APPROXIMATING SPECTRAL INVARIANTS OF HARPER OPERATORS ON GRAPHS II

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ABSTRACT. We study Harper operators and the closely related discrete magnetic Laplacians (DML) on a graph with a free action of a discrete group, as defined by Sunada [Sun]. The spectral density function of the DML is defined using the von Neumann trace associated with the free action of a discrete group on a graph. The main result in this paper states that when the group is amenable, the spectral density function is equal to the integrated density of states of the DML that is defined using either Dirichlet or Neumann boundary conditions. This establishes the main conjecture in [MY]. The result is generalized to other self adjoint operators with finite propagation speed.

1. INTRODUCTION

In [MY] Mathai and Yates examined the spectral properties of the discrete magnetic Laplacian (DML) on a graph with a free action of a discrete group, as defined by Sunada [Sun]. One of the results proved in [MY] is that when the group is amenable and has a finite fundamental domain on the graph, the spectral density function of the DML — defined using the von Neumann trace associated with the discrete group action — can be approximated almost everywhere by the normalized spectral densities of a series of finite approximations of the DML, consisting of restrictions of the DML to a regular exhaustion of the graph. The latter approximation is known in the physics literature as the integrated density of states of the DML, so that the result can be rephrased as saying that the von Neumann spectral density function of the DML is equal to the integrated density of states of the DML almost everywhere. This partial result was known to the experts, cf. [Bel], [Sh].

It was shown in [MY] that in fact the approximation holds at every point whenever the DML is associated with a rational weight function, and it was conjectured there that the assumption on the weight function was irrelevant. Based on the idea of the middle named author, the conjecture is established here in complete generality, and at the same time a much more straightforward argument is presented for the corresponding result in [MY]. Related arguments can be found for example in [DLMSY], [Eck] and [El], and are known to other authors, in particular Wolfgang Lück and Holger Reich.

Hereafter, let X be a combinatorial graph with finite fundamental domain $\mathcal{F}$ under the free action of an amenable discrete group Γ. Each edge $[e]$ of X has associated with it two oriented edges e and $\overline{e}$ with opposite orientation. While denoting the set of oriented edges by $\mathrm{Edge}\,X$, it will be convenient to work with a

subset E^+ of these edges in which each combinatorial edge has exactly one oriented representative. Unless otherwise stated, functions over X will refer to complex-valued functions over the vertices of X.

By Følner's characterization of amenability (see also [Ad]) there must exist a *regular exhaustion* of Γ: a tower of finite subsets $\Lambda_m \subset \Lambda_{m+1}$, $\bigcup_m \Lambda_m = \Gamma$ satisfying

$$(1.1) \qquad \lim_{k\to\infty} \frac{\#\partial_\delta \Lambda_k}{\#\Lambda_k} = 0 \qquad \forall \delta > 0,$$

where $\partial_\delta \Lambda_k$ denotes the δ-neighborhood of the boundary of Λ_k in Γ in the word metric (with respect to some fixed generating set.)

Let $\{X_m\}$ be a sequence of subgraphs of X, with X_m being the largest subgraph of X contained within the translates by Λ_m of the fundamental domain. Then the X_m form a regular exhaustion of the graph, satisfying

$$(1.2) \qquad \lim_{m\to\infty} \frac{\#\mathrm{Vert}\,\partial_\delta X_m}{\#\mathrm{Vert}\,X_m} = 0 \quad \forall \delta > 0,$$

where $\partial_\delta X_m$ refers to the subgraph which is the δ-neighborhood of the boundary of X_m in X in the simplicial metric.

The discrete magnetic Laplacian Δ_σ is an example of an operator with bounded propagation speed, that is there exists a constant c such that for any function f with support $Y \subset X$, the function $\Delta_\sigma f$ has support in the c-neighborhood of Y. The following theorem, proved in Section 3, provides an approximation for the spectral density function of such operators. This was posed (for the DML) as Conjecture 0.1 of [MY], where a proof was obtained for all points λ only when the operator was the DML associated with a rational weight function σ.

