The spectral measure of certain elements of the complex group ring of a wreath product

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Abstract

We use elementary methods to compute the L^2-dimension of the eigenspaces of the Markov operator on the lamplighter group and of generalizations of this operator on other groups. In particular, we give a transparent explanation of the spectral measure of the Markov operator on the lamplighter group found by Grigorchuk-Zuk [4]. The latter result was used by Grigorchuk-Linnell-Schick-Zuk [3] to produce a counterexample to a strong version of the Atiyah conjecture about the range of L^2-Betti numbers.

We use our results to construct manifolds with certain L^2-Betti numbers (given as convergent infinite sums of rational numbers) which are not obviously rational, but we have been unable to determine whether any of them are irrational.

1 Notation and statement of main result

In this section we introduce notation that will be fixed throughout and will be used in the statement of the main result.

Let U denote a discrete group with torsion.

Let e be a nontrivial projection (so $e = e^* = e^2$, $e \neq 0, 1$) in $\mathbb{C}[U]$. For example, U could be finite and nontrivial, and e could be the 'average' of the elements of U,

$$\operatorname{avg}(U) := \frac{1}{|U|}\sum_{u\in U} u.$$

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This will be the example we shall make the most use of.

Let $W = W(U, e)$ denote the inverse of the coefficient of 1 in the expression of e as a $\mathbb{C}$-linear combination of elements of U. By results of Kaplansky and Zaleskii, W is a rational number greater than 1. For example, if U is finite and nontrivial, and $e = \operatorname{avg}(U)$, then $W = |U|$.

For integers m, n, with $1 \leq m \leq n-1$, let $\lambda_{m,n} := 2\cos(\frac{m}{n}\pi)$.

For any integer $n \geq 2$, let $M_n := \{\lambda_{m,n} \mid 1 \leq m \leq n-1, m \text{ coprime to } n\}$.

We write

$$U \wr \mathbb{Z} := (\oplus_{i\in\mathbb{Z}} U) \rtimes C_\infty,$$

where C_∞ denotes an infinite cyclic group with generator $t = t_U$ which acts on $\oplus_{i\in\mathbb{Z}} U$ by the shift, i.e. $t^{-1}((g_n)_{n\in\mathbb{Z}})t = (g_{n-1})_{n\in\mathbb{Z}}$. For each $u \in U$, let a_u denote $(\dots, 1, u, 1, \dots) \in \oplus_{i\in\mathbb{Z}} U$ where u occurs with index 0. Throughout, we identify u with a_u. Thus U is a subgroup of $U \wr \mathbb{Z}$. Notice that $U \wr \mathbb{Z}$ is generated by t and U.

Set

$$T = T(U, e) := \big(et + t^{-1}e\big) \in \mathbb{C}[U \wr \mathbb{Z}].$$

If U is finite and nontrivial, and $e = \operatorname{avg}(U)$, then T is two times the Markov operator of $U \wr \mathbb{Z}$ with respect to the symmetric set of generators $\{ut, (ut)^{-1} \mid u \in U\}$.

Let $\mathcal{N}(U \wr \mathbb{Z})$ denote the (von Neumann) algebra of bounded linear operators on the Hilbert space $l^2(U \wr \mathbb{Z})$ which commute with right multiplication by elements of $U \wr \mathbb{Z}$. We identify each element x of $\mathbb{C}[U \wr \mathbb{Z}]$ with an element of $l^2(U \wr \mathbb{Z})$ in the natural way, and also with the element of $\mathcal{N}(U \wr \mathbb{Z})$ given by left multiplication by x. Thus $\mathbb{C}[U \wr \mathbb{Z}]$ is viewed as a subset of $l^2(U \wr \mathbb{Z})$ and as a subalgebra of $\mathcal{N}(U \wr \mathbb{Z})$. For $a \in \mathcal{N}(U \wr \mathbb{Z})$ the (regularized) trace of a is defined as

$$\operatorname{tr}_{U\wr\mathbb{Z}}(a) := \langle a(1), 1\rangle_{l^2(U\wr\mathbb{Z})}.$$

Similar notation applies for any group.

Note that, if $a \in \mathcal{N}(U \wr \mathbb{Z})$ leaves invariant $l^2(G)$ for a subgroup G, then we can consider a to be an element of $\mathcal{N}(G)$, and here $\operatorname{tr}_G(a)$ and $\operatorname{tr}_{U\wr\mathbb{Z}}(a)$ coincide.

Note also that, if a lies in $\mathbb{C}[U \wr \mathbb{Z}]$, then $\operatorname{tr}_{U\wr\mathbb{Z}}(a)$ is the coefficient of 1 in the expression of a as a $\mathbb{C}$-linear combination of elements of $U \wr \mathbb{Z}$.

The element (left multiplication by) T of $\mathcal{N}(U \wr \mathbb{Z})$ is self-adjoint. For each $\mu \in \mathbb{R}$, let $\operatorname{pr}_\mu \colon l^2(U \wr \mathbb{Z}) \to l^2(U \wr \mathbb{Z})$ denote the orthogonal projection onto $\ker(T - \mu)$, so $\operatorname{pr}_\mu \in \mathcal{N}(U \wr \mathbb{Z})$. The number

$$\dim_{U\wr\mathbb{Z}} \ker(T - \mu) := \langle \operatorname{pr}_\mu(1), 1\rangle_{l^2(U\wr\mathbb{Z})} = \operatorname{tr}_{U\wr\mathbb{Z}}(\operatorname{pr}_\mu)$$

is called the L^2-multiplicity of μ as an eigenvalue of T.

Our main result is the following.

1.1 Theorem. *With all the above notation, for any $\mu \in \mathbb{R}$,*

$$\dim_{U\wr\mathbb{Z}} \ker(T - \mu) = \begin{cases} \frac{(W-1)^2}{W^n - 1} & \text{if } n \geq 2 \text{ and } \mu \in M_n, \\ 0 & \text{if } \mu \notin \bigcup_{n\geq 2} M_n. \end{cases}$$

Moreover, $l^2(U \wr \mathbb{Z})$ is the Hilbert sum of the eigenspaces of T, i.e. the spectral measure of T off its eigenspaces is zero.

