[W \times X, \partial W \times X] \in H_{2m+1}(W \times X, \partial W \times X; \mathbb{Z}/2)$.

Define $V \subset W \times S^m \times \partial W$ by the corresponding recipe:

$$(q, v, p) \in V \iff (q = p \text{ and } v \text{ not inward pointing at } p) \\ \text{or } (q \neq p, v = \frac{q-p}{|q-p|}),$$

and set $\partial V := p_W^{-1}(\partial W)$. The map $(V, \partial V) \to (W \times \partial W, \partial W \times \partial W)$ induces as above a Cech homology isomorphism

$$H_*(V, \partial V; \mathbb{Z}/2) \to H_*((W, \partial W) \times \partial W; \mathbb{Z}/2).$$

In particular, we have the fundamental class

$$[V, \partial V] \in H_{2m+1}(V, \partial V; \mathbb{Z}/2).$$

1.4 Lemma. *Consider the projection*

$$\alpha\colon (V, \partial V) \to (W \times S^m, \partial W \times S^m).$$

This map is of degree 1*, i.e.*

$$\alpha_*([V, \partial V]) = [(W, \partial W) \times S^m].$$

Proof. To see this, fix q in the interior of W and let $D_q \subset W^\circ$ be a small disc centered at q contained in the interior of W, and with boundary the sphere S_q. Consider the commutative diagram

$$\begin{array}{ccccc}
(W, \partial W) \times \partial W & \longrightarrow & (W, W \setminus D_q^\circ) \times \partial W & \longleftarrow & (D_q, S_q) \times \partial W \\
\uparrow & & \uparrow & & \uparrow \\
(V, \partial V) & \longrightarrow & (V, V \setminus \alpha^{-1}(D_q^\circ \times S^m)) & \longleftarrow & \Phi((D_q, \partial S_q) \times \partial W) \\
\downarrow & & \downarrow & & \downarrow \\
(W, \partial W) \times S^m & \longrightarrow & (W, W \setminus D_q^\circ) \times S^m & \longleftarrow & (D_q, S_q) \times S^m
\end{array}$$

Here, $\Phi\colon W^\circ \times \partial W \to W \times S^m \times \partial W$ is the homeomorphism sending (q, p) to $(q, \frac{q-p}{|q-p|}, p)$. Observe that the second horizontal maps induces isomorphisms in (Cech) homology (with $\mathbb{Z}/2$-coefficients) by excision. The same is true (as above) for the first row of vertical maps, and for the first horizontal map in the first and in the third row by the usual properties of manifolds.

Therefore, to compute the degree of the map $(V, \partial V) \to (W, \partial W) \times S^m$ it suffices to compute the degree of the map

$$\beta\colon (D_q, S_q) \times \partial W \to (D_q, S_q) \times S^m;\ (x, p) \mapsto (x, \frac{x-p}{|x-p|}).$$

Let $D'_q \subset W^\circ$ be a second disc with center q, with boundary S'_q, and containing D_q in its interior. β extends in the evident way to $(D_q, S_q) \times (W \setminus (D'_p)^\circ) \to (D_q, S_p) \times S^{m-1}$. Therefore the two maps on the boundary are cohomologous and in particular have identical degree. Consequently, the degree of β equals the degree of

$$\beta'\colon (D_q, S_q) \times S'_p \to (D_q, S_q) \times S^{m-1}.$$

This is a homeomorphism and therefore has degree 1, hence the same follows for the map $(V, \partial V) \to (W, \partial W) \times S^m$, as claimed. $\square$

We now continue the proof of Theorem 1.3. Consider the diagram

$$\begin{array}{ccc}
(W, \partial W) \times X & \longrightarrow & (W, \partial W) \times \partial W \\
\uparrow & & \uparrow \\
(Y_2, \partial Y_2) & \longrightarrow & (V, \partial V) \\
\downarrow & & \downarrow \\
(Y, \partial Y) & \longrightarrow & (W, \partial W) \times S^m.
\end{array}$$

where the second row of vertical maps is induced from the projection away the ∂W-coordinate. We already know that the first row of horizontal maps induces

Cech homology isomorphisms, and the second column of horizontal maps consists of maps of degree 1. Consequently, by chasing the diagram, $[Y_2, \partial Y_2]$ is mapped to a class $[Y, \partial Y] \in H_{2m+1}(Y, \partial Y; \mathbb{Z}/2)$ with image $[(W, \partial W) \times S^m] \in H_{2m+1}(W \times S^m, \partial W \times S^m; \mathbb{Z}/2)$. □

1.3 The parametrized Borsuk-Ulam theorem

Let now W be a smooth compact manifold with boundary ∂W (possibly $\partial W = \emptyset$) and

$$Y \subset W \times S^m \times \mathbb{R}^m$$

a correspondence. As usual, we denote $\partial Y := p_W^{-1}(\partial W)$.

1.5 Definition. We associate to Y its *Borsuk-Ulam* correspondence $Z \subset W \times \mathbb{R}^m$ by

$$(q, w) \in Z \iff \exists v \in S^m : \ (q, v, w) \in Y \text{ and } (q, -v, w) \in Y.$$

We set $\partial Z := p_W^{-1}(\partial W) \cap Z \subset Z$.

Note that, if $W = \{pt\}$ and Y is the graph of a continuous function $S^m \to \mathbb{R}^m$, then the Borsuk-Ulam theorem states that Z is non-empty, whence the chosen name.

1.6 Theorem. *Let $Y \subset W \times S^m \times \mathbb{R}^m$ be a spherical correspondence with a Cech homology class $[Y, \partial Y] \in H_{2m+1}(Y, \partial Y; \mathbb{Z}/2)$ such that*

$$(p_{W \times S^m})_*([Y, \partial Y]) = [(W, \partial W) \times S^m] \in H_{2m+1}(W \times S^m, \partial W \times S^m; \mathbb{Z}/2);$$

$p_{W \times S^m}$ being induced from the projection $W \times S^m \times \mathbb{R}^m \to W \times S^m$.

Then the associated Borsuk-Ulam correspondence Z has a "fundamental class"

$$[Z, \partial Z] \in H_{m+1}(Z, \partial Z; \mathbb{Z}/2)$$

with

$$(p_W)_*([Z, \partial Z]) = [W, \partial W] \in H_{m+1}(W, \partial W; \mathbb{Z}/2).$$

We want to describe the construction of this fundamental class more precisely (which will of course also be necessary to establish its properties). To do this, we need to establish a relative squaring construction in Cech homology.