Theorem 1.1 (Strong spectral approximation theorem). *Let X be a graph on which there is a free group action by an amenable group Γ, with finite fundamental domain. Let $\{X_m\}_{m=1}^{\infty}$ be the regular exhaustion of X corresponding to a regular exhaustion Λ_m of Γ. Let A be a self-adjoint operator of bounded propagation speed acting on $l^2(X)$, which commutes with an adjoint-closed set of twisted translation operators, and has spectral density function F. Construct finite approximations A_m to A by restricting A to the space of functions supported on X_m, and denote by F_m the normalized spectral density function of A_m,*

$$F_m(\lambda) = \frac{1}{\#\Lambda_m}\#\{eigenvalues \leq \lambda \text{ of } A_m, \text{ counting multiplicity}\}.$$

Then

$$(1.3) \qquad \lim_{m\to\infty} F_m(\lambda) = F(\lambda), \qquad \forall \lambda \in \mathbb{R}.$$

2. The discrete magnetic Laplacian and operators of bounded propagation speed

The DML and the Harper operator are defined by a $U(1)$-valued weight function σ on the edges of the graph X which is *weakly Γ-invariant*. That is, for any edge e, $\sigma(\overline{e}) = \overline{\sigma(e)}$, and for any $\gamma \in \Gamma$ there exists some $U(1)$-valued function s_γ on the vertices of X such that

$$(2.1) \qquad \sigma(\gamma e) = \sigma(e) s_\gamma(\mathfrak{t}(e))\overline{s_\gamma(\mathfrak{o}(e))}$$

where $\mathfrak{o}(e)$ and $\mathfrak{t}(e)$ are the origin and terminus of the edge e respectively. The Harper operator H_σ is then given by

$$(H_\sigma f)(v) := \sum_{\substack{e \in E^+ \\ \mathfrak{t}(e)=v}} \sigma(e)f(\mathfrak{o}(e)) - \sum_{\substack{e \in E^+ \\ \mathfrak{o}(e)=v}} \overline{\sigma(e)}f(\mathfrak{t}(e))$$

and the DML Δ_σ is defined by

$$(\Delta_\sigma f)(v) := \mathcal{O}(v)f(v) - (H_\sigma f)(v),$$

where $\mathcal{O}(v)$ is the valence of the vertex v.

For some s_γ satisfying the weak Γ-invariance property for σ (2.1), one can define the magnetic translation operators T_γ,

$$(2.2) \qquad\qquad (T_\gamma f)(x) = s_\gamma(\gamma^{-1}x)f(\gamma^{-1}x).$$

These form an adjoint-closed set of operators over the space $l^2(X)$ of L^2 functions on the vertices of X, and thus determine a von Neumann algebra $B(l^2(X))^{T_\Gamma}$ consisting of all operators over $l^2(X)$ that commute with the T_γ. This von Neumann algebra has a finite trace Tr_Γ, defined in (2.4). The DML is a self-adjoint operator in this von Neumann algebra, and so one can define its spectral density function F by

$$(2.3) \qquad\qquad F(\lambda) = \mathrm{Tr}_\Gamma E(\lambda)$$

where $E(\lambda) = \chi_{(-\infty,\lambda]}(\Delta_\sigma)$ is the spectral projection of the DML for the interval $(-\infty, \lambda]$.

More generally one can start with an adjoint-closed set of twisted translation operators T_γ of the form (2.2), and then consider a self-adjoint bounded operator A in the von Neumann algebra $B(l^2(X))^{T_\Gamma}$ which — like the DML — has bounded propagation speed.

Definition 2.1. An operator A has *propagation speed bounded by R* if for any function $f \in l^2(X)$, the support of Af is a subset of the R-neighborhood of the support of f in the simplicial metric.

The operator A is *weakly Γ-equivariant* if there exists a Γ-indexed set of functions $t_\gamma \colon \mathrm{Vert}\, X \to U(1)$ defining twisted translation operators T_γ:

$$(T_\gamma f)(x) = t_\gamma(\gamma^{-1}x)f(\gamma^{-1}x),$$

such that A commutes with T_γ and T_γ^* for all $\gamma \in \Gamma$. Then A is an element of the associated von Neumann algebra $B(l^2(X))^{T_\Gamma}$ with trace

$$(2.4) \qquad\qquad \mathrm{Tr}_\Gamma(A) = \sum_{v \in \mathcal{F}} < A\delta_v, \delta_v > .$$