In [4, Corollary 3], Grigorchuk-Zuk proved the case of this result in which U is (cyclic) of order two and $e = \mathrm{avg}(U)$, so $W = 2$. This was used in [3] to give a counterexample to a strong version of the Atiyah conjecture about the range of L^2-Betti numbers. The argument in [4] is based on automata and actions on binary trees, while our proof is based on calculating traces of projections in the group ring $\mathbb{C}[U \wr \mathbb{Z}]$.

2 Preliminary matrix calculations

In this section, we introduce more notation which will be used throughout, and verify some identities which will be used in the proof.

For positive integers i, j, let

$$\alpha_{i,j} := \delta_{|i-j|,1} = \begin{cases} 1 & \text{if } i - j = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each integer $n \geq 2$, let A_n denote the $n-1 \times n-1$ matrix

$$A_n = (\alpha_{i,j})_{1 \leq i,j \leq n-1} = \begin{pmatrix} 0 & 1 & 0 & \dots\dots\dots\dots \\ 1 & 0 & 1 & 0 & \dots\dots\dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots & 0 & 1 & 0 & 1 \\ \dots\dots\dots\dots & & 0 & 1 & 0 \end{pmatrix}.$$

Recall that $\lambda_{m,n}$ denotes $2\cos(\frac{m}{n}\pi)$.

2.1 Lemma. *For each $n \geq 2$, the family of eigenvalues of A_n, with multiplicities, is $\{\lambda_{m,n} \mid 1 \leq m \leq n-1\}$.*

Proof. For a complex number μ different from $0, 1, -1$, one checks immediately by induction on n, and determinant expansion of the first row, that

$$\det(A_n + (\mu + \mu^{-1})I_{n-1}) = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}.$$

Now, for $1 \leq m \leq n-1$, taking $\mu = -e^{\frac{m}{2n}2\pi i}$ shows that $\lambda_{m,n}$ is an eigenvalue of A_n. Since we have $n-1$ distinct eigenvalues for A_n, they all have multiplicity one. □

For $n \geq 2$, A_n is a real symmetric matrix, so there exists a real orthogonal matrix $B_n = (\beta^{(n)}_{i,j})_{1 \leq i,j \leq n-1}$ such that $B_n A_n B_n^*$ is a diagonal matrix D_n; here the diagonal entries are $\lambda_{m,n}$, $1 \leq m \leq n-1$, and we may assume the entries occur in this order, so $D_n = (\delta_{i,j}\lambda_{j,n})_{1 \leq i,j \leq n-1}$. Since $B_n B_n^* = I_{n-1}$ and $B_n A_n = D_n B_n$ we have the identities

$$\sum_{j=1}^{n-1} \beta^{(n)}_{i,j} \beta^{(n)}_{k,j} = \delta_{i,k}, \quad 1 \leq i,k \leq n-1, \tag{2.2}$$

$$\sum_{j=1}^{n-1} \beta^{(n)}_{i,j} \alpha_{j,k} = \lambda_{i,n} \beta^{(n)}_{i,k}, \quad 1 \leq i,k \leq n-1. \tag{2.3}$$

3 Proof of the main result

We shall frequently use the following, which is well known and easy to prove.

3.1 Lemma. *Let G and H be discrete groups, and let $p \in \mathcal{N}(G)$ and $q \in \mathcal{N}(H)$. Embed G and H in the canonical way into $G \times H$, so p and q become elements of $\mathcal{N}(G \times H)$. Then*

$$\mathrm{tr}_{G\times H}(pq) = \mathrm{tr}_G(p) \cdot \mathrm{tr}_H(q). \quad \square$$

We need even more notation.

For each $i \in \mathbb{Z}$, we define, in $\mathbb{C}[U \wr \mathbb{Z}]$, $e_i := t^{-i}et^i$ and $f_i := 1 - e_i$.

It is easy to see that all the e_i, f_j are projections which commute with each other; moreover,

$$\mathrm{tr}_{U\wr\mathbb{Z}}(e_i) = \mathrm{tr}_{U\wr\mathbb{Z}}(e) = \frac{1}{W} \text{ and } \mathrm{tr}_{U\wr\mathbb{Z}}(f_i) = 1 - \frac{1}{W}. \tag{3.2}$$

For $n \geq 2$, let $q_n := f_1e_2e_3 \cdots e_{n-2}e_{n-1}f_n$. It is clear that q_n is a projection. Moreover, the factors lie in $\mathbb{C}[t^{-i}Ut^i]$, $1 \leq i \leq n$, so, by Lemma 3.1,

$$\mathrm{tr}_{U\wr\mathbb{Z}}(q_n) = \mathrm{tr}_{U\wr\mathbb{Z}}(f_1)\,\mathrm{tr}_{U\wr\mathbb{Z}}(e_2) \cdots \mathrm{tr}_{U\wr\mathbb{Z}}(e_{n-1})\,\mathrm{tr}_{U\wr\mathbb{Z}}(f_n).$$

By (3.2),

$$\mathrm{tr}_{U\wr\mathbb{Z}}(q_n) = (1 - \frac{1}{W})^2(\frac{1}{W})^{n-2} = \frac{(W-1)^2}{W^n}. \tag{3.3}$$

3.4 Lemma. *If $1 \leq m < n$ and $1 \leq m' < n'$ then*

$$q_{n'}t^{-m'}t^mq_n = \delta_{n,n'}\delta_{m,m'}q_n.$$

Proof. Note that $t^mq_nt^{-m} = f_{1-m}e_{2-m} \cdots e_{n-m-1}f_{n-m}$, and this is a projection. Thus

$$(t^mq_nt^{-m} \mid n \geq 2, 1 \leq m < n) = (f_{-i}e_{-i+1} \cdots e_{j-1}f_j \mid -i \leq 0, 1 \leq j).$$

This is a family of pairwise orthogonal projections, since, if $-i, -i' \leq 0, 1 \leq j, j'$, then either $(i,j) = (i',j')$, or the product of $f_{-i}e_{-i+1} \cdots e_{j-1}f_j$ and $f_{-i'}e_{-i'+1} \cdots e_{j'-1}f_{j'}$ is zero since it contains a factor $e_\alpha f_\alpha = 0$ for at least one $\alpha \in \{-i, -i', j, j'\}$. Since t is invertible, the result follows. $\square$

