REMARKS ON A CONJECTURE OF GROMOV AND LAWSON

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Gromov and Lawson conjectured in [GL2] that a closed spin manifold M of dimension n with fundamental group π admits a positive scalar curvature metric if and only if an associated element in $KO_n(B\pi)$ vanishes. In this note we present counter examples to the 'if' part of this conjecture for groups π which are torsion free and whose classifying space is a manifold with negative curvature (in the Alexandrov sense).

1. Introduction

In their influential paper [GL2] Gromov and Lawson proposed the following conjecture.

1.1. ***Gromov-Lawson Conjecture [GL2].*** *Let M be a smooth, compact manifold without boundary of dimension $n \geq 5$ with fundamental group π. Then M admits a positive scalar curvature metric if and only if $p \circ D([M,u]) = 0 \in KO_n(B\pi)$.*

Here $u\colon M \to B\pi$ is the map classifying the universal covering of M, and $[M,u] \in \Omega_n^{spin}(B\pi)$ is the element in the spin bordism group represented

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by the pair (M,u). The maps D and p are natural transformations between generalized homology theories:

$$\Omega_n^{spin}(X) \xrightarrow{D} ko_n(X) \xrightarrow{p} KO_n(X),$$

referred to as 'spin bordism', 'connective KO-homology' and 'periodic KO-homology', respectively. As the name suggests, $KO_n(\quad)$ is periodic in the sense that $KO_n(X) \cong KO_{n+8}(X)$; moreover,

$$KO_n(\text{point}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{if } n \equiv 1,2 \mod 8 \\ 0 & \text{otherwise} \end{cases}.$$

The homology theory $ko_n(\quad)$ is connective; i.e. $ko_n(\text{point}) = 0$ for $n < 0$. Moreover, the natural transformation p induces an isomorphism $ko_n(\text{point}) \cong KO_n(\text{point})$ for $n \geq 0$ (this does *not* hold with point replaced by a general space X).

We remark that the assumption $n \geq 5$ (which wasn't present in [GL2]) should be added since Seiberg-Witten invariants show that the conjecture is false for $n = 4$ even if the group π is trivial. In their paper [GL2] Gromov and Lawson prove the 'only if' part of Conjecture 1.1 for some groups π, but later Rosenberg [Ro1] noticed that the 'only if' statement does not hold e.g. for finite cyclic groups, since any lens space M admits a metric with positive scalar curvature, but $p \circ D([M,u])$ is non-trivial. He proposed the following variant of Conjecture 1.1.

1.2. ***The Gromov-Lawson-Rosenberg Conjecture [Ro3].*** *Let M be a smooth, compact manifold without boundary of dimension $n \geq 5$ with fundamental group π. Then M admits a positive scalar curvature metric if and only if $\alpha([M,u]) = 0 \in KO_n(C_r^*\pi)$.*

Here α is the following composition

$$\Omega_n^{spin}(B\pi) \xrightarrow{D} ko_n(B\pi) \xrightarrow{p} KO_n(B\pi) \xrightarrow{A} KO_n(C_r^*\pi), \qquad (1.3)$$

where A is the *assembly map*, whose target is the KO-theory of the (reduced) real group C^*-algebra $C_r^*\pi$ (this is a norm completion of the real group ring $\mathbb{R}\pi$ and is equal to the latter for finite π).

Rosenberg proved the 'only if' part of this Conjecture in [Ro2] by interpreting the image of $[M,u] \in \Omega_n^{spin}(B\pi)$ under α as a 'fancy' type of index of the Dirac operator on M and showing that this index vanishes if M admits a metric of positive scalar curvature. We note that if A is injective for a group π (which can be shown for many groups π and is expected to hold for *all* torsion free groups according to the so called strong Novikov-Conjecture), both conjectures are equivalent.

All partial results concerning Conjectures 1.1 and 1.2 discussed so far have to do with the 'only if' part or equivalently, with finding *obstructions* (like the index obstruction α) against the existence of positive scalar curvature metrics. For a few groups π *constructions* have been found of sufficiently many positive scalar curvature metrics to prove the 'if' part as well and hence to prove the Gromov-Lawson-Rosenberg Conjecture 1.2. This includes the trivial group [St1], cyclic groups and more generally all finite groups π with periodic cohomology [BGS]. However, a counter example has been found for $\pi = \mathbb{Z}/3 \times \mathbb{Z}^4$, $n = 5$ [Sch]. So far, no counter example to the Gromov-Lawson-Rosenberg Conjecture has been found for *finite* groups π (which actually is the class of groups the conjecture was originally formulated for; cf. [Ro3], Conjecture 0.1 and §3).

In this paper we address the question of whether the Gromov-Lawson Conjecture and/or Gromov-Lawson-Rosenberg Conjecture might be true for *torsion free groups*. Alas, the answer is 'no' in both cases; more precisely:

Theorem 1.4. *There are finitely generated torsion free groups π for which the 'if' part of the Gromov-Lawson Conjecture 1.1 (and hence also the 'if' part of the Gromov-Lawson-Rosenberg Conjecture 1.2) is false. In other words, there is a closed spin manifold M of dimension $n \geq 5$ with fundamental group π and $p \circ D([M, u]) = 0$ which does not admit a positive scalar curvature metric.*

Still, one might hope to save these conjectures at least for groups π satisfying some additional geometric conditions, say of the kind that guarantee that the assembly map A is an isomorphism. We remark that according to the Baum-Connes Conjecture the assembly map is an isomorphism for all torsion free groups π, and we also remark that the two Conjectures become equivalent if A is injective. For these groups in particular the 'only if' part of the conjectures is true.

