Equivariant geometric K-homology for compact Lie group actions

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Abstract

Let G be a compact Lie-group, X a compact G-CW-complex. We define equivariant geometric K-homology groups $K_*^G(X)$, using an obvious equivariant version of the (M, E, f)-picture of Baum-Douglas for K-homology. We define explicit natural transformations to and from equivariant K-homology defined via KK-theory (the "official" equivariant K-homology groups) and show that these are isomorphism.

1 Introduction

K-homology is the homology theory dual to K-theory. For index theory, concrete geometric realizations of K-homology are of relevance, as already pointed out by Atiyah [2]. In an abstract analytical setting, such a definition has been given by Kasparov [11]. About the same time, Baum and Douglas [3] proposed a very geometric picture of K-homology (using manifolds, bordism, and so on), and defined a simple map to analytic K-homology. This map was "known" to be an isomorphism. However, a detailed proof of this was only published in [4].

The relevance of a geometric picture of K-homology extends to equivariant situations. Kasparov's analytic definition of K-homology immediately does allow for such a generalization, and this is considered to be the "correct" definition. The paper [4] is a spin-off of work on a Baum-Douglas picture for Γ-equivariant

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K-homology, where Γ is a discrete group acting properly on a Γ-CW-complexs. This requires considerable effort because of the difficulty to find equivariant vector bundles in this case. Emerson and Meyer give a very general geometric description even of bivariant equivariant K-theory, provided enoug such vector bundles exist —compare [5].

In the present paper, we give a definition of G-equivariant K-homology for the case that G is a compact Lie group, in terms of the "obvious" equivariant version of the (M, E, ϕ)-picture of Baum and Douglas. Our main result is that these groups indeed are canonically isomorphic to the standard analytic equivariant K-homology groups. The main point of the construction is its simplicity, we were therefore not interested in utmost generality.

In the case of a compact Lie group, equivariant vector bundles are easy to come by, and therefore the work is much easier than in the case of a discrete proper action. We will in part follow closely the work of [4], and actually will omit detailed descriptions of the equivariant generalizations where they are obvious. In other parts, however, we will deviate from the route taken in [4] and give actually simpler constructions. Much of our theory is an equivariant (and more geometric) version of a general theory of Jakob [8]. These constructions have no generalization to proper actions of discrete groups and were therefore not used in [4]. Moreover, we will use the full force of Kasparov's KK-theory in some of our analytic arguments. The diligent reader is then asked to supply full arguments where necessary.

2 Equivariant geometric K-homology

Let G be a compact Lie group, (X, Y) be a compact G-CW-pair with a G-homotopy retraction $(X, Y) \xrightarrow{j} (W, \partial W) \xrightarrow{q} (X, Y)$. We require that $(W, \partial W)$ is a smooth G-spin^c manifold with boundary. G-homotopy retraction means that qj is G-homotopy equivalent to the identity (and the homotopy preserves Y).

2.1 Lemma. *Every finite G-CW-pair, more generally every compact G-ENR and in particular every smooth compact G-manifold (absolute or relative to its boundary) has the required property, i.e. is such a homotopy retraction of a manifold with boundary.*

Proof. This is trivial for a G-spin^c manifold.

The following argument is partly somewhat sketchy, we leave it to the reader to add the necessary details.

In general, by [9], every finite G-CW-complex X has a (closed) G-embedding into a finite dimensional complex linear G-space (using [14]) with an open G-invariant neighborhood U with a G-equivariant retraction $r\colon U \to X$ onto X. Even better, every such G-embeddings admit such a neighborhood retraction, using [6]). In other words, a finite G-CW-complex is a G-ANR. By [1], the converse is true upto G-homotopy equivalence.

A complex G-representation in particular has a G-invariant spin^c-structure, and therefore so has U. Choose a G-invariant metric on U, e.g. the metric induced by a G-invariant Hermitean metric on the G-representation. Let f be the distance to X, a G-invariant map on U. Choose $r > 0$ such that $f^{-1}([0, r])$ is compact. This is possible since X is compact: choose r smaller than the distance from X to the complement of U. Choose a smooth G-invariant approximation g

to f, i.e. g has to be sufficiently close to f in the chosen metric. To construct g, we can first choose a non-equivariant approximation and then average it to make it G-invariant. Choose a regular value $0 < r' < r$ such that $V := f^{-1}((-\infty, r'])$ is a neighborhood of X and is a compact manifold (necessarily a G-manifold) with boundary. Its double W is a G-manifold with inclusion $i\colon X \to W$ (into one of the two copies) and with retraction $W \to X$ obtained as the composition of the "fold map" and the retraction r (restricted to V).

This covers the absolute case.

If (X, Y) is a G-CW-pair, choose an embedding j of X into some linear G-space E of real dimension n (with spin^c-structure), and a G-invariant distance function. The distance to Y then gives a G-invariant function $h\colon X \to [0, \infty)$ with $h(x) = 0$ if and only if $x \in Y$. Consider $X \cup_Y X$ with the obvious $\mathbb{Z}/2$-action by exchanging the two copies of X, and G-action by using the given action on both halves. Extend h to a $G \times \mathbb{Z}/2$-equivariant map to $\mathbb{R}$ with $\mathbb{Z}/2$-action given by multiplication with -1 (and with trivial G-action). Let $q\colon X \cup_Y X \to X$ be the folding map. Taking the product of $j \circ q$ with $h\colon X \cup_Y X \to \mathbb{R}$ (with trivial G-action on $\mathbb{R}$), we obtain a $G \times \mathbb{Z}/2$-embedding of $X \times_Y X$ into $E \times \mathbb{R}$.

Construct now the $G \times \mathbb{Z}/2$-neighborhoood retract U^+ and the manifold W^+ for this embedding as above. By construction, there is a well defined $\mathbb{R}$-coordinate r for all points in these neighborhoods and also in W^+ (a priori only a continuous function). The subset $\{r = 0\}$ consisting exactly of the $\mathbb{Z}/2$-fixed points. The $\mathbb{Z}/2$-action on W^+ is smooth. For each $\mathbb{Z}/2$-fixed point $x \in U^+$, (being an open subset of $E \times \mathbb{R}$ with $\mathbb{Z}/2$-action fixing E and acting as -1 on $\mathbb{R}$) $T_x U^+ \cong \mathbb{R}^n \oplus \mathbb{R}_-$ as $\mathbb{Z}/2$-representation (where $\mathbb{R}$ denotes the trivial $\mathbb{Z}/2$-representation and $\mathbb{R}_-$ denotes the non-trivial $\mathbb{Z}/2$-representation). The same is then true for any $\mathbb{Z}/2$-submanifold with boundary of codimension 0, and also for a double of such a manifold, like W^+.

Because of this special structure of the $\mathbb{Z}/2$-fixed points it follows that $W := W^+/\mathbb{Z}/2$ obtains the structure of a G-manifold with boundary, here homeomorphic to the subset $\{r \geq 0\}$ (as this is a fundamental domain for the action of $\mathbb{Z}/2$). The boundary of $W = W^+/\mathbb{Z}/2$ is exactly the (homeomorphic) image of the fixed point set $\{r = 0\}$. The $G \times \mathbb{Z}/2$-equivariant retraction of W^+ onto $X \cup_Y X$ descends to a G-equivariant retraction of W onto $X = X \cup_Y X/\mathbb{Z}/2$; the $\mathbb{Z}/2$-equivariance of the retraction implies that ∂W, the image of the fixed point set is mapped under this retraction to Y (the image of the $\mathbb{Z}/2$-fixed point set of $X \cup_Y X$), so we really get a retraction of the pair $(W, \partial W)$ onto (X, Y). □

2.2 Definition. A *cycle for geometric equivariant K-homology* is a triple (M, E, f), where

(1) M is a G-spin^c manifold (possibly with boundary)

(2) E is a G-equivariant Hermitean vector bundle on M

(3) $f\colon M \to X$ is a continuous G-equivariant map such that $f(\partial M) \subset Y$.

Here, a G-spin^c-manifold is a spin^c-manifold with a given spin^c structure — given as in [4, Section 4] in terms of a complex spinor bundle for TM, now with a G-action lifted to and compatible with all the structure.

We define isomorphism of cycles (M, E, f) in the obvious way, given by maps which preserve all the structure (in particular also the G-action).

The set of isomorphism classes becomes a monoid under the evident operation of disjoint union of cycles, we write this as $+$. This addition is obviously commutative.