Invariant homology squaring

1.7 Lemma. *Let $(X, A) \subset W$ be a subset of an m-dimensional manifold W. On $X \times X$, we have the involution τ with $\tau(x, y) = (y, x)$. Consider the quotient pair $\widehat{(X, A)}$ with space $(X \times X)/\tau$ and with subspace $X \times A \cup A \times X \cup D$, where $D := \{(x, x) \mid x \in X\}$ is the diagonal.*

Then there is a natural *map in Cech homology with coefficients in $\mathbb{Z}/2$*

$$H_k(X, A; \mathbb{Z}/2) \to H_{2k}(\widehat{(X, A)}; \mathbb{Z}/2); x \mapsto \hat{x}$$

with the following properties:

(1) *Let $(M,\partial M)\subset (W,\partial W)$ be an embedded submanifold, such that $\partial M = M\cap\partial W$ with transversal intersection. Let $x=[M,\partial M]\in H_{\dim M}(M,\partial M;\mathbb{Z}/2)$ be the fundamental class.*

Then, if U is a neighborhood of the diagonal in $W\times W$ which is a codimension zero manifold with boundary ∂U and with interior U°, and such that ∂U meets $M\times M$ transversally, then $\widehat{(M,\partial M)}\setminus p(U^\circ)$ is a manifold with boundary and therefore has a fundamental class. Since the neighborhoods U with the above properties are cofinal among all neighborhoods of U, and since the fundamental classes are mapped to each other under the corresponding inclusions, we get a well defined fundamental class $[\widehat{M,\partial M}]\in H_{2\dim M}((\widehat{M,\partial M});\mathbb{Z}/2)$. Our construction gives $\widehat{x}=[\widehat{M,\partial M}]$.

(2) *Let $(Y,B)\subset W$ be a second subset and $y\in H_l(Y,B;\mathbb{Z}/2)$. Then*

$$\widehat{x\cap y}=\widehat{x}\cap\widehat{y}\in H_{2(k+l-m)}((X\cap Y,\widehat{X\cap B}\cup A\cap Y);\mathbb{Z}/2)),$$

where we note that $(X\cap Y,\widehat{X\cap B}\cup A\cap Y)=\widehat{(X,A)}\cap\widehat{(Y,B)}$.

Proof. Since Cech homology is the inverse limit over the homology of open neighborhoods, we have in reality to construct homology classes relative to an open neighborhood U of (the image of) the diagonal. By excision, we therefore have to construct homology classes in $X\times X\setminus D$ relative $U\setminus D$ (and its image under the quotient map).

Let now $\sum_{i=1}^n\sigma_i$ be a representative by singular chains of x. Upto subdivision, we can assume that for all i,j either the image of $\sigma_i\times\sigma_j$ is contained entirely in U or does not meet the diagonal D at all.

Then we define $\hat{x}:=\sum_{i<j}p_*(\sigma_i\times\sigma_j)$. Technically speaking, we have to subdivide $\sigma_i\times\sigma_j$ into simplices. This can be done in an arbitrary fashion, since we are dealing with $\mathbb{Z}/2$-coefficients, not even the orientations play a role. Note that $p_*(\sigma_i\times\sigma_j)=p_*(\sigma_j\times\sigma_i)$ (when we subdivide appropriately, since we quotient out by τ). Locally and away from the diagonal, therefore $\sum_{i<j}p_*(\sigma_i\times\sigma_j)=\sum_{i\neq j}p_*(\sigma_i\times\sigma_j)$, and therefore away from the diagonal the chain is closed. Since we compute homology relative to U, we therefore have constructed a cycle.

One checks immediately that its cohomology class (actually, even the cycle itself) does not depend on the ordering of the simplices of σ chosen above.

Similarly, if $\sum_j mu_j$ is a second representative, which after subdivision satisfies the same "smallness" condition as $\sum_i\sigma_i$, take a $(k+1)$-chain $\sum\tau_k$ such that $\partial(\sum\tau_k)=\sum_i\sigma_i+\sum_j\mu_j$, and assume again that the simplices τ_k are small as above. We have to show that $\hat{\sigma}$ and $\hat{\mu}$ are cohomologous.

Clearly it suffices to treat the case where τ consists of just one simplex, and to argue by induction.

Then $\sum_i p_*(\sigma_i\times\tau)$ is a $(2k+1)$-chain in the quotient such that

$$\partial(\sum_i p_*(\sigma_i\times\tau))=\sum_i p_*(\sigma_i\times(\sigma-\mu))=\hat{\sigma}-\hat{\tau},$$

using that σ and τ differ only in the simplices o the boundary of the simplex τ (and that we can use compatible orderings, with those boundary of τ simplices maximal), and using that we compute homology relative U and that additional summands or products of parts of the boundary of τ, which by our smallness assumption lie in U.

The fact that the homology class is well defined for a fixed neighborhood of $\widehat{(X,A)}$ implies at the same time that we get a well defined Cech homology class, when passing to the limit. Moreover, it is immediately clear that the construction is natural.

In case X is a manifold (possibly with boundary) and x is the fundamental class, $\hat{x}$ is by construction the fundamental class of $\widehat{X}$.

The second property follows from the description of the intersection product by a suitable intersection of chains. **(Kommentar: This should be more elaborated, it is only needed in Section 1.4.)** $\square$

Using the construction of Lemma 1.7, we can describe explicitly how the fundamental class $[Z,\partial Z]$ is obtained:

1.8 Theorem. *Consider the class*

$$\widehat{[Y,\partial Y]} \in H_{4m+2}(\widehat{(Y,\partial Y)};\mathbb{Z}/2),$$

with $\widehat{(Y,\partial Y)} \subset W\times \widehat{S^m\times} \mathbb{R}^m = (W\times S^m\times\mathbb{R}^m)\times(W\times S^m\times\mathbb{R}^m)/\tau$. *Inside the space, we also find the "antidiagonal"* $\Delta := \{[q,v,w,q,-v,w]\}$, *with* $\Delta \cong W\times\mathbb{R}P^m\times\mathbb{R}^m$. *Observe that* Δ *is disjoint from the singular set* $p(D)$, *the image of the diagonal under the projection.*

Δ *has a fundamental class* $[\Delta,\partial\Delta] \in H_{3m+1}(\Delta,\partial\Delta;\mathbb{Z}/2)$, *where* $\partial W \cong \partial W\times S^m\times\mathbb{R}^m$, *and where we use locally finite homology (because of the non-compactness of* $\mathbb{R}^m$*).*

Define $(Y_2,\partial Y_2) := \widehat{(Y,\partial Y)}\cap(\Delta,\partial\Delta)$. *Observe that this is the correspondence inside* $(\Delta,\partial\Delta)\cong(W,\partial W)\times\mathbb{R}P^m\times\mathbb{R}^m$ *given by* $\{(q,\pm v,w) \mid (q,v,w),(q,-v,w)\in Y\}$.