Example 1.5. ***Examples of groups π for which the assembly map is an isomorphism***

(1) Any countable group π in the class $\mathcal{C} = \cup_{n \in \mathbb{N}} \mathcal{C}_n$, where the class $\mathcal{C}_0$ consists just of the trivial group, and $\mathcal{C}_n$ is defined inductively as the class of groups which act on trees with all isotropy subgroups belonging to $\mathcal{C}_{n-1}$ [Oy].

(2) *The fundamental group of a closed manifold which is CAT(0)-cubical complex, i.e. a piecewise Euclidean cubical complex which is non-positively curved in the sense of Alexandrov (cf.* (5.1)*)*
(3) *Any word hyperbolic group in the sense of Gromov, in particular fundamental groups of strictly negatively curved manifolds (in the Riemannian sense or in the sense of Alexandrov).*

In the second case the claim that the assembly map is an isomorphism follows from combining work of Niblo and Reeves [NR] and Higson and Kasparov [HK]. Niblo and Reeves [NR], compare [Ju, p. 158] show that any group acting properly on a CAT(0)-cubical complex has the Haagerup property (in other words, is a-T-menable). Higson and Kasparov prove that for such groups the Baum-Connes Conjecture holds.

The third case is proved by Mineyev and Yu [MY] (reducing to some deep results of Lafforgue [L1,L2]). Note that the fundamental group of a negatively curved manifold is known to be word hyperbolic.

We note that the papers cited above deal with the complex version of the Baum-Connes Conjecture, whereas we are interested in the real version. However, it is a "folk theorem" that the complex isomorphism implies the real isomorphism, compare e.g. [Ka].

Unfortunately, making these kinds of assumptions about the group π does not save Conjectures 1.1 or 1.2:

1.6. ***Addendum to Theorem 1.4.*** *The group π in Theorem 1.4 may be chosen to be in the class $\mathcal{C}$ and to be the fundamental group of a closed manifold which is a CAT(0)-cubical complex. Alternatively, we may choose π to be the fundamental group of a closed manifold which is negatively curved in the Alexandrov sense.*

With this 'negative' result showing that Conjectures 1.1 and 1.2 are false even for very "reasonable" groups π, the challenge is to find a necessary and sufficient condition for a general M to admit a positive scalar curvature metric. To the authors' knowledge, at this point there is not even a *candidate* for this condition.

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Preparation of the final version of this paper was made difficult by the style file provided by the publisher. Not only did the company require the authors to do the publishers work of final typesetting (as we are used to),

on top of this it ask to use undocumented style files which produce some very strange results. Only after lengthy discussions, and changes in these style file made by the authors, acceptable appearance could be achieved.

2. Outline of the proof

As a first step we rephrase Conjectures 1.1 and 1.2. The key for this is the following result of Gromov-Lawson [GL1] (see also [RS]):

Theorem 2.1. *Let M be a spin manifold of dimension $n \geq 5$, and let $u\colon M \to B\pi$ be the classifying map of its universal covering. Then M admits a positive scalar curvature metric if and only if $[M, u] \in \Omega_n^{spin,+}(B\pi)$.*

Here for any space X, the group $\Omega_n^{spin,+}(X)$ is by definition the following subgroup of $\Omega_n^{spin}(X)$:

$$\Omega_n^{spin,+}(X) = \left\{ [N, f] \;\middle|\; \begin{array}{c} N \text{ is an } n\text{-dimensional spin manifold} \\ \text{with positive scalar curvature metric,} \\ f\colon N \to X \end{array} \right\}$$

It should be emphasized that while the 'only if' portion of the above theorem is of course tautological, the 'if' part is *not*; it can be rephrased by saying that if M is bordant (over $B\pi$) to some spin manifold N with a positive scalar curvature metric, then M itself admits a positive scalar curvature metric.

We claim that for a finitely presented group π the Gromov-Lawson Conjecture 1.1 (resp. Gromov-Lawson-Rosenberg Conjecture 1.2) is equivalent to the first (resp. second) of the following equalities for $n \geq 5$

$$\Omega_n^{spin,+}(B\pi) = \ker p \circ D \tag{2.2}$$

$$\Omega_n^{spin,+}(B\pi) = \ker \alpha \tag{2.3}$$

By the Bordism Theorem 2.1 it is clear that these equations imply Conjectures 1.1 resp. 1.2. Conversely, if π is a finitely presented group, then a surgery argument shows that *every* bordism class in $\Omega_n^{spin}(B\pi)$, $n \geq 5$ can be represented by a pair (M, u), where M is a manifold with fundamental group π, and $u\colon M \to B\pi$ is the classifying map of the universal covering of M. This shows that the Gromov-Lawson Conjecture 1.1 implies equation (2.2) and that the Gromov-Lawson-Rosenberg Conjecture 1.2 implies equation (2.3). Recall that it is known that $\Omega_n^{spin,+}(B\pi) \subset \ker \alpha$. If the Baum-Connes map A is injective for π, this is implies even that $\Omega_n^{spin,+}(B\pi) \subset \ker p \circ D$.