For each of the subgraphs X_m one can define a finite operator A_m by restricting A to those functions with support on the vertices of X_m; explicitly, $A_m = P_m A i_m$ where P_m is the orthogonal projection onto $l^2(X_m)$ and $i_m \colon l^2(X_m) \to l^2(X)$ is the canonical inclusion. We define the normalized spectral density functions of these restricted operators by

$$(2.5) \qquad F_m(\lambda) = \frac{1}{\#\Lambda_m} \#\{\text{eigenvalues} \leq \lambda \text{ of } A_m, \text{ counting multiplicity}\}.$$

Part (i) of Theorem 2.6 of [MY] relates limits of these F_m to the spectral density function F of A when A is the discrete magnetic Laplacian. The argument deployed

though extends trivially to bounded self-adjoint weakly Γ-equivariant operators of bounded propagation speed generally.

Theorem 2.2 (Weak spectral approximation theorem). *Take X to be a graph as above, and let A be a bounded self-adjoint weakly Γ-equivariant operator over $l^2(X)$ of bounded propagation speed, with restrictions A_m to the regular exhaustion X_m of X. Define the following limits of the normalized spectral density functions,*

$$
\begin{aligned}
\overline{F}(\lambda) &= \limsup_{m\to\infty} F_m(\lambda) \\
\underline{F}(\lambda) &= \liminf_{m\to\infty} F_m(\lambda) \\
\overline{F}^+(\lambda) &= \lim_{\delta\to+0} \overline{F}(\lambda+\delta) \\
\underline{F}^+(\lambda) &= \lim_{\delta\to+0} \underline{F}(\lambda+\delta).
\end{aligned}
$$
(2.6)

Then

(2.7) $$ F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda), \qquad \forall \lambda \in \mathbb{R}. $$

Proof. Refer to the proof of Theorem 2.6 of [MY]. $\qquad\square$

As a direct consequence, one has that at every point λ at which F is continuous, that is, for every $\lambda \notin \mathrm{spec}_{\mathrm{point}} A$,

(2.8) $$ \lim_{m\to\infty} F_m(\lambda) = F(\lambda). $$

In particular, equation (2.8) holds for all but at most a countable set of points. In the following section a new argument is presented that extends this result to all λ.

3. The strong spectral approximation result

A consequence of Theorem 2.2 is that if the jump of F at a discontinuity λ can be approximated by the jumps at λ of the piece-wise constant normalized spectral density functions F_m, then the limit of $F_m(\lambda)$ exists and equals $F(\lambda)$.

We first need an elementary lemma, whose proof is omitted.

Lemma 3.1. *Given two sequences $\{a_i\}$ and $\{b_i\}$ with $a_i \le b_i$ for all i, it follows that $\sup_i a_i \le \sup_i b_i$ and $\inf_i a_i \le \inf_i b_i$.*

In particular, if $\{f_i\}$ is a sequence of monotonically increasing functions, then $\liminf_{i\to\infty} f_i$ and $\limsup_{i\to\infty} f_i$ are also monotonically increasing.

The corollary then follows from Theorem 2.2.

Corollary 3.2. *Denote by $D(\lambda)$ the jump in F at λ,*

$$ D(\lambda) = \lim_{\delta\to+0} F(\lambda) - F(\lambda-\delta), $$

and similarly let $D_m(\lambda)$ be the jump in the normalized spectral density functions F_m,

$$ D_m(\lambda) = \lim_{\delta\to+0} F_m(\lambda) - F_m(\lambda-\delta). $$

Then for any $\lambda \in \mathbb{R}$,

(3.1) $$ \lim_{m\to\infty} D_m(\lambda) = D(\lambda) \implies \lim_{m\to\infty} F_m(\lambda) = F(\lambda). $$

Proof. Fix some $\epsilon > 0$ and $\lambda \in \mathbb{R}$, and suppose that $\lim_{m \to \infty} D_m(\lambda) = D(\lambda)$. Then we will show that $F(\lambda) - 3\epsilon < F_m(\lambda) < F(\lambda) + \epsilon$ for all sufficiently large m.