More details about spin^c-structures can be found in [4, Section 4]. All statements there have obvious G-equivariant generalizations.

2.3 Definition. If (M, E, ϕ) is a K-cycle for (X, Y), then its *opposite* is the K-cycle $(-M, E, \phi)$, where $-M$ denotes the manifold M equipped with the opposite spin^c-structure.

2.4 Definition. A *bordism* of K-cycles for the pair (X, Y) consists of the following data:

(i) A smooth, compact G-manifold L, equipped with a G-spin^c-structure.

(ii) A smooth, Hermitian G-vector bundle F over L.

(iii) A continuous G-map $\Phi\colon L \to X$.

(iv) A smooth map G-invariant map $f\colon \partial L \to \mathbb{R}$ for which ± 1 are regular values, and for which $\Phi[f^{-1}[-1,1]] \subseteq Y$.

The sets $M_+ = f^{-1}[+1, +\infty)$ and $M_- = f^{-1}(-\infty, -1]$ are manifolds with boundary, and we obtain two K-cycles $(M_+, F|_{M_+}, \Phi|_{M_+})$ and $(M_-, F|_{M_-}, \Phi|_{M_-})$ for the pair (X, Y). We say that the first is bordant to the opposite of the second.

2.5 Definition. Let M be a G-spin^c-manifold and let W be a G-spin^c-vector bundle of even dimension over M. Denote by $\mathbf{1}$ the trivial, rank-one real vector bundle (with fiberwise trivial G-action). The direct sum $W \oplus \mathbf{1}$ is a G-spin^c-vector bundle, and the total space of this bundle is equipped with a G-spin^c structure in the canonical way, as in [4, Definition 5.6].

Let Z be the unit sphere bundle of the bundle $W \oplus \mathbf{1}$ with bundle projection π. Observe that an element of Z has the form (t, w) with $w \in W$, $t \in [-1, 1]$ such that $t^2 + |w|^2 = 1$. The subset $\{t = 0\}$ is canonically identified with the unit sphere bundle of W, $\{t \geq 0\}$ is called the "northern hemisphere", $\{t \leq 0\}$ the "southern hemisphere". The map $s\colon M \to Z; m \mapsto (0, z(m))$ is called the *north pole section*, where $z\colon M \to W$ is the zero section. Since Z is the boundary of the disk bundle, we may equip it with a natural G-spin^c-structure by first restricting the given G-spin^c-structure on the total space of $W \oplus \mathbf{1}$ to the disk bundle, and then taking the boundary of this spin^c-structure to obtain a spin^c-structure on the sphere bundle.

We construct a bundle F over Z via *clutching*: if S_W is the spinor bundle of W (a bundle over M), then F is obtained from $\pi^* S^*_{W,+}$ over the northern hemisphere of Z and $\pi^* S^*_{W,-}$ over the southern hemisphere of Z by gluing along the intersection, the unit sphere bundle of W, using Clifford multiplication with the respective vector of W. One can show that this bundle is isomorphic to $S^*_{V,+}$, the dual of the even-graded part of the $\mathbb{Z}/2$-graded bundle S_V. The *modification* of a K-cycle (M, E, ϕ) associated to the bundle W is the K-cycle $(Z, F \otimes \pi^* E, \phi \circ \pi)$.

2.6 Definition. We define an equivalence relation on the set of isomorphism classes of cycles of Definition 2.2 as follows. It is generated by the following three elementary steps:

(1) **direct sum is disjoint union**. Given (M, E_1, f) and (M, E_2, f),

$$(M, E_1, f) + (M, E_2, f) \sim (M, E_1 \oplus E_2, f).$$

(2) **bordism**. If there is a bordism of K-cycles (L, F, Φ) as in Definition 2.4 with boundary the two parts (M_1, E_1, f_1) and $-(M_2, E_2, f_2)$, we set $(M_1, E_1, f_1) \sim (M_2, E_2, f_2)$.

(3) **modification**. If $(Z, F \otimes \pi^* E, \phi \circ \pi)$ is the modification of a K-cycle (M, E, ϕ) associated to the G-spin^c bundle W, then $(Z, F \otimes \phi^* E, \phi \circ \pi) \sim (M, E, \phi)$.

2.7 Definition. For a pair (X, Y) as above, we define the *equivariant geometric K-homology* $K_*^{G,geom}(X, Y)$ as the set of isomorphism classes of cycles as in Definition 2.2, modulo the equivalence relation of Definition 2.6.

Disjoint union of K-cycles provides a structure of $\mathbb{Z}/2\mathbb{Z}$-graded abelian group, graded by the parity of the dimension of the underlying manifold of a cycle.

2.8 Lemma. *Given a compact G-spin^c-manifold M with boundary, a G-map $f\colon (M, \partial M) \to (X, Y)$ and a class $x \in K_G^0(M)$, we get a well defined element $[M, x, f] \in K_*^{G,geom}(X, Y)$ by representing $x = [E] - [F]$ with two G-vector bundles E, F over M and setting*

$$[M, x, f] := [M, E, f] - [M, F, f] \in K_*^{G,geom}(X, Y).$$

In the opposite direction, we can assign to each triple (M, E, ϕ) a triple $(M, [E], \phi)$ with $[E] \in K_G^0(M)$ the K-theory class represented by E.

Proof. We have to check that this construction is well defined, i.e. we have to check that $[E \oplus H] - [F \oplus H]$ gives the same geometric K-homology class, but this follows from the relation “direct sum-disjoint union”. □

2.9 Remark. Lemma 2.8 allows to use a geometric picture of equivariant K-homology (for a compact Lie group G) where the bundle E is replaced by a K-theory class x; and all other definitions are translated accordingly.

2.10 Definition. $K_*^{G,geom}$ is a $\mathbb{Z}/2$-graded functor from pairs of G-spaces to abelian groups. Given $g\colon (X, Y) \to (X', Y')$, the transformation $g_*\colon K_*^{G,geom}(X, Y) \to K_*^{G,geom}(X', Y')$ is given by $g_*[M, E, f] := [M, E, g \circ f]$. An inspection of our equivalence relation shows that this is well defined, and it is obviously functorial.

Moreover, we define a boundary homomorphism

$$\partial\colon K_*^{G,geom}(X, Y) \to K_{*-1}^{G,geom}(Y, \emptyset); [M, E, f] \mapsto [\partial M, E|_{\partial M}, f|_{\partial M}].$$

Again, we observe directly from the definitions that this is compatible with the equivalence relation, natural with respect to maps of G-pairs and a group homomorphism.

Our main Theorem 3.1 shows that we have (for the subcategory of compact G-pairs which are retracts of G-spin^c manifolds) explicit natural isomorphisms to $K_*^{G,an}$. In particular, we observe that on this category $K_*^{G,geom}$ with the above structure is a G-equivariant homology theory.

3 Equivariant analytic K-homology

For G a compact group and (X, Y) a compact G-CW-pair, analytic equivariant K-homology and analytic equivariant K-theory are defined in terms of bivariant KK-theory:

$$K_*^{G,an}(X,Y) := KK_*^G(C_0(X \setminus Y), \mathbb{C}); \qquad K_G^*(X,Y) := KK_*^G(\mathbb{C}, C_0(X,Y)).$$

Of course, it is well known that $K_G^0(X,Y)$ is naturally isomorphic to the Grothendieck group of G-vector bundle pairs over X with a isomorphism over Y. Moreover, most constructions in equivariant K-homology and K-theory can be described in terms of the Kasparov product in KK-theory.