Using this translation, our main result Theorem 1.4 is a consequence of the following slightly more precise result.

Theorem 2.4. *For $5 \leq n \leq 8$, there are finitely presented torsion free groups π such that $\Omega_n^{spin,+}(B\pi) \subsetneq \ker p \circ D$.*

To prove this statement, we produce a bordism class $x \in \Omega_n^{spin}(B\pi)$ in the kernel of $p \circ D$ and show that it cannot be represented by a manifold of positive scalar curvature. At present, three methods are known to show that a manifold M does not admit a positive scalar curvature metric: the index-theory of the Dirac operator on M, the Seiberg-Witten invariants, and the stable minimal hypersurface method pioneered by Schoen and Yau [SY]. For the case at hand, the first two methods are useless: the index of any manifold M representing x vanishes due to our assumption $p \circ D(x) = 0$ and the Seiberg-Witten invariants of M are only defined for 4-dimensional manifolds. As explained in [Sch, Proof of Cor. 1.5], a corollary of the stable minimal hypersurface method is the following result [Sch, Cor. 1.5].

Theorem 2.5. *Let X be a space, and let $H_n^+(X;\mathbb{Z})$ be the subgroup of $H_n(X;\mathbb{Z})$ consisting of those elements which are of the form $f_*[N]$, where N is an oriented closed manifold of dimension n which admits a positive scalar curvature metric, and f is a map $f\colon N \to X$. Let*

$$\alpha\cap\colon H_n(X;\mathbb{Z}) \to H_{n-1}(X;\mathbb{Z})$$

be the homomorphism given by the cap product with a class $\alpha \in H^1(X;\mathbb{Z})$. Then for $3 \leq n \leq 8$ the homomorphism $\alpha\cap$ maps $H_n^+(X;\mathbb{Z})$ to $H_{n-1}^+(X;\mathbb{Z})$.

We remark that a better regularity result for hypersurfaces with minimal volume representing a given homology class proved by Smale [Sm] makes it possible to include the case $n = 8$ in the above result (see [JS, §4] for a detailed explanation). The case $n = 8$ was not covered in [Sch].

Corollary 2.6. *Let X be the classifying space of a discrete group π. Assume that $x \in \Omega_n^{spin}(X)$ is a bordism class for $5 \leq n \leq 8$, satisfying the condition*

$$\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap H(x) \neq 0 \in H_2(X;\mathbb{Z}) \tag{2.7}$$

for some cohomology classes $\alpha_1, \ldots, \alpha_{n-2} \in H^1(X;\mathbb{Z})$ (here $H\colon \Omega_n^{spin}(X) \to H_n(X;\mathbb{Z})$ is the Hurewicz map given by sending a bordism class $[M, f]$ to $f_*[M]$, where $[M] \in H_n(M;\mathbb{Z})$ is the fundamental class of M). Then x is not in the subgroup $\Omega_n^{spin,+}(X)$.

Proof. Assume $x \in \Omega_n^{spin,+}(X)$. Then $H(x) \in H_n^+(X;\mathbb{Z})$ (by definition of $H_n^+(X;\mathbb{Z})$); applying Theorem 2.5 first to the cohomology class α_1, then α_2, e.t.c., we conclude $\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap H(x) \in H_2^+(X;\mathbb{Z})$. This is the desired contradiction, since the element $\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap H(x)$ is assumed to be non-zero, while the group $H_2^+(X;\mathbb{Z})$ is trivial: by the Gauss-Bonnet Theorem, the only closed oriented 2-manifold N with positive scalar curvature are disjoint unions of 2-spheres; however any map f from such a union to the classifying space of a discrete group is homotopic to the constant map, which implies $f_*[N] = 0$. □

We note that Corollary 2.6 implies Theorem 2.4, provided we can find a bordism class x in the kernel of $p \circ D$ satisfying condition 2.7. Whether there is such a bordism class is in general a pretty hard question; fortunately the following homological condition is much easier to check, and, as Theorem 2.9 below shows, it implies the existence of a spin bordism class x in the kernel of $p \circ D$ with property (2.7).

2.8. ***Homological condition.*** *There are (co)homology classes*

$$\alpha_1, \ldots, \alpha_{n-2} \in H^1(X;\mathbb{Z}) \quad \text{and } z \in H_{n+5}(X;\mathbb{Z})$$

such that

$$\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap \delta P^1 \rho(z) \neq 0 \in H_2(X;\mathbb{Z})$$

Here

- $\rho\colon H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/3)$ *is mod* 3 *reduction,*
- $P^1\colon H_*(X;\mathbb{Z}/3) \longrightarrow H_{*-4}(X;\mathbb{Z}/3)$ *is the homology operation dual to the degree* 4 *element* P^1 *of the mod* 3 *Steenrod algebra (see [SE, Chapter VI, section 1]), and*
- $\delta\colon H_*(X;\mathbb{Z}/3) \to H_{*-1}(X;\mathbb{Z})$ *is the Bockstein homomorphism associated to the short exact coefficient sequence* $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/3$ *(i.e. the boundary homomorphism of the corresponding long exact homology sequence).*

In section 3 we will use the Atiyah-Hirzebruch spectral sequences converging to $\Omega_*^{Spin}(X)$ resp. $KO_*(X)$ to prove the following result.