From this we see that, projecting away the $\mathbb{R}P^m$*-component, we get*

$$(Y_2,\partial Y_2)\xrightarrow{\pi}(Z,\partial Z)\to(W,\partial W),$$

where the second map is the projection onto W, *and the composition is induced by the projection* $W\times\mathbb{R}P^m\times\mathbb{R}^m\to W$.

We define now

$$[Z,\partial Z] := \pi_*\left(\widehat{[Y,\partial Y]}\cap[\Delta,\partial\Delta]\right).$$

We now prove the two theorems. In the second theorem, we explicitly construct the class $[Z,\partial Z]$. It only remains to prove that it is mapped to $[W,\partial W]$ under the projection map.

Now, we use naturality of the intersection product: instead of first taking the intersection (inside $\widehat{(Y,\partial Y)}$) of $\widehat{[Y,\partial Y]}$ with $[\Delta,\partial\Delta]$ and then project to W, we can first send $\widehat{(Y,\partial Y)}$ under the inclusion i to $W\times\widehat{S^m\times}\mathbb{R}^m$, then intersect with $[\Delta,\partial\Delta]$ and then project to W.

Now observe that the construction of $\hat{x}$ also is natural, therefore

$$i_*(\widehat{[Y,\partial Y]}) = i_*(\widehat{[Y,\partial Y]}) = [W,\partial W]\widehat{\times[S^m}\times\{0\}].$$

The latter follows from the fact that $i_*[Y,\partial Y]$ is mapped to $[W,\partial W]\times[S^m]$ under the projection $W\times S^m\times\mathbb{R}^m\to W\times S^m$, and the latter map is a homotopy equivalence.

Finally, by the main property of the hat-construction, $[(W,\widehat{\partial W)\times S}^m\times\{0\}]$ is the fundamental class of the embedded manifold $(W,\widehat{\partial W)\times S}^m\times\{0\}$ (which is a manifold away from the intersection with the diagonal, but the class is a class relative to the diagonal). A homotopy equivalent embedding is given by the set

$$E:=\{[q,v,v_0,q',v',v_0'] \mid q,q'\in W;\ v,v'\in S^m; v=(v_0,v_m), v'=(v_0',v_m') \text{ with } v_0,v_0'\in\mathbb{R}^m\}$$

where we have to take classes of points under the involution τ. An elementary calculation shows that E and Δ intersect transversally, and therefore the class $[(W,\widehat{\partial W)}\times S^m]\cap[\Delta,\partial\Delta]$ (or rather its image in $H_*(\Delta,\partial\Delta;\mathbb{Z}/2)$ or $H_*((W,\widehat{\partial W)\times S}^m\times\mathbb{R}^m;\mathbb{Z}/2))$ is represented by the fundamental class of the intersection manifold. Now one immediately computes

$$E\cap\Delta=\{[q,e_{m+1},0,q,-e_{m+1},0] \mid q\in W;\ e_m=(0,\cdots,0,1)\in S^m\subset\mathbb{R}^m\},$$

i.e. the intersection is represented by an embedded copy of W which is mapped identically to W under the projection map to W. In particular, the image of $[Z,\partial Z]$ in $H_{m+1}(W,\partial W;\mathbb{Z}/2)$ is exactly $[W,\partial W]$, as claimed.

1.9 Remark. Above, at some point locally finite homology and the fundamental class of $\mathbb{R}^m$ in this theory was used. Instead, one could have used the fundamental class of (D^m,S^{m-1}) since all our correspondences are compact and therefore lie in the interior of a sufficiently large sphere.

1.4 Functoriality of the Borsuk-Ulam class

In this part, we explain that our construction in Section 1.3 satisfies an additional naturality property. To explain what is meant by this, assume that inside ∂W, we have a bounded region $(N,\partial N)$, i.e. a submanifold of codimension zero with boundary (which is locally embedded like a half space in Euclidian space).

Let $Y\subset W\times S^m\times\mathbb{R}^m$ be a correspondence. We can restrict this correspondence to $U\in N\times S^m\times\mathbb{R}^m$, with further restriction $\partial U\in\partial N\times S^m\times\mathbb{R}^m$, by simply intersecting with the corresponding subsets. Note that $N\times S^m\times\mathbb{R}^m$ carries a fundamental class $[N,\partial N]\times[S^m\times\mathbb{R}^m]\in H_{3m}((N\times\partial N)\times S^m\times\mathbb{R}^m;\mathbb{Z}/2)$ (again, we have to use locally finite homology).

Define now the fundamental class $[U,\partial U]:=[Y,\partial Y]\cap[(N\times\partial N)\times S^m\times\mathbb{R}^m]\in H_{2m}(U,\partial Y;\mathbb{Z}/2)$. By the naturality of the intersection product, the image of $[U,\partial U]$ in $H_*((N\times\partial N)\times S^m\times\mathbb{R}^m;\mathbb{Z}/2)$ is the intersection

$$[W,\partial W]\times[S^m\times\{0\}]\cap[N\times\partial N]\times[S^m\times\mathbb{R}^m]=[N\times\partial N]\times[S^m],$$

which is mapped to the fundamental class of $[N,\partial N]\times[S^m]$ under the projection. Therefore, $[U,\partial U]$ really is a fundamental class of the correspondence U in our sense.

We call $[U,\partial U$ the restriction of $[Y,\partial Y]$ to the subcorrespondence U.

1.10 Theorem. *$[V,\partial V]$ is the restriction of $[Z,\partial Z]$ to V.*

Proof. This is a direct consequence of the construction and Property (2) of Lemma 1.7, which says that the hat-construction (and therefore the construction of the Borsuk-Ulam class) is compatible with intersection with the subsets corresponding to N, i.e. with restriction to the corresponding subsets. **(Kommentar: This is rather sketchy and can be expanded, if the result is useful at all. I would guess that it should be at the heart of any glueing result.)** □

1.5 The particular case of a graph of a function

(Kommentar: This section is obsolete and will eventually be removed, it shows the first attempts to get the hands on the this circle of questions. The exposition is not complete.)