Notice that, for $1 \leq m < n$,

$$\begin{aligned} T(t^mq_n) &= ett^mq_n + t^{-1}et^mq_n \\ &= t^{m+1}e_{m+1}q_n + t^{m-1}e_mq_n \\ &= t^{m+1}(1 - \delta_{m,n-1})q_n + t^{m-1}(1 - \delta_{m,1})q_n. \end{aligned}$$

Hence

$$T(t^mq_n) = \sum_{i=1}^{n-1} \alpha_{m,i}t^iq_n. \tag{3.5}$$

For $1 \leq m \leq n-1$, define $r_{m,n} := \sum_{i=1}^{n-1} \beta_{m,i}^{(n)}t^iq_n$ and $p_{m,n} := r_{m,n}r_{m,n}^*$. Observe that, if we identify the ith standard basis vector with t^iq_n, $1 \leq i \leq n-1$, then $r_{m,n}$ is an eigenvector of A_n with eigenvalue $\lambda_{m,n}$. Moreover, we have just checked that T acts like A_n on the span of the $t^m q_n$. This partially explains why the $r_{m,n}$ give rise to pairwise orthogonal projections with image contained in the eigenspace of T for the eigenvalue $\lambda_{m,n}$, which is essentially the statement of the following lemma.

3.6 Lemma. $(p_{m,n} \mid n \geq 2, 1 \leq m \leq n-1)$ *is a family of pairwise orthogonal projections in* $\mathbb{C}[U \wr \mathbb{Z}]$ *which is complete, that is,* $\sum_{n\geq 2}\sum_{m=1}^{n-1} \operatorname{tr}_{U\wr\mathbb{Z}}(p_{m,n}) = 1$. *Moreover, if* $1 \leq m \leq n-1$*, then* $T(p_{m,n}) = \lambda_{m,n} p_{m,n}$.

Proof. Let $1 \leq m \leq n-1$ and $1 \leq m' \leq n'-1$.

Here

$$r^*_{m,n} = q^*_n \sum_{i=1}^{n-1} (t^i)^* \beta^{(n)*}_{m,i} = q_n \sum_{i=1}^{n-1} t^{-i} \beta^{(n)}_{m,i}.$$

Thus

$$\begin{aligned} r^*_{m',n'} r_{m,n} &= q_{n'} \sum_{j=1}^{n'-1} t^{-j} \beta^{(n')}_{m',j} \sum_{i=1}^{n-1} \beta^{(n)}_{m,i} t^i q_n \\ &= \delta_{n,n'} q_n \sum_{i=1}^{n-1} \beta^{(n)}_{m',i} \beta^{(n)}_{m,i} \text{ by Lemma 3.4} \\ &= \delta_{n,n'} q_n \delta_{m,m'} \text{ by (2.2)}. \end{aligned}$$

It follows that the $p_{m,n}$ are pairwise orthogonal.

Moreover,

$$\begin{aligned} \operatorname{tr}_{U\wr\mathbb{Z}}(p_{m,n}) &= \operatorname{tr}_{U\wr\mathbb{Z}}(r_{m,n} r^*_{m,n}) = \operatorname{tr}_{U\wr\mathbb{Z}}(r^*_{m,n} r_{m,n}) \\ &= \operatorname{tr}_{U\wr\mathbb{Z}}(q_n) = \frac{(W-1)^2}{W^n} \text{ by (3.3)}. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{n\geq 2}\sum_{m=1}^{n-1} \operatorname{tr}_{U\wr\mathbb{Z}}(p_{m,n}) &= \sum_{n\geq 2}\sum_{m=1}^{n-1} \frac{(W-1)^2}{W^n} = \sum_{n\geq 2} (n-1)\frac{(W-1)^2}{W^n} \\ &= \sum_{n\geq 1} n \frac{(W-1)^2}{W^{n+1}} = (1-\frac{1}{W})^2 \sum_{n\geq 1} n (\frac{1}{W})^{n-1} = 1, \end{aligned}$$

since, for $|x| < 1$, $\sum_{n\geq 1} n x^{n-1} = (\sum_{n\geq 0} x^n)' = (\frac{1}{1-x})' = \frac{1}{(1-x)^2}$.

Also,

$$\begin{aligned} T(r_{m,n}) &= T(\sum_{j=1}^{n-1}\beta^{(n)}_{m,j}t^j q_n) = \sum_{j=1}^{n-1}\beta^{(n)}_{m,j}T(t^j q_n) \\ &= \sum_{j=1}^{n-1}\beta^{(n)}_{m,j}\sum_{k=1}^{n-1}\alpha_{j,k}t^k q_n \text{ by (3.5)} \\ &= \sum_{k=1}^{n-1}(\sum_{j=1}^{n-1}\beta^{(n)}_{m,j}\alpha_{j,k})t^k q_n \\ &= \sum_{k=1}^{n-1}\lambda_{m,n}\beta^{(n)}_{m,k}t^k q_n \text{ by (2.3)} \\ &= \lambda_{m,n}r_{m,n}. \end{aligned}$$

Thus $T(r_{m,n}) = \lambda_{m,n}r_{m,n}$, and, on right multiplying by $r^*_{m,n}$, we see $T(p_{m,n}) = \lambda_{m,n}p_{m,n}$. □

We have now 'diagonalized' T in the sense that we have decomposed $l^2(U\wr\mathbb{Z})$ into the Hilbert sum of subspaces of the form $p_{m,n}(l^2(U\wr\mathbb{Z}))$ on which T acts as multiplication by the scalar $\lambda_{m,n}$.

Hence, for each $\mu\in\mathbb{R}$, $\ker(T-\mu)$ is the Hilbert sum of those $p_{m,n}(l^2(U\wr\mathbb{Z}))$ such that $\lambda_{m,n}=\mu$. Thus either $\ker(T-\mu)=0$ or $\mu=\lambda_{m_0,n_0}$ for some m_0, n_0 with $1\le m_0\le n_0-1$.