Theorem 2.9. *Let X be a space which satisfies the homological condition 2.8 for $5 \leq n \leq 8$. Then there is a bordism class $x \in \Omega_n^{spin}(X)$ in the kernel of $p \circ D$ satisfying condition (2.7) above.*

Putting these results together, we obtain the following corollary.

Corollary 2.10. *Let X be a space satisfying the homological condition 2.8 for some n with $5 \leq n \leq 8$. If X is the classifying space of some discrete group π, then $\Omega_n^{spin,+}(B\pi) \subsetneq \ker p \circ D$.*

For a given space X it is not hard to check the homological condition 2.8, provided we know enough about the (co)homology of X. For example, in section 4 we will prove the following result by a straightforward calculation.

Proposition 2.11. *For $n \geq 2$, let Γ_n be the cartesian product of 2 copies of $\mathbb{Z}/3$ and $n-2$ copies of $\mathbb{Z}$. Then the classifying space $B\Gamma_n$ satisfies the homological condition 2.8.*

In particular, Corollary 2.10 then shows that the Gromov-Lawson Conjecture 1.1 does not hold for the group Γ_n for $5 \leq n \leq 8$. This is very similar to the result of one of the authors [Sch] who constructed a 5-dimensional spin manifold with fundamental group $\pi = \mathbb{Z}/3 \times \mathbb{Z}^4$ whose index obstruction $\alpha([M,u]) \in KO_5(C^*\pi)$ is trivial, but which does not admit a metric of positive scalar curvature. However, the example in [Sch] does *not* provide a counter example to the Gromov-Lawson Conjecture 1.1, since it can be shown that $p \circ D([M,u]) \in KO_5(B\pi)$ is *non-trivial*.

As explained in the introduction, it is more interesting to find *torsion free* groups π for which the conjecture goes wrong. It seems conceivable that experts might know explicit examples of torsion free groups and enough about their (co)homology to conclude that the homological condition 2.8 is satisfied (or the cohomological condition 4.2, which is stronger, but easier to check).

Lacking this expertise, we argue more indirectly to prove Theorem 2.4. We use a construction of Baumslag, Dyer and Heller [BDH], who associate a discrete group π to any connected CW complex X and show that there is a map $B\pi \to X$ which is an isomorphism in homology. Moreover, if X is a finite CW complex, then $B\pi$ has the homotopy type of a finite CW complex. In particular, the group π is finitely presented. We would like to mention that originally Kan and Thurston described a similar construction [KT]. However, their groups are usually *not* finitely presented and hence not suitable for our purposes.

To prove Theorem 2.4 we let X be any finite CW complex satisfying the condition 2.8, e.g., the $n+5$-skeleton of $B\Gamma_n$ and let π be the discrete group obtained from X via the Baumslag-Dyer-Heller procedure. Then π is a discrete group which is finitely presented and torsion free since its classifying space $B\pi$ is homotopy equivalent to a finite CW complex. Moreover, $B\pi$

satisfies the homological condition 2.8 and hence Corollary 2.10 implies Theorem 2.4.

Applying more sophisticated 'asphericalization procedures' due to Davis-Januskiewicz [DJ] respectively Charney-Davis [CD], we can produce groups π satisfying the geometric conditions mentioned in Addendum 1.6. This is explained in section 5.

3. A spectral sequence argument

The goal of this section is the proof of Theorem 2.9. We consider the Atiyah-Hirzebruch spectral sequence (AHSS for short)

$$E^2_{p,q}(X) = H_p(X; \Omega^{spin}_q) \Longrightarrow \Omega^{spin}_{p+q}(X). \tag{3.1}$$

We recall that the Hurewicz map $H\colon \Omega^{spin}_n(X) \longrightarrow H_n(X;\mathbb{Z})$ is the *edge homomorphism* of this spectral sequence; i.e. H is equal to the composition

$$\Omega^{spin}_n(X) \twoheadrightarrow E^\infty_{n,0}(X) \hookrightarrow E^2_{n,0}(X) = H_n(X;\mathbb{Z}).$$

This shows that in order to produce a bordism class $x \in \Omega^{spin}_n(X)$ satisfying condition (2.7) it suffices to produce a homology class $y \in H_n(X;\mathbb{Z})$ such that

$$y \in H_n(X;\mathbb{Z}) = E^2_{n,0} \quad \text{is an infinite cycle for the AHSS (3.1)} \tag{3.2}$$

$$\alpha_1 \cap \dots \cap \alpha_{n-2} \cap y \neq 0 \tag{3.3}$$

In order to find y we will use the map of spectral sequences

$$E^r_{p,q}(X;\Omega^{Spin}) \longrightarrow E^r_{p,q}(X;KO).$$

induced by the natural transformation $p \circ D\colon \Omega^{Spin}_n(X) \to KO_n(X)$. To guarantee that the element x produced this way is in the kernel of $p \circ D$ requires an addition argument to be given later.

To simplify the analysis of these spectral sequences, from now on we localize all homology theories at the prime 3, which has the effect of replacing all homology groups as well as the groups in the AHSS converging to them by the corresponding localized groups; i.e. their tensor product with $\mathbb{Z}_{(3)} = \{\frac{a}{b} \mid b \text{ is prime to } 3\}$. Note that we continue to write e.g. $H_*(X;\mathbb{Z})$, but mean the localized group.