3.1 Analytic Poincaré duality

The key idea we employ to describe the relation between geometric and analytic K-homology is Poincaré duality in the setting of equivariant KK-theory developped by Kasparov [12]. An orientation for equivariant K-theory is given by a G-spin^c-structure. This Poincaré duality was in fact originally stated by Kasparov for general oriented manifolds by using the Clifford algebra $C_\tau(M)$. But for a manifold M with a G-spin^c-structure, the Clifford algebra $C_\tau(M)$ used in [12] is G-Morita equivalent to $C_0(M)$, the Morita equivalence being implemented by the sections of the spinor bundle. We refer to [12] (see also [16]) for the definition of the representable equivariant K-theory group $RK_G^*(X)$ of a locally compact G-space X. We only recall here that the cycles are given by the cycles (E, ϕ, T) for Kasparov's bivariant K-theory group $\mathrm{KK}_*^G(C_0(X), C_0(X))$ such that the representation ϕ of $C_0(X)$ on E is the one of the $C_0(X)$-Hilbert structure. By forgetting this extra requirement, we get an obvious homomorphism $\iota_X\colon RK_G^*(X) \to \mathrm{KK}_*^G(C_0(X), C_0(X))$. Hence it makes sense to take the Kasparov product with elements in $K_G^*(C_0(X)) = \mathrm{KK}_*^G(C_0(X), \mathbb{C})$ and this gives rise to a product

$$RK_G^*(X) \times K_G^*(C_0(X)) \to K_G^*(C_0(X));\ (x,y) \mapsto \iota_X(x) \otimes y.$$

Recall that for any G-spin^c-manifold M, there is a fundamental class $[M] \in K_G^{\dim(M)}(C_0(M))$ associated to the Dirac element of the G-spin^c-structure on M. Moreover, if N is an open G-invariant subset of M, then $[N]$ is the restriction of $[M]$ to N, i.e the image of $[M]$ under the morphism $K_G^{\dim(M)}(C_0(M)) \to K_G^{\dim(M)}(C_0(N))$ induced by the inclusion $C_0(N) \hookrightarrow C_0(M)$. This is the obvious equivariant generalization of [4, Theorem 3.5], compare also the discussion of [7, Chapters 10,11]

3.1 Theorem. *Given any G-spin^c-manifold M, the Kasparov product with the class $[M]$ gives an isomorphism*

$$\mathcal{PD}_M\colon RK_G^*(M) \xrightarrow{\cong} K^G_{\dim M - *}(C_0(M));\ x \mapsto \iota_M(x) \otimes [M].$$

3.2 Remark.

(1) For a compact space, the equivariant K-theory and the equivariant representable K-theory coincide. In particular, for a compact G-spin^c-manifold M, the Poincaré duality can be stated as an isomorphism

$$\mathcal{PD}_M\colon K_G^*(M) \xrightarrow{\cong} K^{G,an}_{\dim M - *}(M),$$

and moreover, for any complex G-vector E on M, $\mathcal{PD}_M([E])$ is the class in $K^{G,an}_{\dim M+*}(M) = KK_G^{\dim M+*}(C_0(M), \mathbb{C})$ associated to the Dirac operator D_M^E on M with coefficient in the complex vector bundle E.

(2) Recall that representable equivariant K-theory is a functor which is invariant with respect to G-homotopies. In particular, if M is a compact G-spin^c-manifold with boundary ∂M, then M is G-homotopy equivalent to its interior $M \setminus \partial M$ and thus we get a natural identification $K_G^*(M) \cong RK_G^*(M \setminus \partial M)$ given by restriction to $M \setminus \partial M$ of the $C(M)$-structure. In view of this, the Poincaré duality for the pair $(M, \partial M)$ can be stated in the following way

$$\mathcal{PD}_M \colon K_G^*(M) \xrightarrow{\cong} K^{G,an}_{\dim(M)-*}(M, \partial M),$$

For a compact G-space X and a closed G-invariant subset Y of X, let us denote by $\iota_{X,Y}$, the composition

$$K_G^*(X) \cong RK_G^*(X) \to RK_G^*(X \setminus Y) \xrightarrow{\iota_{X,Y}} KK_*^G(C_0(X \setminus Y), C_0(X \setminus Y)),$$

where the first map is induced by the inclusion $X \setminus Y \hookrightarrow X$. Then, with this notations and under the identification $K_G^*(M) \cong RK_G^*(M \setminus \partial M)$, we get for any x in $K_G^*(M)$ that $\mathcal{PD}_M(x) = \iota_{M,\partial M}(x) \otimes [M \setminus \partial M]$.

3.3 Definition. We are now in the situation to define the natural isomorphisms

$$\alpha \colon K_*^{G,geom}(X,Y) \to K_*^{G,an}(X,Y)$$
$$\beta \colon K_*^{G,an}(X,Y) \to K_*^{G,geom}(X,Y).$$

To define α, let (M, E, f) be a cycle for geometric K-homology, with E a complex G-vector bundle on M. Then we set

$$\alpha([M,E,f]) := f_*(\mathcal{PD}_M([E])).$$

To define β, given $x \in K_k^{G,an}(X,Y)$, choose a retraction $(X,Y) \xrightarrow{j} (M, \partial M) \xrightarrow{p} (X,Y)$ with M a compact G-spin^c manifold with $\dim(M) \equiv k \pmod 2$ (such a manifold exists by assumption, if the parity is not correct just take the product with S^1 with trivial G-action). Then set

$$\beta(x) := [M, \mathcal{PD}_M \sigma^{-1}(j_*(x)), p].$$

3.4 Lemma. *The transformation α is compatible with the relation "direct sum—disjoint union" of the definition of $K_*^{G,geom}(X,Y)$. Under the assumption that α is well defined, it is a homomorphism.*

Proof.

$$\begin{aligned}\alpha([M,E,f] + [N,F,g]) &= \alpha([M \amalg N, E \amalg F, f \amalg g]) \\ &= (f \amalg g)_*(\mathcal{PD}([E]) \oplus \mathcal{PD}([F])) = f_*(\mathcal{PD}([E])) + g_*(\mathcal{PD}([F])).\end{aligned}$$

This implies both assertions, as $\mathcal{PD}$ and f_* are both homomorphisms. $\square$

To prove that both maps are well defined and indeed inverse to each other we need a few more properties of Poincaré duality which we collect in the sequel. These statements are certainly well known, for the convenience of the reader we give proofs of most of them in an appendix.

We first relate Poincaré duality to the Gysin homomorphism, and also describe vector bundle modification in terms of the Gysin homomorphism.

Let $f\colon M \to N$ be a smooth G-map between two compact G-spin^c-manifolds without boundary. We use, as a special case of [13, Section 4.3] (see also [17, Section 7.2]), the Gysin element $f_!$ in $KK^G_{\dim M - \dim N}(C(M), C(N))$. It has the functoriality property that if $f\colon M \to N$ and $g\colon N \to N'$ are two smooth G-maps between compact G-spin^c-manifolds, then $f_! \otimes g_! = (g \circ f)_!$. We will also need the corresponding construction for manifolds with boundary. We recall all this in the appendix.

3.5 Lemma. *An equivalent description of vector bundle modification, using Remark 2.9, is given as follows:*

Let (M, x, ϕ) be a triple for $K^{geom,G}_*(X, Y)$, with $x \in K^0_G(M)$, and let W be a G-spin^c vector bundle over M of even rank. Let $\pi\colon Z \to M$ be the underlying G-manifold of the modification with respect to W. Recall from definition 2.5 that $s\colon M \to Z$ is the north pole section and that the bundle F is obtained via clutching. Then the triple $(Z, s_!(x), \phi \circ \pi)$ is equivalent to $(Z, \pi^* x \otimes [F], \phi \circ \pi)$ in $K^{G,geom}_*(X, Y)$, i.e. represents the vector bundle modification of x.

Proof. Note first that the bundle B obtained in the same way as $S^*_{v,+}$, but by gluing $\pi^* S^*_{W,-}$ on both hemispheres with the identity along the unit sphere bundle of W is just $\pi^* S_{W,-}$ and therefore extends to the disk bundle. Therefore $(Z, \pi^* x \otimes B, \phi \circ \pi)$ is $\emptyset$-bordant and represents $0 \in K^{geom,G}_*(X, Y)$.

Observe that the normal bundle of $s\colon M \to Z$ is isomorphic to W. We now establish that $s_!([E]) = \pi^*[E] \otimes \big([S^*_{v,+}] - [B]\big)$ for any complex G-vector bundle E over M, using the definition of $s_!$ as given in Section A.1. We use the notation of A.1. Let us start with $E = M \times \mathbb{C}$. We denote by $\mathcal{E}_Z$ the $C(Z)$-module of sections of $S^*_{v,+}$. By viewing W as a tubular neighborhood of M in the northern hemisphere of Z, the operator $T_W\colon q^*_W \xi^+_W \to q^*_W \xi^-_W$ of A.1 is a compact pertubation of the restriction to $C_0(W)$ of the $C(Z)$-linear map $T'_Z\colon \mathcal{E}_Z \to \pi^* \xi^-_W$, whose restriction to the northern hemisphere is given by pontwise Clifford multiplication and restriction to the southern hemisphere is the identity. Pointwise action of T'_Z provides a map of $C(Z)$-module

$$\{f\colon [0,1] \to \mathcal{E}_Z,\ f(0) \in q^*_W \xi^+_W\} \longrightarrow \{f\colon [0,1] \to \pi^* \xi^-_W,\ f(0) \in q^*_W \xi^-_W\}$$

and we get in this way a homotopy between $s_!([1])$ and the class in $K^0(Z)$ of the K-cycle corresponding to $T'_Z\colon \mathcal{E}_Z \to \pi^* \xi^-_W$. But since Z is compact and T'_Z is indeed induced by an isomorphism of vector bundles, we can forget this isomorphism and we finally get $s_!([1]) = [S^*_{v,+}] - [B]$. Now if we want to take the complex vector bundle E into account, we have to perform the above construction with coefficient in E, i.e replace $q^*_W \xi^+_W$ by $q^*_W(E) \otimes_{C_0(W)} q^*_W \xi^+_W$ and $\mathcal{E}_Z$ by $\pi^*(E) \otimes_{C(Z)} \mathcal{E}_Z$.