We now consider the latter case. Here, for all (m,n), $\lambda_{m,n}=\mu$ if and only if $\frac{m}{n}=\frac{m_0}{n_0}$. We may assume that m_0 and n_0 are coprime, so $\mu\in M_{n_0}$. Also, $\lambda_{m,n}=\mu$ if and only if $(m,n)=(im_0,in_0)$ for some $i\ge 1$. Thus $\ker(T-\mu)$ is the Hilbert sum of the $p_{im_0,in_0}(l^2(U\wr\mathbb{Z}))$ with $i\ge 1$; hence

$$\begin{aligned} \dim_{U\wr\mathbb{Z}}(\ker(T-\lambda_{m_0,n_0})) &= \sum_{i\ge 1}\dim_{U\wr\mathbb{Z}}(p_{im_0,in_0}(l^2(U\wr\mathbb{Z}))) \\ = \sum_{i\ge 1}\operatorname{tr}_{U\wr\mathbb{Z}}(p_{im_0,in_0}) &= \sum_{i\ge 1}\frac{(W-1)^2}{W^{in_0}} = \frac{(W-1)^2}{W^{n_0}-1}. \end{aligned}$$

Theorem 1.1 now follows.

3.7 Remarks. The hypothesis in Theorem 1.1 that U has torsion could be weakened to the assumption that $\mathbb{C}[U]$ has a nontrivial projection; however, if U is torsion-free, it is conjectured, and known in many cases, that $\mathbb{C}[U]$ does not contain any nontrivial projections.

It easy to show that the hypothesis in Theorem 1.1 that e is a nontrivial projection in $\mathbb{C}[U]$ can be weakened to the assumption that e is a nontrivial projection in $\mathcal{N}(U)$; here, the hypothesis that U has torsion should be weakened to the assumption that U is nontrivial. □

4 Direct products of wreath products

We now produce even more unusual examples by taking direct products of the groups studied so far.

4.1 Theorem. *Let U and V be groups with torsion, and $G = (U \wr \mathbb{Z}) \times (V \wr \mathbb{Z})$. Let e be a nontrivial projection in $\mathbb{C}[U]$ and f a nontrivial projection in $\mathbb{C}[V]$. Let $X = (\mathrm{tr}_U(e))^{-1}$ and $Y = (\mathrm{tr}_V(f))^{-1}$, so $X > 1$, $Y > 1$. Let $T = T(U, e) \in \mathbb{C}[U \wr \mathbb{Z}] \subset \mathbb{C}[G]$, and $S = T(V, f) \in \mathbb{C}[V \wr \mathbb{Z}] \subset \mathbb{C}[G]$. Then*

$$\begin{aligned} &\dim_G(\ker(T - S)) \\ &= (X-1)^2(Y-1)^2\Big(\sum_{m\geq 1}\sum_{n\geq 1} \frac{\gcd(m,n)}{X^m Y^n}\Big) - (X-1)(Y-1). \end{aligned} \tag{4.2}$$

Proof. By Lemma 3.6, there is a complete family $(p_{m,n} \mid n \geq 2, 1 \leq m < n)$ of pairwise orthogonal projections in $\mathbb{C}[U \wr \mathbb{Z}]$, such that, if $1 \leq m < n$, then $T(p_{m,n}) = \lambda_{m,n} p_{m,n}$, and, by (3.3), $\mathrm{tr}_{U\wr\mathbb{Z}}(p_{m,n}) = \frac{(X-1)^2}{X^n}$.

Similarly, there is a complete family $(q_{m,n} \mid n \geq 2, 1 \leq m < n)$ of pairwise orthogonal projections in $\mathbb{C}[V \wr \mathbb{Z}]$ such that, if $1 \leq m < n$, then $S(q_{m,n}) = \lambda_{m,n} q_{m,n}$, and $\mathrm{tr}_{V\wr\mathbb{Z}}(q_{m,n}) = \frac{(Y-1)^2}{Y^n}$.

By Lemma 3.1, there is a complete family

$$(p_{m,n} q_{m',n'} \mid n, n' \geq 2, 1 \leq m < n, 1 \leq m' < n')$$

of pairwise orthogonal projections in $\mathbb{C}[G]$, such that, if $1 \leq m < n$ and $1 \leq m' < n'$ then

$$T(p_{m,n} q_{m',n'}) = \lambda_{m,n} p_{m,n} q_{m',n'} \text{ and } S(p_{m,n} q_{m',n'}) = \lambda_{m',n'} p_{m,n} q_{m',n'},$$

and

$$\mathrm{tr}_G(p_{m,n} q_{m',n'}) = \frac{(X-1)^2}{X^n} \frac{(Y-1)^2}{Y^{n'}}.$$

Thus $l^2(G)$ is the Hilbert sum of the subspaces of the form $p_{m,n} q_{m',n'}(l^2(G))$ where $T - S$ acts as multiplication by the scalar $\lambda_{m,n} - \lambda_{m',n'}$.

Hence $\ker(T - S)$ is the Hilbert sum of the $p_{m,n} q_{m',n'}(l^2(G))$ such that $\lambda_{m,n} = \lambda_{m',n'}$.

Therefore,

$$\dim_G(\ker(T - S)) = \sum_{n\geq 1}\sum_{n'\geq 1} b(n, n') \frac{(X-1)^2}{X^n} \frac{(Y-1)^2}{Y^{n'}}$$

where $b(n, n')$ is the number of pairs (m, m') such that $1 \leq m < n$, $1 \leq m' < n'$, and $\frac{m}{n} = \frac{m'}{n'}$. But such pairs correspond bijectively to the fractions of the form $\frac{m_0}{\gcd(n,n')}$, $1 \leq m_0 < \gcd(n, n')$. Thus $b(n, n') = \gcd(n, n') - 1$. Hence

$$\begin{aligned} \dim_G(\ker(T - S)) &= \sum_{n\geq 1}\sum_{n'\geq 1} \frac{(\gcd(n,n') - 1)(X-1)^2(Y-1)^2}{X^n Y^{n'}} \\ &= \sum_{n\geq 1}\sum_{n'\geq 1} \frac{\gcd(n,n')(X-1)^2(Y-1)^2}{X^n Y^{n'}} - \sum_{n\geq 1}\sum_{n'\geq 1} \frac{(X-1)^2(Y-1)^2}{X^n Y^{n'}}. \end{aligned}$$

Since $\sum_{n\geq 1} \frac{1}{X^n} = X^{-1}\frac{1}{1-X^{-1}} = \frac{1}{X-1}$, the result follows. □

4.3 Remarks. Recall that, for any positive integer n, $\phi(n)$ denotes the number of primitive nth roots of unity, so $|M_n| = \phi(n)$.