In particular, the coefficient ring KO_* is now the ring of polynomials $\mathbb{Z}_{(3)}[b]$ with generator $b \in KO_4$. This implies that in the AHSS converging to $KO_*(X)$ only the rows $E^r_{p,q}$ for $q \equiv 0 \mod 4$ are possibly non-trivial. In particular, the first differential that can be non-trivial is d_5. To finish the proof of Theorem 2.9 we will need the following result, which identifies d_5 with a homology operation. This is certainly well known among experts; since we failed to find an explicit reference in the literature, for completeness we include a proof of this result below.

Lemma 3.4. *In the AHSS converging to $KO_*(X)$ localized at 3, the differential*

$$d_5\colon E^5_{p,q} = E^2_{p,q} = H_p(X;\mathbb{Z}) \longrightarrow E^5_{p-5,q+4} = E^2_{p-5,q+4} = H_{p-5}(X;\mathbb{Z})$$

for $q \equiv 0 \mod 4$ is up to sign the composition

$$H_p(X;\mathbb{Z}) \xrightarrow{\rho} H_p(X;\mathbb{Z}/3) \xrightarrow{P^1} H_{p-4}(X;\mathbb{Z}/3) \xrightarrow{\delta} H_{p-5}(X;\mathbb{Z}). \tag{3.5}$$

For the notation, compare 2.8.

Proof of Theorem 2.9. Assume that the space X satisfies the homological condition 2.8; i.e. there are (co)homology classes $\alpha_1 \dots, \alpha_{n-2} \in H^1(X;\mathbb{Z})$ and $z \in H_{n+5}(X;\mathbb{Z})$ with

$$\alpha_1 \cap \dots \cap \alpha_{n_2} \cap y \neq 0 \qquad \text{for} \qquad y \overset{\text{def}}{=} \delta \circ P^1 \circ \rho(z).$$

We observe that y has property 3.3 by our assumption on z; moreover, it also has property 3.2 (i.e. it survives to the E^∞-term of the AHSS converging to $\Omega^{spin}_*(X)$) by the following argument. Recall that $\Omega^{spin}_n \to KO_n$ is an isomorphism for $0 \le n < 8$. Therefore, the map of spectral sequences

$$E^r_{p,q}(X;\Omega^{Spin}) \longrightarrow E^r_{p,q}(X;KO)$$

is an isomorphism on the rows $q = 0, 4$ for $r = 2$ and hence for $r = 3, 4, 5$ (all the other rows in the range $-4 < q < 8$ are trivial since we work localized at the prime 3). Lemma 3.4 shows that the differential

$$d_5\colon E^5_{n+5,-4}(X;KO) \longrightarrow E^5_{n,0}(X;KO)$$

sends $z \in H_{n+5}(X;\mathbb{Z}) = E^2_{n+5,-4}(X;KO) = E^5_{n+5,-4}(X;KO)$ to $y \in E^5_{n,0}(X;KO)$ or to $-y$. In particular, y is in the kernel of d_5 in the spectral sequence converging to $KO_*(X)$, and hence also in the kernel of d_5 in the spectral sequence converging to $\Omega^{Spin}_*(X)$. The next possibly non-trivial differential

$$d_9\colon E^9_{n,0}(X;\Omega^{Spin}) \to E^9_{n-9,8}(X;\Omega^{Spin})$$

is trivial due to our assumption $n \le 8$.

This shows that there is a bordism class $x \in \Omega^{Spin}_n(X)$ with $H(x) = y$. Next we want to show that with a careful choice of x we can also arrange for

$p \circ D(x) = 0$. We note that the relation $d_5(z) = y$ in the spectral sequence converging to $KO_*(X)$ implies that $p \circ D(x)$ is zero in

$$E^{\infty}_{n,0}(X; KO) = F_n KO_n(X)/F_{n-1}KO_n(X).$$

Here, $F_k H_*(X)$ denotes the n-the term in the Atiyah-Hirzebruch filtration for $H_*(X)$ (which gives rise to the Atiyah-Hirzebruch spectral sequence), i.e. is the image of $H_*(X^{(k)})$ in $H_*(X)$, where $X^{(k)}$ is the k-skeleton of X.

The relation $d_r(z) = y$ does *not* imply that $p \circ D(x)$ is zero, only that $p \circ D(x)$ lies in the filtration $n-1$ subgroup $F_{n-1}KO_n(X) \subset KO_n(X)$.

We want to show that replacing x by $x' = x - x''$ for a suitable $x'' \in F_{n-1}\Omega^{Spin}_n(X)$ produces an element with the desired properties $H(x') = H(x)$ and $p \circ D(x') = 0$. We note that the first condition is satisfied because H sends elements of

$$F_{n-1}\Omega^{spin}_n(X) = \operatorname{im}\left(\Omega^{spin}_n(X^{(n-1)}) \longrightarrow \Omega^{spin}_n(X)\right)$$

to zero, since $H_n(X^{(n-1)}; \mathbb{Z}) = 0$. To obtain $p \circ D(x') = 0$, we need to be able to choose x'' such that $p \circ D(x'') = p \circ D(x)$; in other words, it suffices to show that the map

$$p \circ D \colon F_{n-1}\Omega^{spin}_n(X) \to F_{n-1}KO_n(X)$$

is surjective for $n \leq 8$. The argument is the following. The map $\Omega^{spin}_q \to KO_q$ is an isomorphism for $0 \leq q < 8$ and surjective for $q = 8$. It follows that the map of spectral sequences