We finally get $s_!(E) = \pi^* E \otimes \big([S^*_{v,+}] - [B]\big)$. The fact that B is $\emptyset$-bordant implies for the geometric K-homology cycles, as desired, that

$$[Z, s_!(E), \phi \circ \pi] = [Z, \pi^* E \otimes S^*_{v,+}, \phi \circ \pi] \in K^{geom,G}_*(X, Y).$$

□

From now on, we will use the following notation: if $f\colon M \to N$ is a G-map between G-spin^c and E is a complex vector bundle over M, then $f!E$ will stand for the element $f![E]$ of $K_G^{*+n-M}(N)$. It is well known that the Gysin map and functoriality in K-homology are intertwined by Poincaré duality. This is the key for proving that α is compatible with vector bundle modification, using the description of the latter given in Lemma 3.5. We will prove the next assertion in A.2.

3.6 Lemma. *Let $f\colon M \to N$ be a G-map between G-spin^c manifolds with $m = \dim M$ and $n = \dim N$, possibly with boundary. Assume that $f(\partial M) \subset \partial N$. Then we have the following commutative diagram*

$$\begin{array}{ccc} K_G^*(M) & \xrightarrow{\mathcal{PD}_M} & K_{m-*}^{G,an}(M,\partial M) \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ K_G^{*+n-m}(N) & \xrightarrow{\mathcal{PD}_N} & K_{m-*}^{G,an}(N,\partial N). \end{array}$$

3.7 Lemma. *The transformation α of Definition 3.3 is compatible with vector bundle modification.*

Proof. The assertion is a direct consequence of Lemma 3.5 and Lemma 3.6. Explicitly, if (M,E,f) is a cycle for $KK_*^{G,geom}(X,Y)$ and $(Z, s_!(E), f\circ\pi)$ the result of vector bundle modification according to Lemma 3.5, then

$$\begin{aligned} \alpha(Z, s_!(E), f\circ\pi) &= f_*\pi_*\mathcal{PD}_Z(s_!(E)) \\ &\overset{\text{Lemma 3.6}}{=} f_*\pi_*s_*\mathcal{PD}_M(E) \overset{\pi\circ s=\mathrm{id}}{=} f_*\mathcal{PD}_M(E) \\ &= \alpha(M,E,f). \end{aligned}$$

□

We now recall that, in the usual long exact sequences in K-homology, the boundary of the fundamental class is the fundamental class, or, formulated more casually: the boundary of the Dirac is Dirac of the boundary. To deal with bordisms of manifolds with boundary, we actually need a slightly more general version as follows, which we prove in Appendix A.4.

3.8 Lemma. *Let L be a G-spin^c manifold with boundary ∂L, let M be a G-invariant submanifold of ∂L with boundary ∂M such that $\dim M = \dim L - 1$ and let $\partial \in KK_1^G(C_0(M\setminus\partial M), C_0(L\setminus\partial L))$ be the boundary element associated to the exact sequence*

$$0 \to C_0(L\setminus\partial L) \to C_0((L\setminus\partial L)\cup(M\setminus\partial M)) \to C_0(M\setminus\partial M) \to 0.$$

Then $[\partial]\otimes[L\setminus\partial L] = [M\setminus\partial M]$.

3.9 Corollary. *With notation of Lemma 3.8, the following diagram commutes*

$$\begin{array}{ccc} K_G^*(L) & \longrightarrow & K_G^*(M) \\ \big\downarrow{\scriptstyle \mathcal{PD}_L} & & \big\downarrow{\scriptstyle \mathcal{PD}_M} \\ K_{\dim L-*}^{G,an}(L,\partial L) & \xrightarrow{\partial\otimes} & K_{\dim L-*-1}^{G,an}(M,\partial M), \end{array}$$

where the top arrow is induced by the inclusion $i : M \hookrightarrow L$.

Proof. Fix $x \in K^*_G(L)$ and denote by $x|_M$ the image of x under the homomorphism $K^*_G(L) \to K^*_G(M)$ induced by the inclusion $M \hookrightarrow L$. Then we get

$$\begin{aligned} \partial \otimes \mathcal{PD}_L(x) &= \partial \otimes \iota_{L,\partial L}(x) \otimes [L, \partial L] \\ &= \iota_{M,\partial M}(x|_M) \otimes \partial \otimes [L \setminus \partial L] \\ &= \iota_{M,\partial M}(x|_M) \otimes [M \setminus \partial M] \\ &= \mathcal{PD}_M(x|_M), \end{aligned}$$

where the second equality should be a well known consequence of the naturality of boundaries and is proved in Lemma A.8 and where the third equality holds by Lemma 3.8. □

3.10 Lemma. *The transformation α is compatible with the bordism relation of $K^{G,geom}_*(X,Y)$, i.e. let (L,F,ϕ,f) be a bordism for a G-CW-pair (X,Y). Then, with notations of Definition 2.4,*

$$\begin{aligned} \alpha(M^+, F|_{M^+}, \phi|_{M^+}) &= (\phi|_{M^+})_* \mathcal{PD}_{M^+}([F|_{M^+}]) \\ &= -(\phi|_{M^-})_* \mathcal{PD}_{M^-}([F|_{M^-}]) = \alpha(M^-, F|_{M^-}, \phi|_{M^-}). \end{aligned}$$

Proof. If we set $M = M^- \amalg M^+$, this amounts to prove that $(\phi|_M)_* \mathcal{PD}([F|_M]) = 0$ in $K^{G,an}_*(X,Y) = KK^G_*(C_0(X \setminus Y), \mathbb{C})$. But this a consequence of Corollary 3.9, together with naturallity of boundaries in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C_0(X \setminus Y) & \longrightarrow & C_0(X \setminus Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_0(L \setminus \partial L) & \longrightarrow & C_0(L \setminus f^{-1}([-1,1])) & \longrightarrow & C_0(M \setminus \partial M) & \longrightarrow & 0 \end{array},$$

where the middle and right vertical arrows are induced by ϕ. □

We are now in the situation to state and prove our main theorem.

3.11 Theorem. *The transformations α and β of Definition 3.3 are well defined and inverse to each other natural transformations for G-homology theories.*

Proof. Lemmas 3.4, 3.7, and 3.8 together imply that α is a well defined homomorphism. If we fix, for given (X,Y) the manifold $(M,\partial M)$ which retracts to (X,Y) (or rather two such manifods, one for each parity of dimensions), then β also is well defined. As soon as we show that β is inverse to α we can conclude that it does not depend on the choice of $(M,\partial M)$.

It is a direct consequence of the construction (and of naturality of K-homology) that α is natural with respect to maps $g\colon (X,Y) \to (X',Y')$.

Corollary 3.9 implies that α is compatible with the boundary maps of the long exact sequence of a pair, and therefore a natural transformation of homology theories (strictly speaking, we really know that $K^{G,an}_*$ is a homology theory only after we know that α is an isomorphism).

We now prove that $\alpha \circ \beta = \mathrm{id}$. Fix $x \in K^{G,an}_*(X,Y)$. Then

$$\begin{aligned} \alpha(\beta(x)) = \alpha([M, \mathcal{PD}^{-1}(j_*(x)), p]) &= p_*(\mathcal{PD} \circ \mathcal{PD}^{-1}(j_*(x))) \\ &= p_* j_*(x) = x \end{aligned}$$

The proof of $\beta \circ \alpha = \mathrm{id}$ is given in the next section. □

4 Normalization of geometric cycles

The goal of this section is to prove that $\beta\circ\alpha\colon K_*^{G,geom}(X,Y)\to K_*^{G,geom}(X,Y)$ is the identity whenever (X,Y) is a compact G-pair with a retraction $(X,Y)\xrightarrow{j}(N,\partial N)\xrightarrow{p}(X,Y)$. Since α and β are natural, the above map is a direct summand of $\beta\circ\alpha\colon K_*^{G,geom}(M,\partial M)\to K_*^{G,geom}(M,\partial M)$. It therefore suffices to show that $\beta\circ\alpha=\mathrm{id}$ for G-spin^c manifolds.