By the right continuity and monotonicity of F, there exists a $\delta > 0$ such that $F(x) < F(\lambda) + \frac{\epsilon}{2}$ for all $\lambda \leq x < \lambda + \delta$. The F_m approach F at all points of continuity of F; as F has at most a countable number of discontinuities there exists some $x_0 \in [\lambda, \lambda + \delta)$ where $\lim_{m \to \infty} F_m(x_0) = F(x_0) < F(\lambda) + \frac{\epsilon}{2}$. So for m greater than some M_1, it follows that $F_m(x_0) < F(\lambda) + \epsilon$. The F_m are monotonically increasing, so

$$(3.2) \qquad \exists M_1 \text{ such that} \quad F_m(\lambda) < F(\lambda) + \epsilon \quad \forall m > M_1.$$

Now $F(\lambda) = D(\lambda) + \lim_{\delta \to +0} F(\lambda - \delta)$, so we can choose d such that $F(\lambda - d) + D(\lambda) > F(\lambda) - \epsilon$. As the F_m are monotonically increasing functions, $\underline{F}$ is also by Lemma 3.1. By Theorem 2.2, $\underline{F}^+(\mu) = F(\mu)$ for all μ. Therefore $\underline{F}(\mu + \delta) > F(\mu)$ for all $\delta > 0$. Letting $\mu = \lambda - d$ and $\delta = \frac{d}{2}$ gives $\underline{F}(\lambda - \frac{d}{2}) > F(\lambda - d)$ and so for sufficiently large m, $F_m(\lambda - \frac{d}{2}) > F(\lambda - d) - \epsilon$.

From $F_m(\lambda) \geq F_m(\lambda - \frac{d}{2}) + D_m(\lambda)$ and $F(\lambda - d) > F(\lambda) - D(\lambda) - \epsilon$,

$$(3.3) \qquad \begin{aligned} \exists M_2 \text{ such that} \quad F_m(\lambda) &> F(\lambda - d) - \epsilon + D_m(\lambda) \\ &> F(\lambda) + D_m(\lambda) - D(\lambda) - 2\epsilon \\ &\geq F(\lambda) - |D_m(\lambda) - D(\lambda)| - 2\epsilon \quad \forall m > M_2. \end{aligned}$$

By supposition, $\lim_{m \to \infty} D_m(\lambda) = D(\lambda)$. In particular we can pick an M greater than M_1 and M_2 such that $|D_m(\lambda) - D(\lambda)| < \epsilon$ for all $m > M$. Therefore

$$\exists M \text{ such that} \quad m > M \implies F(\lambda) + \epsilon > F_m(\lambda) > F(\lambda) - 3\epsilon,$$

and thus

$$\lim_{m \to \infty} D_m(\lambda) = D(\lambda) \implies \lim_{m \to \infty} F_m(\lambda) = F(\lambda).$$

$\square$

The following theorem establishes this approximation of the jumps. Its proof is modelled on the argument in [El].

Theorem 3.3. *The jump $D(\lambda)$ of F at λ is the limit of the jumps of the normalized spectral density functions:*

$$\lim_{m \to \infty} D_m(\lambda) = D(\lambda) \quad \forall \lambda \in \mathbb{R}.$$

Proof. Note that the jumps can be expressed in terms of the dimensions of kernels,

$$(3.4) \qquad \begin{aligned} D(\lambda) &= \dim_\Gamma \ker(A - \lambda), \\ D_m(\lambda) &= \frac{1}{\#\Lambda_m} \dim \ker(A_m - \lambda). \end{aligned}$$

Consider subsets Y_m of X_m consisting of the r-interior of X_m, where A has propagation speed bounded by r.

For each X_m define $\dim_{X_k}$ for a subspace W of $l^2(X)$ by

$$\dim_{X_k} W = \frac{1}{\#\Lambda_k} \sum_{x \in X_k} \langle P_W \delta_x, \delta_x \rangle$$

where P_W is the orthogonal projection onto W. This has the following properties for subspaces W, V of $l^2(X)$:

(1) $W \perp V \implies \dim_{X_k}(W \oplus V) = \dim_{X_k} W + \dim_{X_k} V$,

(2) $W \subset V \implies \dim_{X_k} W \le \dim_{X_k} V$,

(3) $P_W \in B(l^2(X))^{T_\Gamma} \implies \dim_{X_k} W = \dim_\Gamma W$,

(4) $W \subset l^2(X_k) \implies \dim_{X_k} W = \frac{1}{\#\Lambda_k} \dim W$, regarding $l^2(X_k)$ as a subspace of $l^2(X)$.