$$E^r_{p,q}(X; \Omega^{Spin}) \longrightarrow E^r_{p,q}(X; KO)$$

is a surjection for $r = 2$, $0 \leq q \leq 8$. Since all the differentials of the domain spectral sequence in the range $0 < q \leq 8$, $n = p + q \leq 8$ are trivial, the above map is also surjective for $r = 3, \ldots, \infty$ in that range. The groups $E^{\infty}_{p,q}(X; \Omega^{spin})$, $n = p + q \leq 8$, $q > 0$, are the associated graded groups for the filtered groups $F_{n-1}\Omega^{spin}_n(X)$ (resp. $F_{n-1}KO_n(X)$). This shows that the map $F_{n-1}\Omega^{spin}_n(X) \to F_{n-1}KO_n(X)$ is surjective for $n \leq 8$ and finishes the proof of Theorem 2.9. □

Proof of Lemma 3.4. Multiplication by the periodicity element $b \in KO_4$ produces a homotopy equivalence of spectra $\Sigma^4 KO \cong KO$, which in turn induces an isomorphism of spectral sequences $E^r_{p,q}(X; KO) \cong E^r_{p,q+4}(X; KO)$. Hence it suffices to prove the corresponding statement for the differential $d_5 \colon E^5_{p,0}(X; KO) \to E^5_{p-5,4}(X; KO)$.

Given an integer k, let $KO\langle k\rangle \to KO$ be the $(k-1)$-*connected cover* of KO. Up to homotopy equivalence, $KO\langle k\rangle$ is characterized by the properties that $\pi_n(KO\langle k\rangle) = 0$ for $n < k$ and that the induced map

$$\pi_n(KO\langle k\rangle) \longrightarrow \pi_n(KO)$$

is an isomorphism for $n \geq k$. The spectrum $KO\langle 0\rangle$ is also known as the *connective real K-theory spectrum* and is usually denoted ko. Given a second integer $l \geq k$, let $KO\langle k, l\rangle$ be the part of the Postnikov tower for KO, whose homotopy groups $\pi_n(KO\langle k, l\rangle)$ are trivial for $n < k$ or $n > l$ and are isomorphic to $\pi_n(KO)$ for $k \leq n \leq l$ (this isomorphism is induced by a map $KO\langle k\rangle \to KO\langle k, l\rangle$). In particular, $KO\langle k, k\rangle$ has only one possibly non-trivial homotopy group and hence can be identified with the Eilenberg-MacLane spectrum $\Sigma^k H\pi_k(KO)$ (here HA for an abelian group A is the Eilenberg-MacLane spectrum characterized by $\pi_0(HA) = A$ and $\pi_n(HA) = 0$ for $n \neq 0$).

We note that the maps $KO\langle 0\rangle \to KO$ and $KO\langle 0\rangle \to KO\langle 0, 4\rangle$ induce isomorphisms of AHSS's in the range $0 \leq q \leq 4$ for $r = 2, 3, 4, 5$ (since we work localized at 3, all rows for $q \neq 0 \mod 4$ are trivial, and hence the first possibly non-trivial differential is d_5). The AHSS-term $E^r_{p,q}(X; KO\langle 0, 4\rangle)$ has only two non-trivial rows and hence degenerates to a long exact sequence

$$\begin{aligned} \ldots \longrightarrow KO\langle 0,4\rangle_n(X) \longrightarrow E^5_{n,0} = H_n(X;\mathbb{Z}) \xrightarrow{d_5} \\ E^5_{n-5,4} = H_{n-5}(X;\mathbb{Z}) \longrightarrow KO\langle 0,4\rangle_{n-1}(X) \longrightarrow \ldots. \end{aligned}$$

This can be identified with the long exact homotopy sequence induced by the Puppe sequence

$$\begin{aligned} KO\langle 4,4\rangle \wedge X \longrightarrow KO\langle 0,4\rangle \wedge X \longrightarrow KO\langle 0,0\rangle \wedge X = H\mathbb{Z} \wedge X \\ \xrightarrow{f\wedge 1} \Sigma KO\langle 4,4\rangle \wedge X = \Sigma^5 H\mathbb{Z} \wedge X \end{aligned}$$

The homotopy class of $f\colon H\mathbb{Z} \to \Sigma^5 H\mathbb{Z}$ can be interpreted as a cohomology class in $H^5(H\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/3$ (we've localized at 3). The generator of $H^5(H\mathbb{Z};\mathbb{Z})$ is given by applying $\delta \circ P^1 \circ \rho$ to the generator of $H^0(H\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}_{(3)}$. This shows that d_5 is a *multiple* of the homology operation (3.5).