1.11 Theorem. *Let W be an oriented manifold with boundary $M = \partial W$ of dimension $m+1$, and $f\colon W \times S^n \to \mathbb{R}^n$ a continuous map.*

Define

$$\begin{aligned}
\tilde{X}_S &:= \{(p,v) \in W \times S^n \mid f(p,v) = f(p,-v)\}\\
X_W &:= \{(p,w) \in W \times \mathbb{R}^n \mid \exists v \in S^n \text{ with } f(p,v) = w = f(p,-v)\}\\
\tilde{X}_t &:= \{(p,v,w) \in W \times S^n \times \mathbb{R}^n \mid f(p,v) = w = f(p,-v)\}.
\end{aligned}$$

Observe that the information contained in the S^n coordinate in X_S and X_t is redundant, since v and $-v$ always by definition occur together. Therefore, we also define the following less redundant spaces, using the antipodal action of $\mathbb{Z}/2$ on S^n.

$$\begin{aligned}
X_S &:= \tilde{X}_S/(\mathbb{Z}/2) \subset W \times \mathbb{R}P^n\\
X_t &:= \tilde{X}_t/(\mathbb{Z}/2) \subset W \times \mathbb{R}P^n \times \mathbb{R}^n
\end{aligned}$$

We denote with ∂X_W etc. the subsets which live over ∂W, e.g. $\partial X_W := \{(p,w) \in X_W \mid p \in \partial W\}$.

Then there is a (Cech) homology class $[X] \in H_{m+1}(X_W, \partial X_W; \mathbb{Z}/2)$ which projects to the fundamental class in $H_m(W, \partial W; \mathbb{Z}/2)$.

In particular, the boundary of $[X_W]$ in the long exact sequence of the pair $(X_W, \partial X_W)$ is a class $[\partial X_W] \in H_m(\partial X_W; \mathbb{Z}/2)$ which projects to the fundamental class in $H_m(\partial W; \mathbb{Z}/2)$. Because this is a long exact sequence, the image of $[\partial X_W]$ in $H_m(X_w; \mathbb{Z}/2)$ vanishes.

1.12 Remark. We will prove the stronger result that the cohomology classes are images of corresponding cohomology classes in X_t and ∂X_t, respectively.

The result is a strong form of the Borsuk-Ulam theorem because of the following observation: If for some point $p \in W$ $X_S \cap \{p\} \times \mathbb{R}P^n$ was empty (and the assertion of the Borsuk-Ulam theorem is, that this will never happen) then we had a factorization

$$\begin{tikzcd}
\partial X_W \arrow[r, "i_W"] \arrow[d, "p_\partial"] & X_W \arrow[r, equal] \arrow[d] & X_W \arrow[d] \\
\partial W \arrow[r, "i"] & W \setminus \{p\} \arrow[r] & W.
\end{tikzcd}$$

Since p is an interior point in the manifold with boundary W, $i_*[\partial W] \neq 0 \in H_m(W \setminus \{p\}; \mathbb{Z}/2)$. On the other hand, $[\partial X_W]$ is mapped to zero in $H_m(X_W; \mathbb{Z}/2)$, and therefore also in $H_m(W \setminus \{p\}; \mathbb{Z}/2)$. This is the desired contradiction.

To see, that $i_*[\partial W] \neq 0$, consider the following diagram, where the rows are the exact sequences of the pairs $(W, \partial W)$ and $(W, W \setminus \{p\})$, respectively:

$$\begin{tikzcd}
\arrow[r] & H_{m+1}(W) \arrow[r] \arrow[d, "="] & H_{m+1}(W, \partial W) \arrow[r] \arrow[d] & H_m(\partial W) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H_{m+1}(W) \arrow[r] & H_{m+1}(W, W \setminus \{p\}) \arrow[r] & H_m(W \setminus \{p\}) \arrow[r] & {}
\end{tikzcd}$$

By every manifold is a homology manifold, the fundamental class of $(W, \partial W)$ is mapped to the generator of $H_{m+1}(W, W \setminus \{p\})$. It can not be mapped further to $0 \in H_m(W \setminus \{p\})$, because then it would have an inverse image in $H_{m+1}(W)$, which is zero.

We now prove Theorem 1.11. To do this, observe that X_t is the graph of a continuous function over X_s. As such, both are homeomorphic, and it therefore suffices to construct corresponding homology classes for X_S.

We start with a construction:

Define $g\colon W \times \mathbb{R}P^n = W \times S^n/(\mathbb{Z}/2) \to (S^n \times \mathbb{R}^n)/(\mathbb{Z}/2)$ be given by

$$g(p, v) := (v, f(p, v) - f(p, -v)).$$

Note that this is well defined: if v is replaced by $-v$, $(v, f(p, v) - f(p, -v))$ is replaced by $-(v, f(p, v) - f(p, -v))$. Here, the action of $\mathbb{Z}/2$ on $S^n \times \mathbb{R}^n)$ is given by multiplication with -1. Since $W \times \mathbb{R}P^n$ is compact, we can assume that the image of g is contained in the interior of $(S^n \times D^n)/(\mathbb{Z}/2)$. Note that $\mathbb{Z}/2$ acts freely on $S^n \times D^n$, therefore the quotient is a manifold with boundary.

Example: if $n = 1$, this quotient is homeomorphic to the Möbius strip.

Since the radial contraction of D^n to the origin is equivariant under point reflection at the origin, the projection $(S^n \times D^n)/(\mathbb{Z}/2) \to \mathbb{R}P^n$ is a homotopy equivalence, with inverse the inclusion $S^n/(\mathbb{Z}/2) \to (S^n \times D^n)\mathbb{Z}/2; [v] \mapsto [v, 0]$.

2 Borsuk-Ulam with higher dimensional range

The classical Borsuk-Ulam theorem deals with maps from S^n to $\mathbb{R}^n$. For the applications to game theory we want to consider, the setting is slightly different; we have families of maps from S^n to $\mathbb{R}^{n+2}$, but with an additional fullness property. How this comes about, we recall the following construction of [?].

2.1 Definition. Let K be a finite set and *absK* the simplex spanned by K. Assume that a family $\mathcal{L} := \{L\}$ of subsets of K (i.e. of faces of $|K|$) is given with the following properties:

(1) $\bigcup_{L \in \mathcal{L}} L = K$

(2) whenever $L_1, L_2 \in \mathcal{L}$ with $L_1 \subset L_2$ then $L_1 = L_2$

Note that the second condition means that the situation is interesting only if $K \notin \mathcal{L}$.

Set $J := [-1, 1]$. For each $L \in \mathcal{L}$, a correspondence $F_L \subset |L| \times J^L$ shall be given such that the following properties are satisfied.