For $X > 1$, $Y > 1$, the double infinite sum occurring in (4.2) has an expession as a single infinite sum,

$$\sum_{m\geq 1}\sum_{n\geq 1}\frac{\gcd(m,n)}{X^m Y^n} = \sum_{k\geq 1}\frac{\phi(k)}{(X^k-1)(Y^k-1)},$$

since

$$\sum_{k\geq 1}\frac{\phi(k)}{(X^k-1)(Y^k-1)} = \sum_{k\geq 1}\phi(k)\sum_{i\geq 1}X^{-ik}\sum_{j\geq 1}Y^{-jk} = \sum_{m\geq 1}\sum_{n\geq 1}\frac{a(m,n)}{X^m Y^n}$$

where

$$a(m,n) = \sum_{\{k\geq 1: k|m, k|n\}}\phi(k) = \sum_{k|\gcd(m,n)}\phi(k) = \gcd(m,n).$$

It follows that

$$\dim_G(\ker(T-S)) = (X-1)^2(Y-1)^2\sum_{k\geq 2}\frac{\phi(k)}{(X^k-1)(Y^k-1)}. \qquad \square$$

5 L^2-Betti numbers

We previously observed that, by results of Kaplansky and Zaleskii, the traces of projections in complex, or rational, group algebras are rational numbers in the interval $[0,1]$. In order to maximize the scope of Theorem 4.1 for producing examples of L^2-Betti numbers, we need the following result which shows that the traces of projections in rational group algebras are *precisely* the rational numbers in the interval $[0,1]$. We write C_n for a cyclic group of order n, written multiplicatively, with generator $t = t_n$.

5.1 Lemma. *Let q be a rational number in the interval $[0,1]$. Then there is an expression $q = \frac{m}{n}$ where the denominator has the form $n = 2^r s$ with s odd and $2^r \geq s-1$, and, for any such expression, $\mathbb{Q}[C_n]$ contains some projection e with trace q, and $ne \in \mathbb{Z}[C_n]$.*

Proof. By multiplying the numerator and denominator of q by a sufficiently high power of 2, we see that q has an expression of the desired type. Now consider any expression $q = \frac{m}{n}$ where $n = 2^r s$ with s odd and $2^r \geq s-1$.

We first show, by induction on r, that, if $0 \leq c \leq 2^r$, then $\mathbb{Q}[C_{2^r}] = \mathbb{Q}[t \mid t^{2^r} = 1]$ has an ideal whose dimension over $\mathbb{Q}$ is c. Since the orthogonal complement is then an ideal of dimension $2^r - c$ over the rationals, it amounts to the same if we consider only $c \leq 2^{r-1}$. For $r = 0$, we can take the zero ideal; thus, we may assume that $r \geq 1$ and the result holds for smaller r. Now $\mathbb{Q}[C_{2^r}]$ has a projection $e = \frac{1+t^{2^{r-1}}}{2}$; this is $\operatorname{avg}(U)$ for the subgroup U of order 2 in C_{2^r}. As rings

$$e\mathbb{Q}[C_{2^r}] \simeq \mathbb{Q}[C_{2^r}]/(1-e) \simeq \mathbb{Q}[C_{2^{r-1}}].$$

By the induction hypothesis, the latter has an ideal of dimension c over $\mathbb{Q}$, and viewed in $e\mathbb{Q}[C_{2^r}]$ this is an ideal of $\mathbb{Q}[C_{2^r}]$. This completes the proof by induction. Hence, if $0 \le c \le 2^r$, then $\mathbb{Q}[C_{2^r}]$ has a projection $e(c)$ with $\mathrm{tr}_{C_{2^r}}(e(c)) = \frac{c}{2^r}$.

Let $f = \mathrm{avg}(C_s) \in \mathbb{Q}[C_s]$, so $\mathrm{tr}_{C_s}(f) = \frac{1}{s}$, and $\mathrm{tr}_{C_s}(1-f) = \frac{s-1}{s}$.

By identifying

$$\mathbb{Q}[C_n] = \mathbb{Q}[C_n^s \times C_n^{2^r}] = \mathbb{Q}[C_{2^r} \times C_s],$$

we see that, for $0 \le c \le 2^r$, we have projections $e(c)f$ and $e(c)(1-f)$ in $\mathbb{Q}[C_n]$, with traces $\frac{c}{2^r}\frac{1}{s} = \frac{c}{n}$ and $\frac{c}{2^r}\frac{s-1}{s} = \frac{c(s-1)}{n}$, respectively, by Lemma 3.1.

We claim there exist integers a, b with $0 \le a, b \le 2^r$ such that $a + (s-1)b = m$. We know that $0 \le m \le n = 2^r s$. If $m \ge 2^r(s-1)$, then $m \in [2^r(s-1), 2^r s]$, and we can take $b = 2^r$ and $a = m - (s-1)b = m - 2^r(s-1) \in [0, 2^r]$. If $m < 2^r(s-1)$, then, by the division algorithm, $m = (s-1)b + a$ with $0 \le b < 2^r$, and $0 \le a \le s-2 < 2^r$. This proves the claim.

Now let $e = e(a)f + e(b)(1-f)$, a sum of orthogonal projections. Thus, e is a projection and

$$\mathrm{tr}_{C_n}(e) = \mathrm{tr}_{C_n}(e(a)f) + \mathrm{tr}_{C_n}(e(b)(1-f)) = \frac{a}{n} + \frac{b(s-1)}{n} = \frac{a + b(s-1)}{n} = \frac{m}{n},$$

as desired.

It remains to show that e lies in $\frac{1}{n}\mathbb{Z}[C_n]$, but it is well known that this holds for all the idempotents of $\mathbb{Q}[C_n]$. Alternatively, it is straightforward to check that all the projections involved in the foregoing proof have the right denominators. □

We now obtain the following special case of Theorem 4.1.