To show that it is a *non-trivial* multiple (i.e. either the element itself or its negative, since the group is isomorphic to $\mathbb{Z}/3/Z$), it suffices to show that the differential is non-trivial for some space or spectrum X. We choose

$X = H\mathbb{Z}/3$, and consider the AHSS

$$E^r_{p,q}(H\mathbb{Z}/3; ko) \Longrightarrow ko_{p+q}(H\mathbb{Z}/3) = \pi_{p+q}(ko \wedge H\mathbb{Z}/3) = H_{p+q}(ko; \mathbb{Z}/3).$$

It is well known that $H^*(ko; \mathbb{Z}/3) \cong A/\left(AQ_1 + A\beta\right)$. For this, see [AP, Prop. 2.3]; that proposition applies to the spectrum $X = KO\langle 4\rangle$, which localized at 3 by Bott-periodicity may be identified with $\Sigma^4 ko$. Here A is the mod 3 Steenrod algebra, $\beta \in A_1$ is the Bockstein ($A_n \subset A$ consists of the elements of degree n), and $Q_1 \in A_5$ is the commutator of $P^1 \in A_4$ and β. In particular, since βP^1 and $P^1\beta$ form a basis of A_5, the cohomology group $H^5(ko; \mathbb{Z}/3)$ and hence the group $H_5(ko; \mathbb{Z}/3) = ko_5(H\mathbb{Z}/3)$ is trivial. It follows that the non-trivial elements in

$$E^2_{5,0}(H\mathbb{Z}/3; ko) = H_5(H\mathbb{Z}/3; \mathbb{Z}) \cong \mathbb{Z}/3$$

do not survive to the E^∞-term. For dimensional reasons the only possibly non-trivial differential is d_5. This finishes the proof of Lemma 3.4. □

4. Construction of spaces satisfying the homological condition 2.8

The goal of this section is the construction of spaces satisfying the homological condition 2.8; in particular, we will prove Proposition 2.11, which claims that the classifying space $B\Gamma_n$ satisfies this condition, where Γ_n is the product of two copies of $\mathbb{Z}/3$ and $n-2$ copies of $\mathbb{Z}$. For this calculation it is convenient to pass to cohomology. The following lemma will give a *cohomological* condition which implies the homological condition 2.8.

To state the lemma, we first need some notation. Let $\rho\colon H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}/3)$ be mod 3 reduction and let $\beta\colon H^*(X; \mathbb{Z}/3) \to H^{*+1}(X; \mathbb{Z}/3)$ the mod 3 Bockstein homomorphism, the boundary map of the long exact cohomology sequence induced by the short exact coefficient sequence $\mathbb{Z}/3 \xrightarrow{\times 3} \mathbb{Z}/3^2 \longrightarrow \mathbb{Z}/3$. We recall that β is the composition $\rho\delta$, where $\delta\colon H^*(X; \mathbb{Z}/3) \to H^{*+1}(X; \mathbb{Z})$ is the integral Bockstein. Let Q_1 be the degree 5 element of the mod 3 Steenrod algebra which is the commutator $Q_1 = [P^1, \beta]$.

Lemma 4.1. *Let X be a space and assume there are cohomology classes $\alpha_1, \ldots, \alpha_{n-2} \in H^1(X; \mathbb{Z})$ and $\zeta \in H^2(X; \mathbb{Z}/3)$ such that*

$$\rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \beta Q_1 \zeta \neq 0 \in H^{n+6}(X; \mathbb{Z}/3). \tag{4.2}$$

Then X satisfies the homological condition 2.8.

Proof. The assumption of the lemma implies that there is a homology class $y \in H_{n+6}(X;\mathbb{Z}/3)$ such that the Kronecker product

$$\langle \rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \beta Q_1 \zeta, y \rangle$$

is non-zero. We calculate, using that Q_1 and β are graded derivations and that the cohomological β and Q_1 are dual to the homological versions,

$$\begin{aligned}
0 &\neq \langle \rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \beta Q_1 \zeta, y \rangle \\
&= \langle \rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup Q_1 \zeta, \beta y \rangle \qquad (\text{since } \beta\rho = 0) \\
&= \langle \rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \zeta, Q_1(\beta y) \rangle \qquad (\text{since } Q_1(H^1(X)) = 0) \\
&= \langle \rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \zeta, -\beta P^1(\beta y) \rangle \qquad (\text{since } \beta\beta = 0) \\
&= -\langle \zeta, \rho\left(\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap \delta P^1 \rho(\delta y)\right) \rangle \qquad (\text{since } \beta = \rho\delta)
\end{aligned}$$

This shows that the non-triviality of $\rho(\alpha_1) \cup \cdots \cup \rho(\alpha_{n-2}) \cup \beta Q_1 \zeta$ implies that $\alpha_1 \cap \cdots \cap \alpha_{n-2} \cap \delta P^1 \rho(z)$ for $z = \delta y$ is non-trivial. ☐

Recall the statement of Proposition 2.11:

Proposition 4.3. *For $n \geq 2$, let Γ_n be the Cartesian product of 2 copies of $\mathbb{Z}/3$ and $n-2$ copies of $\mathbb{Z}$. Then the classifying space $B\Gamma_n$ satisfies the homological condition 2.8.*

Proof of Proposition 2.11. We recall that the cohomology ring of

$$B\Gamma_n = \underbrace{S^1 \times \cdots \times S^1}_{n-2} \times B\mathbb{Z}/3 \times B\mathbb{Z}/3$$

is given by

$$H^*(B\Gamma_n;\mathbb{Z}/3) = \Lambda\left(\rho(\alpha_1), \ldots, \rho(\alpha_{n-2}), x_1, x_2\right) \otimes \mathbb{Z}/3[\beta x_1, \beta x_2],$$

where $\alpha_i \in H^1(B\Gamma_n;\mathbb{Z})$ is the pull back of the generator of $H^1(S^1;\mathbb{Z})$ via the projection to the i-th copy of S^1, and x_1 (resp. x_2) is the pull back of the generator of $H^1(B\mathbb{Z}/3;\mathbb{Z}/3)$ via the projection to the first (resp. second) $B\mathbb{Z}/3$-factor of $B\Gamma_n$.