In particular, $\dim_{X_k} l^2(X_k) = \#\mathcal{F}$ and $\dim_{X_k} l^2(Y_k)$ converges to $\#\mathcal{F}$ as k approaches infinity.

Let $i'_m : l^2(Y_m) \to l^2(X_m)$ be the inclusion with $(i'_m f)(x) = 0$ for all x in $X_m \setminus Y_m$. Let $A'_m : l^2(Y_m) \to l^2(X_m)$ be the operator $A_m i'_m$. For any function f in $l^2(Y_m)$, $(A'_m - \lambda i'_m)f = (A_m - \lambda)(i'_m f)$. Therefore as subspaces of $l^2(X)$,

(3.5)
$$\ker(A'_k - \lambda i'_k) \subset \ker(A_k - \lambda),$$
$$\operatorname{im}(A'_k - \lambda i'_k) \subset \operatorname{im}(A_k - \lambda).$$

Let $D'_k(\lambda) = \frac{1}{\#\Lambda_m} \dim \ker(A'_k - \lambda i'_k)$. Then taking $\dim_{X_k}$,

$$D'_k(\lambda) = \dim_{X_k} \ker(A'_k - \lambda i'_k) \le \dim_{X_k} \ker(A_k - \lambda) = D_k(\lambda),$$
$$\dim_{X_k} \operatorname{im}(A'_k - \lambda i'_k) \le \dim_{X_k} \operatorname{im}(A_k - \lambda).$$

One also has for any m,

$$\lim_{k\to\infty} \dim_{X_k} \ker(A'_k - \lambda i'_k) + \dim_{X_k} \operatorname{im}(A'_k - \lambda i'_k) = \lim_{k\to\infty} \dim_{X_k} l^2(Y_k)$$
$$= \#\mathcal{F} = \dim_{X_m} \ker(A_m - \lambda) + \dim_{X_m} \operatorname{im}(A_m - \lambda).$$

Consequently,

(3.6)
$$\lim_{k\to\infty} D_k(\lambda) - D'_k(\lambda) = 0.$$

Recall that $A_k = P_k A i_k$, where P_k is the projection onto $l^2(X_k)$ and i_k is the canonical inclusion of $l^2(X_k)$ into $l^2(X)$. In terms of A then, $A'_k = P_k A i_k i'_k$. By the bounded propagation speed of A, the support of $A i_k i'_k f$ is contained within X_k for any f in $l^2(Y_k)$ and hence as in (3.5),

$$\ker(A'_k - \lambda i'_k) \subset \ker(A - \lambda),$$
$$\operatorname{im}(A'_k - \lambda i'_k) \subset \operatorname{im}(A - \lambda).$$

The kernel and image of $A - \lambda$ are invariant under the twisted translations T_γ, and so for these subspaces, $\dim_{X_k}$ equals $\dim_\Gamma$. It follows that

$$D'_k(\lambda) = \dim_{X_k} \ker(A'_k - \lambda i'_k) \le \dim_\Gamma \ker(A_k - \lambda) = D(\lambda),$$
$$\dim_{X_k} \operatorname{im}(A'_k - \lambda i'_k) \le \dim_\Gamma \operatorname{im}(A_k - \lambda),$$

and

$$\lim_{k\to\infty} \dim_{X_k} \ker(A'_k - \lambda i'_k) + \dim_{X_k} \operatorname{im}(A'_k - \lambda i'_k)$$
$$= \#\mathcal{F} = \dim_\Gamma \ker(A - \lambda) + \dim_\Gamma \operatorname{im}(A - \lambda).$$

Therefore

$$\lim_{k\to\infty} D'_k(\lambda) = D(\lambda),$$

which together with Equation (3.6) gives the required limit

$$\lim_{k\to\infty} D_k(\lambda) = D(\lambda).$$

$\square$

Theorem 1.1 follows from Theorem 3.3 and Corollary 3.2.

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