Fix now (M,x,f) a cycle for $K_*^{geom}(M,\partial M)$ as above, with x in $K^0_G(M)$. Then

$$\beta(\alpha[M,x,f])=[N,\mathcal{PD}^{-1}f_*\mathcal{PD}(x),\mathrm{id}]\overset{Lemma 3.6}{=}[N,f_!x,\mathrm{id}].$$

4.1 Theorem. *Let $h\colon(M,\partial M)\hookrightarrow(N,\partial N)$ be the inclusion of a G-spin^c submanifold, E a complex G-vector bundle on M (or more generally an element of $K^0_G(M)$) and let $\phi\colon(N,\partial N)\to(X,Y)$ be a G-equivariant continuous map, where (X,Y) is a G-space. Let ν be the normal bundle of h. Fix the trivial complex line bundle on N. Then the vector bundle modification of $(M,E,\phi\circ h)$ “along” $\mathbb{C}\oplus\nu$ (with its canonical spin^c-structure) and of $(N,h_!E,\phi)$ “along” $\mathbb{C}\times N$ are bordant. In particular,*

$$[M,E,\phi\circ h]=[N,h_!E,\phi]\in K_*^{G,geom}(X,Y).$$

Proof. We just have to write down the bordism. Recall the construction of vector bundle modification (of N along $\mathbb{C}\times N$): we consider $\mathbb{C}\times\mathbb{R}\times N$, equip this with the standard Riemannian metric, and consider the unit disc bundle $D^3\times N$ with its boundary $S^2\times N$ within this bundle. It comes with a canonical “north-pole inclusion” $i\colon N\to S^2\times N$, and the modificaton is $(S^2\times N,\phi\circ\mathrm{pr}_N,i_!h_!E)$.

Fix $\epsilon>0$ small enough and an embedding of ν into N as tubular neighborhood of M. Fix a G-invariant Riemannian metric on ν. Then the ϵ-disk bundle and the ϵ-sphere bundle of $\nu\oplus\mathbb{C}\oplus\mathbb{R}$ are contained in $D^3\times N$, and if we remove the ϵ-disk bundle we get a manifold W with two boundary components, being $S^2\times N$ and the sphere bundle of $\nu\oplus\mathbb{C}\oplus\mathbb{R}$, i.e. the underlying manifold S of the modification of $(M,\phi\circ h,E)$, with its north pole embedding $i_M:M\to S$.

Observe that we have an obvious embedding e (using the $\mathbb{R}$-coordinate of the vector bundle) of $M\times[\epsilon,1]$ into W.

We actually get cartesian diagrams

$$\begin{array}{ccc} M & \xrightarrow{i_1} & M\times[\epsilon,1] \\ \big\downarrow{\scriptstyle i\circ h} & & \big\downarrow{\scriptstyle e} \\ S^2\times N & \xrightarrow{j} & W \end{array}\quad,\quad \begin{array}{ccc} M & \xrightarrow{i_\epsilon} & M\times[\epsilon,1] \\ \big\downarrow{\scriptstyle i_M} & & \big\downarrow{\scriptstyle e} \\ S & \xrightarrow{j'} & W \end{array}\quad.$$

Consider $e_!(\mathrm{pr}_M^*E)$ on W, with $\mathrm{pr}_M\colon M\times[\epsilon,1]\to M$ the obvious map. We claim that $(W,\phi\circ\mathrm{pr},e_!(\mathrm{pr}_M^*E))$ is a bordism (in the sense of cycles for geometric K-homology) between the two cycles we consider.

Obviously, the boundary has the right shape, and $\phi\circ\mathrm{pr}\colon W\to X$ restricts on $S^2\times N$ to the correct map. The restriction $\phi\circ\mathrm{pr}|_S$ is homotopic to the map of the vector bundle modification of $(M,E,\phi\circ h)$ (one has to compose with the projection from the normal bundle to M, composed with the inclusion h); an easy modification of $\phi\circ\mathrm{pr}$ will produce a true bordism.

The final point concerns the K-theory class. Here we use the compatibility of pullback and push-down along cartesian products, i.e. that $j^*\circ e_!=(i\circ h)_!\circ$

$(i_1)^*$ and $(i_M)_! \circ (i_\epsilon)^* = (j')^* e_!$. The restriction $j^*(e_!(\mathrm{pr}_M^* E))$ of $e_!(\mathrm{pr}_M^* E)$ to $S^2 \times N$ therefore is equal to $j^*(e_!(\mathrm{pr}_M^* E)) = (i \circ h)_!(i_1^* \mathrm{pr}_M^*)E = i_! h_! E$, and the restriction of $e_!(\mathrm{pr}_M^* E)$ to S is equal to $(j')^* e_!(\mathrm{pr}_M^* E) = (i_M)_! i_\epsilon^* \mathrm{pr}_M^* E = i_{M!}E$. The claim is proved. □

We now finish the proof that $\beta(\alpha(M, E, f)) = [N, f_! E, \mathrm{id}] = [M, E, f]$. For this, choose a finite dimensional G-representation V and a G-embedding $j_V \colon M \to V$ (this is possible because G is a compact Lie group and M is compact, compare e.g. [14]). Observe that j_V is G-homotopic to the constant map with value 0. Embed V into its one-point compactification V^+, a sphere (it can also be realized as the unit sphere in $V \oplus \mathbb{R}$). By composition we obtain a G-embedding $j \colon M \to V^+$ which is still homotopic to the constant map $c \colon M \to V^+$ with value 0.

We obtain an embedding $M \xrightarrow{(f,j)} N \times V^+$, with $\mathrm{pr}_N \circ (f, j) = f$.

By Theorem 4.1 therefore

$$[M, E, f] = [N \times V^+, (f, j)_! E, \mathrm{pr}_N]. \tag{4.2}$$

On the other hand, $(f, j) \colon M \to N \times V^+$ is G-homotopic to $(f, c) \colon M \to N \times V^+$. Lemma 3.6 shows that $(f, j)_! E$ depends only on the homotopy class of the map. Therefore

$$[N \times V^+, (f, j)_! E, \mathrm{pr}_N] = [N \times V^+, (f, c)_! E, \mathrm{pr}_N]. \tag{4.3}$$

Finally, $(f, c) = (\mathrm{id}_N, c) \circ f$, and $(\mathrm{id}_N, c) \colon N \to N \times V^+$ is an embedding with $\mathrm{pr}_N \circ (\mathrm{id}_N, c) = \mathrm{id}_N$. Using functoriality of the Gysin homomorphism and Theorem 4.1 again, we obtain

$$[N, f_! E, \mathrm{id}] = [N \times V^+, (f, c)_! E, \mathrm{pr}_N]. \tag{4.4}$$

This finishes the proof of our main theorem.

A Analytic Poincaré duality and Gysin maps

A.1 Construction of the Gysin element for closed manifolds

Let $f \colon M \to N$ be a smooth G-map between two compact G-spin^c manifolds without boundary. We describe the construction of the (functorial) Gysin element $f! \in KK^G_{\dim M - \dim N}(C(M), C(N))$.

By using the composition rule and since every smooth G-map $f \colon M \to N$ between compact G-spin^c-manifolds can be written as the composition of the embedding $M \hookrightarrow M \times N;\ x \mapsto (x, f(x))$ with the canonical projection $\pi_2 \colon M \times N \to N$, it will be enough for our purpose to describe the Gysin elements associated to an equivariant embedding and to π_2.

For the projection π_2, the Gysin element is $\pi_2! = \tau_{C(N)}([M])$, where $\tau_{C(N)}([M])$ is the element of $KK^G_{\dim M}(C(M \times N), C(N)) \cong KK^G_{\dim M}(C(M) \otimes C(N), C(N))$ obtained from $[M] \in KK^G_{\dim M}(C(M), \mathbb{C})$ by tensoring with $C(N)$. Notice that in the special case of the map $f \colon M \to \{*\}$, $f_! = [M]$.