We calculate for $j = 1, 2$

$$Q_1(x_j) = [P^1, \beta]x_j = P^1 \beta x_j - \beta P^1 x_j = (\beta x_j)^3.$$

Since β and Q_1 are graded derivations, it follows that

$$\beta Q_1(x_1 x_2) = \beta\left((\beta x_1)^3 x_2 - x_1 (\beta x_2)^3\right) = (\beta x_1)^3 \beta x_2 - \beta x_1 (\beta x_2)^3 \neq 0.$$

It follows that the α_i, $i = 1, \ldots, n-2$ and $\zeta = x_1 x_2$ satisfies the cohomological condition (4.2). ☐

5. Asphericalization procedures

In this section we prove Addendum 1.6 to our main theorem claiming that the groups π for which we can construct counter examples to the Gromov-Lawson Conjecture 1.1 may be chosen to have classifying spaces which are manifolds which are non-positively (resp. negatively) curved in the Alexandrov sense. This means the following.

Definition 5.1. *A* length space *is a metric space where any two points can be joined by a geodesic. A length space X is called a (locally)* CAT(r)-space *for $r \in \mathbb{R}$, if given a triangle T in a sufficiently small open set of X and a vertex x of T, the distance from x to the opposite edge is not more than the corresponding distance in a comparison triangle in a simply connected manifold with constant curvature equal to r. A* comparison triangle *is a triangle with the same side lengths as the given one.*

We say that X is negatively curved in the Alexandrov sense *(resp.* non-positively curved*), if X is a CAT(r)-space with $r < 0$ (resp. $r = 0$).*

This is a generalization of the classical notion in Riemannian geometry: every complete Riemannian manifold of negative curvature is negatively curved in the Alexandrov sense, and correspondingly for non-positive curvature.

To produce discrete groups π whose classifying space is a manifold which is non-positively curved (resp. negatively curved) in the Alexandrov sense, we use asphericalization procedures due to Davis-Januskiewicz [DJ] and Charney-Davis [CD]. An *asphericalization procedure* assigns to every space X in a certain class of spaces an aspherical space BLX together with a map $BLX \to X$ with certain homological properties. Here the notation BLX is chosen to reflect the fact that BLX serves as a classifying space for its fundamental group, for which the notation LX is used.

Typically, the spaces X considered are simplicial (or cubical) cell complexes and BLX is constructed by replacing each n-cell of X by some 'model space', and then gluing together these model spaces to form BLX according to the same combinatorial patters as the cells of X are glued to form X.

If this is done carefully, as in the procedures described below, one can construct metrics on the result which satisfy appropriate curvature conditions. Some constructions even give yield CAT(0)-spaces BLX that can be given the structure of a smooth manifold, however, it is not at all clear whether the metric on BLX can be chosen to be a smooth Riemannian metric.

Baumslag-Dyer-Heller asphericalization

The construction of Baumslag, Dyer and Heller uses as basic building block an acyclic group, i.e. a non-trivial group with trivial integral homology. For suitable choices of this building block, their construction gives explicit descriptions of LX as elements of $\mathcal{C}$. This description uses the combinatorics of the simplicial complex X (compare also the description of a very similar asphericalization procedure in [Bl]). The map $BLX \to X$ they produce induces an isomorphism in homology.

Davis-Januskiewicz asphericalization

The goal of constructing BLX which is non-positively curved can be achieved using an asphericalization procedure of Davis and Januskiewicz [DJ]. If X is a closed n-manifold, their construction produces a new n-manifold BLX which is non-positively curved in the Alexandrov sense, and whose fundamental group LX belongs to the class $\mathcal{C}$. The price to be paid is that the map $BLX \to X$ unlike in the case of the Baumslag-Dyer-Heller procedure in general does *not* induces an isomorphism in homology (necessarily so: e.g. for $X = S^2$ there is no non-positively curved 2-manifold with the same homology as S^2). However, the map $H_*(BLX;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ is still *surjective*.

This is good enough to prove part (1) of Addendum 1.6, The argument is as follows. Let Y be the manifold with boundary obtained as a 'thickening' of the $n+5$-skeleton of $B\Gamma_n$ (cf. Prop. 2.11) in $\mathbb{R}^N$ with N large. Let X be the boundary of Y. We note that the homology of X is isomorphic to the homology of $B\Gamma_n$ in degrees $\leq n+5$ for N sufficiently large. In particular, since $B\Gamma_n$ satisfies the homology condition 2.8, so does the manifold X. Then the surjectivity of the map $H_*(BLX;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ implies that also BLX satisfies the condition, and hence Corollary 2.10 implies that the Gromov-Lawson Conjecture does not hold for $\pi = LX$.

Charney-Davis asphericalization

This procedure is essentially a strengthening of the Davis-Januskiewicz procedure: from a closed n-manifold X it produces an n-manifold BLX which is *negatively curved* in the Alexandrov sense and a map $BLX \to X$ which induces a surjection on homology. However, LX might not belong to the class $\mathcal{C}$.

With the same argument as above, we can then produce a group π for which the Gromov-Lawson Conjecture doesn't hold, which is the fundamental group of a negatively curved manifold. This proves the second part of Addendum 1.6.

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