(1) F_L is *saturated*, i.e. if $(x, v) \in F_L$ with $x \in |L'| \subset |L|$ for some face $L' \subset L$, then a point $(x', v') \in |F_L| \times J^L$ also belongs to F_L, provided it satisfies the following conditions:

   (a) $x' = x$

   (b) every L'-coordinate of v' coincides with the corresponding coordinate of v

   (c) each $L \setminus L'$-coordinate of v' is bigger or equal to the corresponding coordinate of v

(2) F_L has the *spanning property*, i.e. there is a Cech homology class $[F_L, \partial F_L] \in H_{\dim(|L|)}(F_L, \partial F_L)$ which maps to the fundamental class $[|L|, \partial |L|] \in H_{\dim(|L|)}(|L|, \partial |L|)$, where $\partial F_L = F_L \cap (\partial |L| \times J^L)$, and where we use $\mathbb{Z}/2$-coefficients as above. We use the map in Cech homology induced by the projection $p\colon |L| \times J^L \to |L|$, or rather by its restriction to F_L.

We define the *Borsuk-Ulam* correspondence $\Gamma_{\mathcal{L}} \subset |K| \times J^K$ associated to this data as follows:

(1) First, for each $L \in \mathcal{L}$ define $\tilde{F}_L \subset |L| \times J^K$ to be the product of F_L with $J^{K \setminus L}$. If F_L had dimension $\dim(|L|)$, now $\tilde{F}_L$ has dimension $\dim(|K|)$.

(2) $(p, v) \in |K| \times J^K$ belongs to Γ if and only if there is $k \in \mathbb{N}$ and pairwise distinct $L_1, \cdots, L_k \in \mathcal{L}$ and $(p_j, v) \in \tilde{F}_L$ such that p is contained in the convex hull of $p_1, \ldots, p_k$.

   The condition that all the L_j are distinct implies that the inverse image of a given value $v \in J^K$ of Γ in $|K|$ might be a union of several sets homeomorphic to a simplex, without itself being convex or even contractible.

Our goal is to prove that the correspondence Γ itself is saturated **(Kommentar: Check whether saturatedness can be expected)** and has the spanning property, i.e. a "fundamental class" $[\Gamma, \partial\Gamma] \in H_{\dim(|K|)}(\Gamma, \partial\Gamma)$ which projects to $[|K|, \partial |K|]$.

For this, observe that the following simplifications are possible:

(1) We replace $\mathcal{L}$ by a maximal subset $\mathcal{L}' \subset \mathcal{L}$ which covers K. The resulting Borsuk-Ulam correspondence $\Gamma_{\mathcal{L}'}$ will be a subset of $\Gamma_{\mathcal{L}}$, the one for all of $\mathcal{L}$. If the smaller one supports the required fundamental class, the bigger one certainly also will (as image under the map induced by the inclusion). To avoid clutter notation, we will from now on assume that $\mathcal{L}$ doesn't contain a subset which covers all of K.

(2) Write $\mathcal{L} = \{L\} \cup \mathcal{L}'$ with $\mathcal{L}' \subsetneq \mathcal{L}$. Set $L' := \bigcup_{S \in \mathcal{L}'} S \subsetneq K$. The above construction defines a correspondence $F_{L'} := \Gamma_{\mathcal{L}'} \subset |L'| \times J^{L'}$. If we prove (by induction) that it is saturated and satisfies the spanning property, we can use $\{L, L'\}$ together with the correspondences $(F_L, F_{L'})$ to define a corresponding Borsuk-Ulam correspondence Y_2. Since convex hulls can be defined inductively, Y_2 is identical to $Y_{\mathcal{F}}$. By induction, it therefore suffices to prove that $Y_{\mathcal{F}}$ has the spanning property if $\mathcal{F} = \{F_1, F_2\}$ contains exactly two elements.

(3) Instead of the definition of Y as above, we can also define a Borsuk-Ulam correspondence Y' as in Section 1 by first defining the associated spherical correspondence of Definition 1.2 and then taking the associated Borsuk-Ulam correspondence of Definition 1.5. Note that $Y \subset Y'$ is a closed subset, but possibly strictly smaller because Y' also contains points (p, v) where $p = tp_1 + (1-t)p_2$ with $p_1, p_2 \in |F_1|$ or $p_1, p_2 \in |F_2|$, $0 \leq t \leq 1$ and $(p_1, v), (p_2, v) \in L_{F_1}$ or $\in L_{F_2}$, respectively. These additional points Q form another closed subset of Y', whose image under p is contained in the boundary of $|K|$.

    Apply now the Mayer-Vietoris theorem to the decomposition $Y' = Y \cup Q$ to conclude that a fundamental class $[Y', \partial Y']$ exists if and only if a fundamental class $[Y, \partial Y]$ exists. We will therefore work with Y' instead of Y.

(4) We enlarge the correspondence even further in the following way: for *every* subset $S \subset K$, $S \neq K$ choose a value $x_S > 1 = \sup J$ such that all these values are pairwise disjoint. Define over $|S|$ the correspondence $L_S \subset |S| \times J^{K \setminus S}$ with values (p, v), where each $K \setminus S$-component of v is set to x_S. Take the product with $\mathbb{R}^S$, i.e. allow arbitrary values in the remaining coordinates, to obtain a correspondence contained in $|S| \times \mathbb{R}^K$.

    Perform now the construction of Y or Y' as above, with the bigger boundary correspondence. Since all the x_S are pairwise disjoint and lie outside of J, the resulting Y will again be identical to the original one, apart from an additional closed subset which is in the inverse image of $\partial |K|$ under the projection $|K| \times \mathbb{R}^K \to |K|$, and which doesn't effect the spanning property (as explained in (3)). Then form Y' again by taking all the additional correspondences into account. By the arguments of (3), it suffices to prove that this Y' has the spanning property to conclude that the original Y has the spanning property (i.e. a "fundamental class").