5.2 Corollary. *Let p and q be rational numbers with $0 < p, q < 1$. There exist positive integers m and n, and projections*

$$e = e^* = e^2 \in \mathbb{Q}[C_m], \qquad f = f^* = f^2 \in \mathbb{Q}[C_n]$$

with $\mathrm{tr}_U(e) = p$, $\mathrm{tr}_V(f) = q$. Let

$$G(p,q) := (C_m \wr \mathbb{Z}) \times (C_n \wr \mathbb{Z}),$$

$$T := T(U,e) \in \mathbb{C}[U \wr \mathbb{Z}] \subset \mathbb{C}[G], \text{ and } S := T(V,f) \in \mathbb{C}[V \wr \mathbb{Z}] \subset \mathbb{C}[G].$$

Let $Z = Z(p,q) := mn(T - S)$, and let

$$\kappa = \kappa(p,q) := (p^{-1}-1)^2(q^{-1}-1)^2 \sum_{k \ge 2} \frac{\phi(k)}{(p^{-k}-1)(q^{-k}-1)}$$
$$= (p^{-1}-1)^2(q^{-1}-1)^2 \left(\sum_{i \ge 1} \sum_{j \ge 1} \gcd(i,j) p^i q^j \right) - (p^{-1}-1)(q^{-1}-1).$$

Then $Z \in \mathbb{Z}[G]$ and $\dim_G(\ker Z) = \kappa$. □

5.3 Remarks. Let $0 < p, q < 1$ be rational numbers. Let $G = G(p,q)$, $Z = Z(p,q)$ and $\kappa = \kappa(p,q)$ as in Corollary 5.2.

By the Higman Embedding Theorem, any recursively presented group can be embedded in a finitely presented group, so G can be embedded in a finitely presented group H. (Here it is easy to find an explicit suitable finitely presented group; see, for example, [2] or [3, Lemma 3]. This explicit supergroup has the additional nice property of being metabelian, that is, 2-step solvable. Moreover, one can precisely describe its finite subgroups.)

By Corollary 5.2, $Z \in \mathbb{Z}[G] \subseteq \mathbb{Z}[H]$ and $\dim_H(\ker Z) = \dim_G(\ker Z) = \kappa$.

It is then well known how to construct a finite CW-complex or a closed manifold M with $\pi_1(M) \simeq H$ and with third L^2-Betti number κ; see, for example, [3].

Thus $\kappa(p, q)$ is an L^2-Betti number of a closed manifold. It is conceivable that this is a counterexample to Atiyah's conjecture [1] that L^2-Betti numbers of closed manifolds are rational, but we have not been able to decide whether $\kappa(p, q)$ is rational or not. $\square$

5.4 Example. Consider $\kappa(\frac{1}{2}, \frac{1}{2}) = \sum_{k\geq 2} \frac{\phi(k)}{(2^k-1)^2} = 0.1659457149\ldots$. If we sum the first 400 terms, then elementary methods show that the remaining tail is less than 10^{-201}. This allows us to calculate the first 199 terms of the continued fraction expansion of $\kappa(\frac{1}{2}, \frac{1}{2})$. One consequence we find is that if $\kappa(\frac{1}{2}, \frac{1}{2})$ is rational then both the numerator and the denominator exceed 10^{100}. It seems reasonable to assert that $\kappa(\frac{1}{2}, \frac{1}{2})$ is not obviously rational. $\square$

6 Power series

Throughout this section, let $\mathbb{C}((x, y))$ denote the field of (formal) Laurent series in two variables (with complex coefficients).

The expression

$$\Phi(x, y) := \sum_{m\geq 1} \sum_{n\geq 1} \gcd(m, n) x^m y^n$$

arising from (4.2) can be viewed as an element of $\mathbb{C}((x, y))$. By Remarks 5.3, if there exist rational numbers p, q in the interval $(0, 1)$ such that (the limit of) $\Phi(p, q)$ is irrational, then there exists a counterexample to the Atiyah conjecture; so it is of interest to know whether $\Phi(p, q)$ is always rational for such rational numbers p, q. One (traditionally successful) way to show that such an expression is rational would be to show that $\Phi(x, y)$ itself is rational, that is, lies in the subfield $\mathbb{Q}(x, y)$ of rational Laurent series over the rationals. In this section, we will eliminate this possibility by showing that $\Phi(x, y)$ is transcendental over $\mathbb{C}(x, y)$. In fact, we will show the stronger result that the specialization $\Phi(x, x)$ is transcendental over $\mathbb{C}(x)$.

The following result is well known, but we have not found a reference. The proof is left to the reader.

6.1 Lemma. *Suppose that $f \in \mathbb{C}((x))$ is algebraic over $\mathbb{C}(x)$ of degree d. Then the subfield $\mathbb{C}(x, f)$ is closed under the usual derivation operation, $F \mapsto F' = \frac{dF}{dx}$, on $\mathbb{C}((x))$. Moreover, $\mathbb{C}(x, f)$ is a d-dimensional vector space over $\mathbb{C}(x)$, so the $d + 1$ higher-order derivatives $f^{(i)} := (\frac{d}{dx})^i(f)$, $0 \leq i \leq d$, are $\mathbb{C}(x)$-linearly dependent. Hence f satisfies some non-trivial order d differential equation over $\mathbb{C}(x)$.* $\square$

We can now apply this lemma to get a transcendentality criterion.

6.2 Proposition. *Suppose that* $a\colon \mathbb{N} \to \mathbb{C}$, $n \mapsto a(n)$, *has the property that, for each* $N \in \mathbb{N}$, *there exist infinitely many* $m \in \mathbb{N}$ *such that, whenever* $j \in \mathbb{Z}$ *satisfies* $1 \le |j| \le N$,

$$|a(m)| > N\,|a(m+j)|\,.$$

Then the power series $\sum_{n\ge 0} a(n)x^n \in \mathbb{C}((x))$ *does not satisfy any non-trivial differential equation over* $\mathbb{C}(x)$, *so is transcendental over* $\mathbb{C}(x)$.