In order to define the Gysin element associated to an embedding, we first recall the KK-theory description of the Thom isomorphism. For a G-space X and a G-spin^c vector bundle $W \xrightarrow{q_W} X$, the Thom isomorphism

$$K_G^*(X) \xrightarrow{\cong} K_G^{*+\operatorname{rank} W}(W)$$

is implemented by an element $\beta_W \in KK^G_{\operatorname{rank} W}(C_0(X), C_0(W))$ represented by the following K-cycle. Let S_W be the G-spinor bundle associated to the G-spin^c structure on W and let ξ_W be the $C_0(X)$-module of continuous sections of S_W. If we choose a G-spin^c metric on S_W, then ξ_W can be endowed with a G-equivariant $C_0(X)$-Hilbert module structure. Then the pull-back $q_W^*\xi_X$ is the $C_0(W)$-Hilbert module of continuous sections on the pulled-back vector bundle $q_W^* S_W$ and the morphism

$$C_0(X) \to C_b(W);\ f \mapsto f \circ q_W$$

gives rise to an equivariant representation ϕ_W of $C_0(X)$ on $q_W^*\xi_W$. Let

$$T_W : q_W^*\xi_W \to q_W^*\xi_W$$

be the operator defined using the Clifford representation on S_W by

$$T_W.e(v) := \frac{v}{1 + \|v\|} \cdot e(v) \qquad \text{for } e \in q_W^*\xi_W.$$

Then $(\phi_W, q_W^*\xi_W, T_W)$ is a K-cycle for $KK^G_{\operatorname{rank} W}(C_0(X), C_0(W))$ and its class β_W implements by (right) Kasparov product the Thom isomorphism.

Now if $f : M \to N$ is a G-equivariant embedding of G-spin^c-manifolds, let us consider the normal bundle $\nu_M \xrightarrow{q_{\nu_M}} M$ corresponding to this embedding. Then ν_M is a G-spin^c-vector bundle, and if we fix a G-invariant metric on N, ν_M can be viewed as a G-invariant open tubular neighborhood of M in N via the exponential map. This gives rise to an equivariant inclusion $\theta_M : C_0(\nu_M) \hookrightarrow C(N)$ and then

$$f_! = \beta_{\nu_M} \otimes [\theta_M].$$

A.2 Gysin and Poincaré duality

Proof of Lemma 3.6, case $\partial M = \emptyset = \partial N$. Let us denote by $[f]$ the element of $KK^G_*(C(N), C(M))$ corresponding to the morphism $C(N) \to C(M); h \mapsto h \circ f$. Then the commutativity of the diagram amouts to prove that

$$\iota_N(x \otimes f_!) = [f] \otimes \iota_M(x) \otimes f_! \tag{A.1}$$

for all x in $K_G^*(M)$. Namely, using this equality, we have

$$\mathcal{PD}_N(x \otimes f_!) = \iota_N(x \otimes f_!) \otimes [N] = [f] \otimes \iota_M(x) \otimes f_! \otimes [N].$$

Since $[N]$ is the Gysin element corresponding to the map $N \to \{*\}$, we get then that $f_! \otimes [N] = [M]$ and hence that

$$\mathcal{PD}_N(x \otimes f_!) = [f] \otimes \iota_M(x) \otimes [M] = f_*(\mathcal{PD}_M(x)).$$

Let us now prove Equation A.1. Since f can be written as the composition of an embedding and of the projection $\pi_2\colon M\times N\to N$, it is enought by using the functoriality in K-homology and the composition rule for Gysin elements to check this for an embedding and for π_2.

We start with π_2. Fix $x\in K^*_G(M\times N)$. Recall now first that for G-C^*-algebras A,A',B,B' and $z\in KK^G(A,A')$, $z'\in KK(B,B')$ we have the following commutativity of the exterior Kasparov product:

$$\tau_B(z)\otimes\tau_{A'}(z')=\tau_A(z')\otimes\tau_{B'}(z)\in KK^G(A\otimes B,A'\otimes B').$$

(where for a G-C^*-algebras, $\tau_D: KK^G(A,B)\to KK^G(A\otimes D,B\otimes D)$ is tensorising by D). Recall then that $\pi_2!=\tau_{C(N)}([M])$ and $[\pi_2]=\tau_{C(N)}([pt])$, for $pt: M\to\{*\}$, and that we can write

$$\iota_{M\times N}(x)=\tau_{C(M\times N)}(x)\otimes\mu_{M\times N},$$

where $\mu_{M\times N}\colon C(M\times N)\otimes C(M\times N)\to C(M\times N)$ is the multiplication. Then

$$\begin{aligned}
[\pi_2]\otimes\iota_{M\times N}(x)\otimes\pi_2! &= \tau_{C(N)}[pt]\otimes\tau_{C(M\times N)}(x)\otimes[\mu_{M\times N}]\otimes\pi_2!\\
&= \tau_{C(N)}\left([pt]\otimes\tau_{C(M)}(x)\right)\otimes[\mu_{M\times N}]\otimes\pi_2!\\
&= \tau_{C(N)}\left(x\otimes\tau_{C(M\times N)}[pt]\right)\otimes[\mu_{M\times N}]\otimes\pi_2!\\
&= \tau_{C(N)}(x)\otimes\tau_{C(N\times M\times N)}[pt]\otimes[\mu_{M\times N}]\otimes\pi_2!\\
&= \tau_{C(N)}(x)\otimes\tau_{C(M)}([\mu_N])\otimes\tau_{C(N)}([M])\\
&= \tau_{C(N)}(x)\otimes\tau_{C(N\times N)}([M])\otimes[\mu_N]\\
&= \tau_{C(N)}(x\otimes\tau_{C(N)}([M]))\otimes[\mu_N]\\
&= \tau_{C(N)}(x\otimes\pi_2!)\otimes[\mu_N]\\
&= \iota_N(x\otimes\pi_2!).
\end{aligned}$$

Recall that, if x is an element in $K^*_G(M)$, then $\iota_M(x)$ is the element of $KK^G_*(C(M),C(M))$ obtained from any K-cycle representing x by noticing that $C(M)$ being commutative, the right action is also a left action. Since $x\otimes f_!=[p]\otimes\iota_M(x)\otimes f_!$, where $[p]$ is the element of $K^*_G(M)$ corresponding to the inclusion $\mathbb{C}\hookrightarrow C(M)$, we can see that if (ϕ,T,ξ) is a K-cycle representing $\iota_M(x)\otimes f_!$, then $\iota_N(x\otimes f_!)$ can be represented by the K-cycle (ϕ',T,ξ) where ϕ' is equal to the (right) action of $C(N)$ on ξ. Thus we only have to check that (ϕ',T,ξ) and $(\phi\circ f,T,\xi)$ represent the same class in $KK^G_*(C(N),C(N))$.

Since f is an embbeding, $f_!$ can be represent by the K-cycle $(\phi_{\nu_M},q^*_{\nu_M}\xi_{\nu_M},T_{\nu_M})$ where $q^*_{\nu_M}\xi_{\nu_M}$ is viewed as a $C(N)$-Hilbert module via the inclusion $C_0(\nu_M)\hookrightarrow C(N)$ and where ϕ_{ν_M} is the representation induced by $\phi_0\colon C(M)\to C_b(\nu_M); h\mapsto h\circ q_{\nu_M}$. Thus we can choose the K-cycle (ϕ,T,ξ) representing $\iota_M(x)\otimes f_!$ in such a way that

- ξ is in fact a $C_0(\nu_M)$-Hilbert module (by associativity of the Kasparov product);
- T commutes with the action of $C_b(\nu_M)$ viewed as the multiplier algebra of $C_0(\nu_M)$ (use an approximate unit and the continuity of T, observe that $The=heT=hTe$ for all $h\in C_b(\nu_M)$, $e\in C_0(\nu_M)$);

- the $C(M)$-structure is induced by ϕ_0.

The maps $\nu_M \to \nu_M; v \mapsto tv$ for t in $[0,1]$ provide a homotopy between $\phi_0 \circ f$ and the restricton map $C(N) \to C_b(\nu_M)$ and this homotopy commutes with T_{ν_M}. But the restriction map corresponds precisely to the $C(N)$-Hilbert module structure on $q^*_{\nu_M}\xi_{\nu_M}$, and hence we get the result. □

A.3 Gysin and Poincaré duality if $\partial M \neq \emptyset$

Our key tool to study G-manifolds with boundary is the double.