References

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[2] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology.* Princeton University Press, Princeton, New Jersey, 1952. 1.1

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3 Old stuff

More properties of Cech homology

(1) **(Kommentar: This is not needed in the main text and therefore removed)**If, in (7), a finite group G acts freely on W, preserving both pairs (X, A) and (Y, B), then the intersection pairing is natural with

respect to the quotient map, i.e. the following diagram commutes

$$\begin{array}{ccc} H_p(X,A)\otimes H_q(Y,B) & \longrightarrow & H_{p+q-m}(X\cap Y,(A\cap Y)\cup(X\cap B)) \\ \downarrow & & \downarrow \\ H_p(X/G,A/G)\otimes H_q(Y/G,B/G) & \longrightarrow & H_{p+q-m}((X\cap Y)/G,((A\cap Y)\cup(X\cap B))/G) \end{array}$$

(2) The intersection pairing is natural with respect to the long exact sequence of a pair, or more generally of a triple. I.e. if (X,A,A_1) is a compact triple in a manifold W, and (Y,B) is a compact pair, we get the following commuting diagram of long exact sequences:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ H_p(A,A_1)\otimes H_q(Y,B) & \longrightarrow & H_{p+q-m}(A\cap Y,(A_1\cap Y)\cup(A\cap B)) \\ \downarrow & & \downarrow \\ H_p(X,A_1)\otimes H_q(Y,B) & \longrightarrow & H_{p+q-m}(X\cap Y,(A_1\cap Y)\cup(X\cap B)) \\ \downarrow & & \downarrow \\ H_p(X,A)\otimes H_q(Y,B) & \longrightarrow & H_{p+q-m}(X\cap Y,(A\cap Y)\cup(X\cap B)) \\ \downarrow & & \downarrow \\ H_{p-1}(A,A_1)\otimes H_q(Y,B) & \longrightarrow & H_{p+q-1-m}(A\cap Y,(A_1\cap Y)\cup(A\cap B)) \\ \downarrow & & \downarrow \end{array}$$

Here, the horizontal maps are the intersection pairings, and the left vertical exact sequence is the tensor product of the long exact sequence of the triple (X,A,A_1) with the identity on $H_q(Y,B)$. Since we are dealing with $\mathbb{Z}/2$-coefficients, this is indeed an exact sequence.

The right vertical sequence is the exact sequence of the triple $(X\cap Y, A\cap Y\cup X\cap B, A_1\cap Y\cup X\cap B)$, using excision to compare $A\cap Y$ and $A\cap Y\cup X\cap B$.

(Kommentar: Check, whether (2) is in the literature. (1) probably is not; should we give a proof?)

3.1 Theorem. $(Y,\partial Y)\times(Y,\partial Y)$ *is a subset of the manifold with boundary* $(W,\partial W)\times S^m\times\mathbb{R}^m\times(W,\partial W)\times S^m\times\mathbb{R}^m$ *(we straighten angles to obtain a manifold here).*

This manifold also contains the anti-diagonally embedded submanifold with boundary $(\Delta,\partial\Delta)\cong(W,\partial W)\times S^m\times\mathbb{R}^m$ *(meeting the boundary of the product transversally), with embedding*

$$(q,v,w)\mapsto(q,v,w,q,-v,w).$$

Then

$$(Z,\partial Z)=((Y,\partial Y)\times(Y,\partial Y))\cap(\Delta,\partial\Delta).$$

Moreover, Δ *has a fundamental class* $[\Delta,\partial\Delta]\in H_{3m+1}(\Delta,\partial\Delta;\mathbb{Z}/2)$ *(because of the non-compactness of* $\mathbb{R}^m$*, we use locally finite homology here), and from the* *properties of Cech homology we can define the intersection*

$$[Z,\partial Z] := ([Y,\partial Y]\times[Y,\partial Y])\cap[\Delta,\partial\Delta]\in H_{m+1}(Z,\partial Z;\mathbb{Z}/2)$$

(since Z is compact, we don't end up in locally finite Cech homology, but in ordinary Cech homology).

To prove this theorem, we will again have to introduce a number of auxiliary correspondences. Note that the fundamental class $[Z,\partial Z]$ is perfectly well defined from the general properties of the intersection product on Cech homology. We "only" have to check that is has the properties required of a fundamental class. To do this, we need a number of auxiliary lemmas.

Our approach is to perform the intersections inside a manifold where the homology classes represented by the sets we intersect have a nontrivial intersection. For this reason, we translate our obvious intersection with the anti-diagonal in the product into an intersection living over some kind of a projective space.

Before we get there, we need a few preparatory lemmas.

Let $D\subset(W\times S^m\times\mathbb{R}^m)\times(W\times S^m\times\mathbb{R}^m)$ be the diagonal, i.e. $W\cong W\times S^m\times\mathbb{R}^m$ with embedding $(q,v,w)\mapsto(q,v,w,q,v,w)$. Let B be a sufficiently small neighborhood of D which is a domain with smooth boundary. Note that the involution τ which interchanges the factors acts freely on the complement of the interior of B.

3.2 Lemma. *Under the composition*

$$\begin{array}{c} H_{4m+2}((Y,\partial Y)\times(Y,\partial Y);\mathbb{Z}/2 \\ \big\downarrow \\ H_{4m+2}(Y\times Y, Y\times\partial Y\cup\partial Y\times Y\cup(Y\times Y)\cap B;\mathbb{Z}/2) \\ \big\downarrow{\scriptstyle f_*} \\ H_{4m+2}(Y\times Y/\tau,(Y\times\partial Y\cup\partial Y\times Y\cup(Y\times Y)\cap B)/\tau;\mathbb{Z}/2) \end{array}$$

the class $[Y,\partial Y]\times[Y,\partial Y]$ is mapped to 0.

Proof. This follows from the fact that, by choosing a sufficiently small subdivision and using the fact that we compute homology relative to B, we can find singular chain representative for $[Y, boundary Y]\times[Y,\partial Y]$ which is fixed by the involution τ.

Strictly speaking, one has to use the definition of Cech homology as inverse limit of the homology of neighborhoods of $Y\times Y$, and then one has to use suitable decompositions of the product homology class. We also have to use that the particular choice of decomposition doesn't affect the homology class which is represented, and with coefficients in $\mathbb{Z}/2$, not even the orientation is relevant. The details of this are left to the reader.

In any event, after having found such a fixed representative, we observe that its image under f_* will be a formal sum where each summand appears twice. With coefficients in $\mathbb{Z}/2$, it therefore represents 0. □

3.3 Lemma. *Let $(K,\partial K)$ be a pair with a free involution τ. Define $(\hat K,\partial\hat K)$ to be the quotient of this free involution. Then there is a transgression map*

$$H_k(\hat K,\partial\hat K;\mathbb{Z}/2)\to H_k(K,\partial K;\mathbb{Z}/2).$$

A class $x \in H_k(K, \partial K; \mathbb{Z}/2)$ lies in the image of the transgression homomorphism if and only if x is mapped to zero under the projection map

$$H_k(K, \partial K; \mathbb{Z}/2) \to H_k(\hat{K}, \partial\hat{K}; \mathbb{Z}/2).$$

Proof. Note that the projection $K \to \hat{K}$ can be interpreted as bundle projection of an S^0-bundle $S^0 \hookrightarrow K \to \hat{K}$. With coefficients $\mathbb{Z}/q2$, this bundle is orientable, and we therefore get a Gysin sequence

$$H_k(\hat{K}, \partial\hat{K}) \to H_k(K, \partial K) \to H_k(\hat{K}, \partial\hat{K}) \to$$

where the first map is the transgression map and the second map is induced from the projection. The assertion follows.