Proof. Let $f := \sum_{n\ge 0} a(n)x^n \in \mathbb{C}((x))$, and suppose that f satisfies a non-trivial differential equation over $\mathbb{C}(x)$,

$$\sum_{i=0}^{d} q_i f^{(i)} = 0 \tag{6.3}$$

where $q_i \in \mathbb{C}(x)$, not all zero. By multiplying through by a common denominator, we may assume that all the q_i lie in $\mathbb{C}[x]$. (Notice it is natural not to have a "constant term" on the right-hand side of (6.3) since it could be eliminated by iterated derivation of the equation.)

Viewing (6.3) as a collection of equations, one for each power x^n, we see that there exists some $N \in \mathbb{N}$, and polynomials $p_k(t) \in \mathbb{C}[t]$ such that

$$\sum_{k=0}^{N} p_k(n)a(n+k) = 0 \text{ for all } n \in \mathbb{N}. \tag{6.4}$$

Choose k_0, with $0 \le k_0 \le N$, and $n_0 \in \mathbb{N}$ such that $|p_{k_0}(n)| \ge |p_k(n)|$ for all $n \ge n_0$, and all k with $0 \le k \le N$. In other words, p_{k_0} eventually dominates all the p_k, $0 \le k \le N$.

It follows from the hypothesis on the $a(n)$ that there exists $m \in \mathbb{N}$ such that $m \ge n_0 + k_0$, and $|a(m)| > N\,|a(m+j)|$ for all $j \in \mathbb{Z}$ with $1 \le |j| \le N$. Now take $n = m - k_0$. Then $n \ge n_0$, and

$$|a(n+k_0)| > \sum_{k=0}^{k_0-1} |a(n+k)| + \sum_{k=k_0+1}^{N} |a(n+k)|\,.$$

Thus

$$\begin{aligned} |p_{k_0}(n)a(n+k_0)| &> \sum_{k=0}^{k_0-1} |p_k(n)a(n+k)| + \sum_{k=k_0+1}^{N} |p_k(n)a(n+k)| \\ &\ge \left|(\sum_{k=0}^{N} p_k(n)a(n+k)) - p_{k_0}(n)a(n+k_0)\right| \\ &= |0 - p_{k_0}(n)a(n+k_0)| \text{ by (6.4).} \end{aligned}$$

This contradiction shows that f does not satisfy any non-trivial differential equation over $\mathbb{C}(x)$, so, by Lemma 6.1, f is not algebraic over $\mathbb{C}(x)$. □

We now record some important results from number theory that we shall require.

6.5 Lemma. *For each positive integer* i, *let* p_i *denote the* i*th prime number. There exists an integer* Q_0 *such that, for all* $Q \ge Q_0$, *the following hold.*

(1) $Q! \leq (\frac{Q}{2})^Q$.

(2) $\frac{3}{4} \leq \frac{p_Q}{Q \log Q} \leq \frac{5}{4}$.

(3) $\prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \geq \frac{1}{Q}$.

Proof. In the following, $f(Q) = o(g(Q))$ means $\lim_{Q\to\infty} f(Q)/g(Q) = 0$, and $f(Q) \sim g(Q)$ means $\lim_{Q\to\infty} f(Q)/g(Q) = 1$.

(1) By Stirling's formula, $Q! \sim \sqrt{2\pi} Q^{Q+\frac{1}{2}} e^{-Q}$, and the latter is $o((\frac{Q}{2})^Q)$, since $e > 2$. One can argue directly that $\sum_{i=1}^{Q} \log i \leq \int_1^{Q+1} \log x \; dx$, so

$$\log Q! \leq (Q+1)\log(Q+1) - Q,$$

so

$$Q! \leq (Q+1)^{Q+1} e^{-Q} = Q^Q (1 + \frac{1}{Q})^Q (Q+1) e^{-Q} = o((\frac{Q}{2})^Q),$$

since $e > 2$.

(2) By the Prime Number Theorem, $p_Q \sim Q \log Q$; see [5, Theorem 8, pages 10, 367].

(3) By Mertens' Theorem, $\prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \sim \frac{e^{-\gamma}}{\log p_Q}$, where γ is Euler's constant; see [5, Theorem 429, page 351]. By the Prime Number Theorem, $\log p_Q \sim \log Q$, so $\prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \sim \frac{e^{-\gamma}}{\log Q}$. Since $\frac{1}{Q} = o(\frac{1}{\log Q})$, we see that $\frac{1}{Q} = o(\prod_{i=1}^{Q} (1 - \frac{1}{p_i}))$.

The result now follows. □

6.6 Theorem. *$\Phi(x,x) = \sum_{m\geq 1} \sum_{n\geq 1} \gcd(m,n) x^{m+n}$ and $\sum_{n\geq 1} \sum_{d|n} \frac{\phi(d)}{d} n x^n$ are transcendental over $\mathbb{C}(x)$.*

Proof. For each positive integer n, let $a(n) := n \sum_{d|n} \frac{\phi(d)}{d}$. Thus

$$\begin{aligned} a(n) &= n \sum_{d|n} \frac{\phi(d)}{d} = \sum_{d|n} \frac{n}{d} \phi(d) = \sum_{d|n} \sum_{\{i:1\leq i\leq n, d|i\}} \phi(d) = \sum_{i=1}^{n} \sum_{\{d:d|i, d|n\}} \phi(d) \\ &= \sum_{i=1}^{n} \gcd(i,n) = \sum_{i=1}^{n} \gcd(i, n-i) = \sum_{i=1}^{n-1} \gcd(i, n-i) + n. \end{aligned}$$

Now

$$\Phi(x,x) = \sum_{i\geq 1} \sum_{j\geq 1} \gcd(i,j) x^{i+j} = \sum_{n\geq 1} \sum_{i=1}^{n-1} \gcd(i, n-i) x^n,$$

so

$$(\sum_{n\geq 1} a(n) x^n) - \Phi(x,x) = \sum_{n\geq 1} n x^n = x(\sum_{n\geq 0} x^n)' = \frac{x}{(1-x)^2}.$$

Thus $\sum_{n\geq 1} a(n) x^n$ and $\Phi(x,x)$ differ by an element of $\mathbb{Q}(x)$, so it suffices to show that $\sum_{n\geq 1} a(n) x^n$ is transcendental over $\mathbb{C}(x)$.