For a manifold X with boundary ∂X, let us define the double $\mathcal{D}X$ of X to be the manifold obtained by identifying the two copies of the boundaries ∂X in $X \amalg X$. To distinguish the two copies, we write $\mathcal{D}X = X \cup X^-$. Let $p_X \colon \mathcal{D}X \to X$ be the map obtain by identifying the two copies of X. Let $\jmath_X \colon X \to \mathcal{D}X$ be the map induced by the inclusion of the first factor of $X \amalg X$, and let us set $g_X = \jmath_X \circ p_X$. It is straightforward to check that if X is a G-spin^c compact manifold with boundary ∂X, then $\mathcal{D}X$ is a G-spin^c compact manifold without boundary. The given orientation or G-spin^c structure on the first copy of X and the negative structure on the second copy X^- together define canonically a G-spin^c structure on $\mathcal{D}X$. Note that $p_X \circ j_X = \mathrm{id}_X$. Therefore, the exact sequence

$$0 \to C_0(X^- \setminus \partial X) \xrightarrow{\iota_X} C(\mathcal{D}X) \xrightarrow{j_X^*} C(X) \to 0$$

has a split, and we get induced split exact sequences in K-theory and K-homology. Note that in general there is not split of ι_X by algebra homomorphisms, but the corresponding split in K-theory and K-homology of course exists nonetheless. We use the corresponding sequence and split with the roles of X and X^- interchanged.

We now state the workhorse lemma for the extension of the treatment of Gysin homomorphism from closed manifolds to manifolds with boundary.

A.2 Lemma. *Poincaré duality for M is a direct summand of Poincaré duality for $\mathcal{D}M$, i.e. the following diagram commutes, if M is a compact G-spin^c manifold with boundary.*

$$\begin{tikzcd}
K_G^*(M) \arrow[r, "p_M^*"] \arrow[d, "\mathcal{PD}_M"] & K^*(\mathcal{D}M) \arrow[r, "j_M^*"] \arrow[d, "\mathcal{PD}_{\mathcal{D}M}"] & K^*(M) \arrow[d, "\mathcal{PD}_M"] \\
K^G_{n-*}(M,\partial M) \arrow[r, "s"] & K^G_{n-*}(\mathcal{D}M) \arrow[r, "(\iota_M)^*"] & K^G_{n-*}(M,\partial M).
\end{tikzcd} \tag{A.3}$$

Here, s is the K-homology split mentioned above, and $n = \dim(M) = \dim(\mathcal{D}M)$.

Proof. This result is certainly well known. For the convenience of the reader, we give a proof of it in this appendix.

We use the following alternative description of the Poincaré duality homomorphism. For a compact manifold M (possibly with boundary) it is the composition

$$KK(\{*\}, M) \xrightarrow{\tau_{C_0(M^\circ)}} KK(M^\circ, M^\circ \times M) \xrightarrow{\mu} KK(M^\circ, M^\circ) \xrightarrow{\otimes[M^\circ]} KK(M^\circ, \{*\}).$$

Here we abbreviate KK for KK_*^G (and ask the reader to add the correct grading), and write $KK(X,Y) = KK(C_0(X), C_0(Y))$ for two spaces X, Y, μ is the map induced by the multiplication $C_0(M\times M^\circ) = C(M)\otimes C_0(M^\circ) \to C_0(M^\circ)$.

Naturality of KK-theory and of the fundamental class (under inclusion of open submanifolds) now gives the following commutative diagram, writing $N = \mathcal{D}M$

$$\begin{tikzcd}
KK(\{*\},M) \arrow[r, "\tau_{C_0(M^\circ)}"] & KK(M^\circ, M^\circ\times M) \arrow[r, "\mu"'] & KK(M^\circ,M^\circ) & \\
KK(\{*\},N) \arrow[u, "j^*"] \arrow[d] \arrow[r, "\tau_{C_0(M^\circ)}"] & KK(M^\circ, M^\circ\times N) \arrow[u, "j^*"] \arrow[d, "\iota_*"] \arrow[r, "\mu"] & KK(M^\circ,M^\circ) \arrow[u, equal] \arrow[d, "\iota_*"] \arrow[r, "\otimes[M^\circ]"'] & KK(M^\circ,\{*\}) \arrow[d, equal] \\
KK(N,N\times N) \arrow[r, "\iota_M^*"] \arrow[d, equal] & KK(M^\circ, N\times N) \arrow[r, "\mu"] & KK(M^\circ,N) \arrow[r, "\otimes[N]"] & KK(M^\circ,\{*\}) \\
KK(N,N\times N) \arrow[r, equal] & KK(N,N\times N) \arrow[u, "\iota^*"] \arrow[r, "\mu"] & KK(N,N) \arrow[u, "\iota^*"] \arrow[r, "\otimes[N]"] & KK(N,\{*\}) \arrow[u, "\iota^*"]
\end{tikzcd}$$

Walking around the boundary of this diagram shows that the right square of (A.3) is commutative.

The commutativity of the left square of (A.3) is more difficult to show, in particular since s is not induced from an algebra homomorphism. However, from what we have just seen we can conclude that

$$\iota^*\mathcal{PD}_{\mathcal{D}M}p^* = \mathcal{PD}_M j^* p^* \overset{p\circ j = \mathrm{id}_M}{=} \mathcal{PD}_M = \iota^* s \mathcal{PD}_M. \tag{A.4}$$

The section s is characterized by the properties $\iota^* s = \mathrm{id}$ and $sp_* = 0$. Therefore, to be allowed to "cancel" ι^* in Equation (A.4) we have to show that $\mathcal{PD}_{\mathcal{D}M}p^*$ maps to the image of s, i.e. to the kernel of p_*: we must show that

$$0 = p_*\mathcal{PD}_N p^* \colon KK(\{*\},M) \to KK(N,\{*\}). \tag{A.5}$$

The relevant groups, namely $K^*(M)$, $K^*(\mathcal{D}M)$, $K_*(\mathcal{D}M)$, $K_*(M^\circ)$ all are $K^*(M)$-modules, and all homomorphisms are $K^*(M)$-module homomorphisms. The module structure on $K^*(M)$ is induced via the ring structure of $K^*(\mathcal{D}M)$ and the map p^*. $K_*(\mathcal{D}M)$ is a $K^*(\mathcal{D}M)$-module via the cap product, and via p^* it therefore also becomes a $K^*(M)$-module; the cap product also gives the $K^*(M)$-module structure on $K_*(M^\circ)$.

As $K^*(M)$ is generated by 1 as a $K^*(M)$-module, Equation (A.5) follows if

$$0 = p_*\mathcal{PD}_{\mathcal{D}M}p^*1 = p_*[\mathcal{D}M].$$

To see this, remember that every double of a manifold with boundary is canonically a boundary, namely $\mathcal{D}M = \partial(Y := (M\times[-1,1]/\sim))$, where the equivalence relation is generated by $(x,t)\sim(x,s)$ is $x\in\partial M$ and $s,t\in[-1,1]$. Observe that this construction is valid in the world of G-spin^c manifolds. Note that $p_M\colon \mathcal{D}M \to M$ extends to $P\colon Y = (M\times[-1,1]/\sim) \to (M\times[0,1]/\sim); (x,t)\mapsto (x,|t|)$. From the long exact sequences of the pairs $(Y,\mathcal{D}M)$ and

$(M\times\{1\}, M\times[0,1]/\sim)$, we have the following commutative diagram

$$\begin{array}{ccc} K_{\dim M+1}(Y^\circ) & \xrightarrow{P_*} & K_{\dim M+1}(C_0((M\times[0,1]/\sim)\setminus M\times\{1\}) = \{0\} \\ \downarrow\partial & & \downarrow\partial \\ K_{\dim M}(\mathcal{D}M) & \xrightarrow{p_*} & K_{\dim M}(M). \end{array}$$

In this diagram, $[Y^\circ]$ is, according to Lemma 3.8, mapped under the boundary to $[\mathcal{D}M]$. Therefore, by naturality, $p_*[\mathcal{D}M] = \partial P_*([Y^\circ])$. However,

$$M\times[0,1]/\sim\setminus M\times\{1\} = M^\circ\times[0,1),$$

and $C_0(M^\circ\times[0,1))$ is G-equivariantly contractible, hence its equivariant K-homology vanishes. The assertion follows. ☐

A.6 Definition. Let now $f\colon M\to N$ be a G-equivariant continuous map between G-spin^c manifolds with boundary such that $f(\partial M)\subset\partial N$. Then we define $f_!\colon K_G^*(M)\to K_G^{*+n-m}(N)$ as the composition

$$f_! = \mathcal{PD}_N^{-1} f_* \mathcal{PD}_M.$$

A.7 Remark. Note that this is consistent with the definition for closed manifolds and smooth maps by the considerations of Section A.2. Lemma 3.6 holds in the general case by definition.