Note that, by passing to the suitable inverse limit, this Gysin sequence also holds for Cech homology with coefficients in $\mathbb{Z}/2$, at least as long as the spaces involved are compact. □

(Kommentar: von hier wirds eher falsch)

More specifically, define $\widehat{Y} \subset (Y \times Y)/\tau$ by

$$\widehat{Y} := \{[x, y] \mid p_{S^m}(x) = -p_{S^m}(y)\},$$

where τ is the involution with $\tau(x, y) = (y, x)$ (we will use this notation for the square of any space with itself), and where p_{S^m} is the canonical map to S^m induced from the projection $W \times S^m \times \mathbb{R}^m \to S^m$. Note that this is a correspondence inside

$$W \times \widehat{S^m \times} \mathbb{R}^m := \{[q, v, w, q', -v, w'] \in (W \times S^m \times \mathbb{R}^m) \times (W \times S^m \times \mathbb{R}^m)/\tau\}.$$

Observe that we can also think of $W \times \widehat{S^m \times} \mathbb{R}^m$ as the quotient of manifold $W \times \mathbb{R}^m \times S^m \times W \times \mathbb{R}^m$ under the free involution $(q, w, v, q', w') \mapsto (q', w', -v, q, w)$, therefore $W \times \widehat{S^m \times} \mathbb{R}^m$ is itself a manifold.

Let $\widehat{W \times S^m}$ be the corresponding construction without the factor $\mathbb{R}^m$. Again this is a manifold, and the projection onto $W \times S^m \times W \times S^m$ induces a homotopy equivalence

$$p_{\widehat{W \times S^m}} : W \times \widehat{S^m \times} \mathbb{R}^m \to \widehat{W \times S^m}.$$

Note that $\widehat{W \times S^m}$ is a $(3m + 2)$-dimensional compact manifold with boundary $\partial\widehat{W \times S^m}$. As usual, let $\partial\widehat{X}$ be the inverse image of $\partial\widehat{W \times S^m}$ in $\widehat{X}$ under the projection map $p_{\widehat{W \times S^m}}$.

We now want to construct from $[Y, \partial Y]$ a "fundamental class"

$$[\widehat{Y}, \partial\widehat{Y}] \in H_{3m+2}(\widehat{X}, \partial\widehat{X}; \mathbb{Z}/2)$$

which maps onto the fundamental class $[\widehat{W \times S^m}, \partial\widehat{W \times S^m}]$ under $(p_{\widehat{W \times S^m}})_*$.

Here, we feel that such a construction should be quite natural and automatic, and moreover well known. Consider, e.g. the case where Y is the graph of a continuous function $W \times S^m \to \mathbb{R}^m$, where $\widehat{Y}$ is then also a manifold with boundary and $p_{\widehat{W \times S^m}}$ a homeomorphism, in particular a map of degree 1. Since we are not aware of a reference for the general case, we supply our own crude proof of the existence of this fundamental class.

Recall, first, that $H_*(X;\mathbb{Z}/2)$ is the inverse limit of $H_*(U;\mathbb{Z}/2)$, where U runs through the system of open neighborhoods of X in $W\times S^m\times\mathbb{R}^m$. Since X is compact, we can and will choose a cofinal system of such neighborhoods which is homotopy equivalent to a finite CW-complex, such that in particular all the groups $H_*(U;\mathbb{Z}/2)$ are finite.

If we construct $\widehat{U}$ as above, we will obtain a cofinal system of open neighborhoods of $\widehat{X}$ in $W\times\widehat{S^m\times}\mathbb{R}^m$.

The open set U are open subsets of $W\times S^m\times\mathbb{R}^m$ and as such are submanifolds with boundary $\partial U=U\cap(\partial W\times S^m\times\mathbb{R}^m)$. Similarly, $\widehat{U}$ is a manifold with boundary $\partial\widehat{U}$.

Recall now from [3] that $\Omega_k^O(U)\to$

The first one is

$$Y\times\tau(Y)\subset(S\times S^m\times\mathbb{R}^m)\times(W\times S^m\times\mathbb{R}^m),$$

defined by

$$(q,v,w,q_2,v_2,w_2)\in Y\times\tau(Y)\iff(q,v,w),(q_2,-v_2,w_2)\in Y.$$

Let $\tau\colon(W\times S^m\times\mathbb{R}^m)\times(W\times S^m\times\mathbb{R}^m)$ be the involution given by

$$\tau(q,v,w,q_2,v_2,w_2)=(q_2,-v_2,w_2,q,-v,w).$$

Note that this is a *free* involution, hence the quotient is still a manifold. We denote this quotient by $W\times\widehat{S^m\times}\mathbb{R}^m$. The obvious projection to $\widehat{W\times S^m}=(W\times S^m\times W\times S^m)/\tau$ is a homotopy equivalence, using the radial (and τ-equivariant) contraction of the $\mathbb{R}^m$-factors.

Observe that $Y\times\tau Y$ is τ-invariant, and therefore its quotient $\widehat{Y}\subset W\times\widehat{S^m\times}\mathbb{R}^m$ defines a compact correspondence in this manifold.

We need the following lemma.

3.4 Lemma. *Let $X\subset W\times S^m\times\mathbb{R}^m$ be a compact correspondence with a fundamental class $[X,\partial X]\in H_{2m+1}(X,\partial X;\mathbb{Z}/2)$, i.e. the image $\alpha_*([X,\partial X])=[(W,\partial W)\times S^m]\in H_{2m+1}(W\times S^m,\partial W\times S^m;\mathbb{Z}/2)$. Define $\hat{X}\subset W\times\widehat{S^m\times}\mathbb{R}^m$ as above. Then there is a fundamental class*

$$[\widehat{X},\partial\widehat{X}]\in H_{4m+2}(\widehat{X},\partial\widehat{X};\mathbb{Z}/2)$$

which maps to $[\widehat{W\times S^m},\partial\widehat{W\times S^m}]\in H_{4m+2}(\widehat{W\times S^m},\partial\widehat{W\times S^m};\mathbb{Z}/2)$ under the map induced from the projection.