By Proposition 6.2, it suffices to show that, for each $N \in \mathbb{N}$, there exist infinitely many $m \in \mathbb{N}$ such that, whenever $j \in \mathbb{Z}$ satisfies $1 \leq |j| \leq N$,

$$|a(m)| > N \, |a(m+j)| \, .$$

We may suppose that N is fixed.

Remember the p_i is the ith prime number. For each $Q \in \mathbb{N}$, let

$$m_Q := \prod_{i=1}^{Q} p_i \prod_{i=1}^{N} p_i^N.$$

We may now suppose that j is fixed with $1 \leq |j| \leq N$, and it suffices to show that

$$\lim_{Q\to\infty} \frac{a(m_Q + j)}{a(m_Q)} = 0.$$

We use the notation of Lemma 6.5, concerning Q_0. Let

$$C_1 = \prod_{i=1}^{Q_0} p_i \prod_{i=1}^{N} p_i^N.$$

Now suppose that Q is an integer with $Q \geq \max\{Q_0, N\}$, let $m = m_Q$ and let $m' = \prod_{i=1}^{Q} p_i$.

We wish to bound $a(m) = m \sum_{d|m} \frac{\phi(d)}{d}$ from below. Recall that, for any positive integer n, $\frac{\phi(n)}{n} = \prod(1-\frac{1}{p})$, where the product is over the distinct prime divisors p of n. Thus $a(m) \geq m \sum_{d|m} \frac{\phi(m)}{m} = m\,\mathrm{d}(m)\frac{\phi(m)}{m}$, where $\mathrm{d}(m)$ denotes the number of divisors d of m. Also, $\frac{\phi(m)}{m} = \prod\limits_{i=1}^{Q}(1-\frac{1}{p_i})$, which, by Lemma 6.5(3), is at least $\frac{1}{Q}$. Thus $a(m) \geq m\,\mathrm{d}(m)\frac{1}{Q}$. Notice that $\mathrm{d}(m) \geq \mathrm{d}(m')$, since m' divides m. From the definition of m', we see that $\mathrm{d}(m') = 2^Q$. Thus

$$a(m) \geq m 2^Q \frac{1}{Q}.$$

We next wish to bound $a(m+j)$ from above. Let $\Omega(m+j)$ be the number, counting multiplicity, of prime factors of m, and let

$$m + j = p_{i_1} p_{i_2} \cdots p_{i_{\Omega(m+j)}}$$

be the factorization of $m+j$ into prime factors. Then $\mathrm{d}(m+j) \leq 2^{\Omega(m+j)}$, and

$$\begin{aligned} a(m+j) &= (m+j) \sum_{d|(m+j)} \frac{\phi(d)}{d} \leq (m+j) \sum_{d|(m+j)} 1 = (m+j)\,\mathrm{d}(m+j) \\ &\leq (m+j)2^{\Omega(m+j)} \leq (m+N)2^{\Omega(m+j)} \leq 2m2^{\Omega(m+j)}. \end{aligned}$$

Consider $1 \leq l \leq \Omega(m+j)$. If $i_l \leq Q$, then p_{i_l} divides m so p_{i_l} divides j. But $1 \leq |j| \leq N$, so $p_{i_l} \leq N$, so $i_l \leq N$. Hence $p_{i_l}^N$ divides m, but $p_{i_l}^N \geq 2^N > N \geq |j|$, so $p_{i_l}^N$ cannot divide j, so cannot divide $m+j$. Thus, the number of i_l which are less than Q is at most N^N. Let $z = z(Q,j)$ denote the number of l such that $i_l \geq Q$, so $\Omega(m+j) \leq z + N^N$, and

$$a(m+j) \leq 2m2^{\Omega(m+j)} \leq 2m2^{z+N^N}.$$

Thus

$$\frac{a(m+j)}{a(m)} \leq \frac{2m2^{z+N^N}}{m2^Q\frac{1}{Q}} = Q2^{z-Q}2^{N^N+1}.$$

Hence it remains to show that $\lim_{Q\to\infty} Q2^{z-Q} = 0$, or equivalently,

$$\lim_{Q\to\infty} Q - z - \log_2 Q = \infty.$$

Since z is the number, counting multiplicity, of prime factors p_{i_l} of $m+j$ with $p_{i_l} \geq p_Q$,

$$p_Q^z \leq m+j \leq m+N \leq 2m.$$

We can write

$$m = \prod_{i=1}^{Q} p_i \prod_{i=1}^{N} p_i^N \leq \prod_{i=1}^{Q_0} p_i \prod_{i=2}^{Q} (\frac{5}{4}\ i\log i) \prod_{i=1}^{N} p_i^N = C_1 \prod_{i=2}^{Q} (\frac{5}{4}\ i\log i)$$
$$\leq C_1 (\frac{5}{4})^Q Q! (\log Q)^Q \leq C_1 (\frac{5}{4})^Q (\frac{Q}{2})^Q (\log Q)^Q,$$

by Lemma 6.5(1). Thus

$$(\frac{3}{4}\ Q\log Q)^z \leq p_Q^z \leq 2m \leq 2C_1(\frac{5}{4})^Q(\frac{Q}{2})^Q(\log Q)^Q.$$

Hence

$$(\frac{3}{4} Q\log Q)^{z-Q} \leq 2C_1(\frac{4}{3})^Q(\frac{1}{2})^Q(\frac{5}{4})^Q = 2C_1(\frac{5}{6})^Q,$$

so $(z-Q)(\log\frac{3}{4} + \log Q + \log\log Q) \leq \log 2C_1 - Q\log(\frac{6}{5})$, and

$$-(Q-z) \leq \frac{\log 2C_1 - Q\log(\frac{6}{5})}{\log\frac{3}{4} + \log Q + \log\log Q} \sim -\log(\frac{6}{5})\frac{Q}{\log Q}.$$

It follows that

$$\lim_{Q\to\infty} Q - z - \log_2 Q \geq \lim_{Q\to\infty} \log(\frac{6}{5})\frac{Q}{\log Q} - \log_2 Q = \infty,$$

as desired. □

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