However, at least in special situations, we can also define the Gysin map geometrically. Let, for example, M be a G-spin^c compact manifold with boundary, let W be a G-spin^c vector bundle over M, let Z be the manifold obtained from vector bundle modification with respect to W and, as above, let $\pi\colon Z\to M$ and $s\colon M\to Z$ be the canonical projection or the "north pole" section of π, respectively. The vector bundle W is the normal vector bundle of M in Z (with respect to the embedding s) and is therefore a G-invariant tubular open neighbourhood of M.

We can then define the Gysin element $s_!\in KK^G(C(M), C(Z))$ associated to s as we did for manifold without boundary by $s_! = \beta_W\otimes[\theta_M]$, where $\theta_M\colon C_0(W)\to C(Z)$ is the morphism induced by the inclusion of W into Z. The Gysin homomorphism can then be defined correspondingly.

With arguments similar to those of Section A.2 we can show that with this definition Lemma 3.6 holds, so that our Definition A.6 is consistent with the geometric one. The proof will also use Lemma A.2, that $\mathcal{PD}_M$ is a direct summand of $\mathcal{PD}_{\mathcal{D}M}$.

A.8 Lemma. *If $i\colon M\hookrightarrow L$ is as in Lemma 3.8, then*

$$\partial\otimes\iota_{L,\partial L}(x) = \iota_{M,\partial M}(i^*x)\otimes\partial \qquad \forall x\in K_G^*(L),$$

where $\partial\in KK(C_0(M^\circ), C_0(L^\circ))$ is the boundary element of the exact sequence of $C_0(L^\circ)\hookrightarrow C_0(L^\circ\cup M^\circ)\twoheadrightarrow C_0(M^\circ)$ as in Lemma 3.8.

Proof. We first recall a KK-description of $\iota_{L,\partial L}$. We abbreviate $L^\circ = L\setminus\partial L$. It is given by the composition

$$KK(\{*\}, L)\xrightarrow{\tau_{C_0(L^\circ)}} KK(L^\circ, L\times L^\circ)\xrightarrow{\otimes\mu} KK(L^\circ, L^\circ)$$

where μ is the multiplication homomorphism. By commutativity of the exterior Kasparov product, we therefore get that $\partial \otimes \iota_{L,\partial L}$ equals the composition

$$KK(\{*\}, L) \xrightarrow{\tau_{M^\circ}} KK(M^\circ, L \times M^\circ) \xrightarrow{\otimes\partial} KK(M^\circ, L \times L^\circ) \xrightarrow{\mu} KK(M^\circ, L^\circ). \tag{A.9}$$

Now observe that we have commutative diagrams of short exact sequences

$$\begin{tikzcd}
C_0(L \times L^\circ) \arrow[r] \arrow[d] & C_0(L \times (L^\circ) \cup M^\circ) \arrow[r] \arrow[d, "="] & C_0(L \times M^\circ) \arrow[d, "i^* \times \mathrm{id}_{M^\circ}"'] \\
C_0(L \setminus M \times M^\circ \cup L \times L^\circ) \arrow[r] \arrow[d, "\mu"] & C_0(L \times (L^\circ \cup M^\circ)) \arrow[r] \arrow[d, "\mu"] & C_0(M \times M^\circ) \arrow[d, "\mu"'] \\
C_0(L^\circ) \arrow[r] & C_0(L^\circ \cup M^\circ) \arrow[r] & C_0(M^\circ)
\end{tikzcd}$$

Using naturality of the boundary map, we observe that the composition of the last two arrows of (A.9) coincides with the composition

$$KK(M^\circ, L \times M^\circ) \xrightarrow{i^*} KK(M^\circ, M \times M^\circ) \xrightarrow{\otimes\mu} KK(M^\circ, M^\circ) \xrightarrow{\otimes\partial} KK(M^\circ, L^\circ).$$

As i^* commutes with the exterior product with $C_0(M^\circ)$, this implies the assertion. $\square$

A.4 Proof of Lemma 3.8

We finish by proving Lemma 3.8. Recall that it states

A.10 Lemma. *Let L be a G-spin^c manifold with boundary ∂L, let M be a G-invariant submanifold of ∂L with boundary ∂M such that $\dim M = \dim L - 1$ and let $\partial \in KK_1^G(C_0(M \setminus \partial M), C_0(L \setminus \partial L))$ be the boundary element associated to the exact sequence*

$$0 \to C_0(L \setminus \partial L) \to C_0((L \setminus \partial L) \cup (M \setminus \partial M)) \to C_0(M \setminus \partial M) \to 0.$$

Then $[\partial] \otimes [L \setminus \partial L] = [M \setminus \partial M]$.

Proof. Using a G-invariant metric on L and a corresponding collar, $(0,1] \times M \setminus \partial M$ can be viewed as a G-invariant open neighborhood of $(L \setminus \partial L) \cup (M \setminus \partial M)$ and with $\{1\} \times M \setminus \partial M \subset \partial L$, moreover, the inclusion $C_0((0,1] \times M \setminus \partial M) \hookrightarrow C_0((\partial L \setminus \partial L) \cup (M \setminus \partial M))$ gives rise to the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_0(L \setminus \partial L) \arrow[r] & C_0((L \setminus \partial L) \cup (M \setminus \partial M)) \arrow[r] & C_0(M \setminus \partial M) \arrow[r] & 0 \\
0 \arrow[r] & C_0((0,1) \times M \setminus \partial M) \arrow[r] \arrow[u] & C_0((0,1] \times M \setminus \partial M) \arrow[r] \arrow[u] & C_0(M \setminus \partial M) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

By naturality of the boundary homomorphism and since by [7, Proposition 11.2.12] (for the non-equivariant case, but the equivariant one follows along identical lines) the restriction of $[L \setminus \partial L]$ to $(0,1) \times M \setminus \partial M$ is $[(0,1) \times M \setminus \partial M]$, the statement of the lemma amouts to show that

$$[\partial'] \otimes [(0,1) \times M \setminus \partial M] = [M \setminus \partial M],$$

where $\partial' \in KK_1^G(C_0(M \setminus \partial M), C_0((0,1) \times M \setminus \partial M))$ is the boundary element associated to the bottom exact sequence of the diagram above. Viewing $M \setminus \partial M$ as an invariant open subset of $\mathcal{D}M$, using naturality of boundaries in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((0,1) \times \mathcal{D}M) & \longrightarrow & C_0((0,1] \times \mathcal{D}M) & \longrightarrow & C_0(\mathcal{D}M) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C_0((0,1) \times M \setminus \partial M) & \longrightarrow & C_0((0,1] \times M \setminus \partial M) & \longrightarrow & C_0(M \setminus \partial M) & \longrightarrow & 0, \end{array}$$

and since the elements $[M \setminus \partial M]$ of $KK_*^G(C_0(M \setminus \partial M), \mathbb{C})$ and $[(0,1) \times M \setminus \partial M]$ of $KK_*^G(C_0((0,1) \times M \setminus \partial M), \mathbb{C})$ are the restrictions of $[\mathcal{D}M]$ to $M \setminus \partial M$ and of and of $[(0,1) \times \mathcal{D}M]$ to $(0,1) \times M \setminus \partial M$, respectively, we can indeed assume without loss of generality that M has no boundary.

Observe now that in the exact sequence in (non-equivariant) K-homology

$$0 \to C_0((0,1)) \to C_0((0,1]) \to C(\{1\}) \to 0$$

by the well known principle that "boundary of Dirac is Dirac" we indeed observe $[\partial''] \otimes [(0,1)] = [\{1\}]$ in $KK_0(C(\{1\}), \mathbb{C})$, compare [7, Propositions 9.6.7, 11.2.15]. We can now take the exterior Kasparov product of the whole situation with with $[M] \in KK_{\dim M}^G(C_0(M), \mathbb{C})$. By naturality of this Kasparov product, we obtain $[\partial'] \otimes [(0,1)] \otimes [M] = [\{1\}] \otimes [M]$. Finally, we know that the fundamental class of a product is the exterior Kasparov product of the fundamental classes, compare again [7, Proposition 11.2.13]; the equivariant situation follows similarly. This implies the desired relation $[\partial'] \otimes [(0,1) \times M] = [M] \in KK_{\dim M}(C_0(M), \mathbb{C})$.

□

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