Here, as usual ∂X and $\partial\widehat{X}$ are the subsets sitting over the boundaries of the manifolds $W\times S^m$ and $\widehat{W\times S^m}$, respectively.

p

Proof. This should be another well known result in Cech homology. Since we are not aware of a reference, we include a proof here.

More specifically, let $[X,\partial X]$ be represented by homology classes

$$[X_U,\partial X_U]\in H_{2m+1}(U,U\cap(\partial W\times S^m\times\mathbb{R}^m);\mathbb{Z}/2),$$

where U runs through the system of (relatively compact) open neighborhoods of $X \subset W \times S^m \times \mathbb{R}^m$. By representation results of Thom [3], each of the homology classes $[X_U, \partial X_U]$ are represented (i.e. the image of the fundamental class) of a (by general position immersed) submanifold X_U with boundary ∂X_U, where X_U meets $\partial U = U \cap \partial W \times S^m \times \mathbb{R}^m$ transversally at the boundary.

Then the sets $U \times \tau(U)$ form a cofinal system of open neighborhoods of $X \times \tau(X) \in (W \times S^m \times \mathbb{R}^m) \times (W \times S^m \times \mathbb{R}^m)$. Each of these are invariant under the involution τ and therefore the quotients define a cofinal system of open neighborhoods $\widehat{U} = (U \times \tau U)/\mathbb{Z}/2$ of $\widehat{X}$.

Note, moreover, that the immersed submanifolds $X_U \times \tau(X_U)$ inherit (since they are locally embedded) the free involution τ: let $i\colon X_U \to U$ be the immersion and $\tau \circ i\colon X_U \to U$ the immersion of "$\tau(X_U)''$. At each multiple point of the immersion $i \times \tau \circ i$, by transversality two leaves of X_U intersect, and for each leaf its image under τ is well defined, which defines the lift of τ from the image $i \times \tau(i)$ to $X_U \times X_U$. Then $\widehat{X_U} := (X_U \times \tau(X_U))/\mathbb{Z}/2$ is an immersed submanifold of $\widehat{U}$ (with boundary meeting the boundary of $\widehat{U}$ transversally). Consequently it represents a homology class $[\widehat{X_U}, \widehat{\partial X_U}] \in H_{2m+1}(\widehat{U}, \partial\widehat{U}; \mathbb{Z}/2)$.

If $i\colon V \hookrightarrow U$ is an inclusion of two of the neighborhoods of X, then $i_*[X_V, \partial X_V] = [X_U, \partial X_U]$, and the homology is (again by the results of Thom) represented by a bordism immersed into U. As above, this can be used to see that $\hat{i}_*[\widehat{X_V}, \widehat{\partial X_V}] = [\widehat{X_U}, \widehat{\partial X_U}] \in H_{2m+1}(\widehat{U}, \partial\widehat{U}; \mathbb{Z}/2)$. Consequently, the classes $[\widehat{X_U}, \widehat{\partial X_U}]$ define a Cech homology class $[\widehat{X}, \partial\widehat{X}] \in H_{2m+1}(\widehat{X}, \partial\widehat{X}; \mathbb{Z}/2)$. The image of this class $[\widehat{X}, \partial\widehat{X}]$ in $H_{2m+1}(W \times S^m, \partial W \times S^m; \mathbb{Z}/2)$ is by definition the image of any of the classes $[\widehat{X_U}, \widehat{\partial X_U}]$.

To compute the latter, let $f\colon X_U \to W \times S^m$ be a regular (either smooth or PL) approximation to map induced by the projection map, and let $q \in W^\circ \times S^m$ be a regular value. Then $\left|f^{-1}(q)\right| \equiv 1(2)$, since the degree of α is 1 (in $\mathbb{Z}/2$-homology).

Approximate by immersed submanifolds, then count regular points. $\square$

3.1 Restriction to boundary

Then we can consider the triple of spaces $(Y, \partial Y, U_2)$ inside $(W, \partial W, N_2) \times S^m \times \mathbb{R}^m$. Associated to it is the following commutative diagram of a piece of the long exact sequences of triples (since we are using coefficients in $\mathbb{Z}/2$, Cech homology is exact):

$$\begin{array}{ccccc} H_{2m+1}(Y, \partial Y) & \longrightarrow & H_{2m}(\partial Y, U_2) & \longrightarrow \\ \downarrow & & \downarrow & \\ H_{2m+1}(W \times S^m, \partial W \times S^m) & \longrightarrow & H_{2m}(\partial W \times S^m, N_2 \times S^m) & \longrightarrow \end{array} \tag{3.5}$$

The horizontal maps are induced by the projection $W \times S^m \times \mathbb{R}^m \to W \times S^m$.

By excision, the groups on the right hand side are isomorphic to

$$\begin{aligned} H_{2m}(U, \partial U; \mathbb{Z}/2) &\xrightarrow{\cong} H_{2m}(\partial Y, U_2; \mathbb{Z}/2) \\ H_{2m}(N \times S^m, \partial N \times S^m; \mathbb{Z}/2) &\xrightarrow{\cong} H_{2m}(\partial W \times S^m, N_2 \times S^m; \mathbb{Z}/2) \end{aligned} \tag{3.6}$$

Moreover, we know that for $W, N, \cdots$ the fundamental class of $W \times S^m$ is mapped under the boundary map in the diagram (3.5) to the fundamental class of $N \times S^m$.

Therefore, if $[Y, \partial Y] \in H_{2m+1}(Y, \partial Y; \mathbb{Z}/2)$ is a fundamental class (i.e. mapped to $[W \times S^m]$), then its image under the boundary map in (3.5) is a fundamental class $[U, \partial U] \in H_{2m}(U, \partial U; \mathbb{Z}/2)$ (using the excision isomorphism (3.6)), i.e. is mapped to the fundamental class of $N \times S^m$.

This we will call the "restriction" of the class $[Y, \partial Y]$ to U, and this is the fundamental class we will use throughout for U.

Let now $(Z, \partial Z) \subset W \times \mathbb{R}^m$ be the associated Borsuk-Ulam correspondence of $(Y, \partial Y)$ with associated fundamental class $[Z, \partial Z]$ as in Theorem 1.6. On the other hand, let $(V, \partial V)$ the Borsuk-Ulam correspondence of $(U, \partial U)$. Then, to the restricted fundamental class $[U, \partial U]$ is associated by Theorem 1.6 a fundamental class $[V, \partial V]$.