The Ore conditions and the lamplighter group (after Warren Dicks)

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Abstract

Let $G = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ be the so called lamplighter group and k a commutative ring. We show that kG does not have a classical ring of quotients (i.e. does not satisfy the Ore conditions). Assume that kG is contained in a ring R in which the element $1 + x$ is invertible, with x a generator of $\mathbb{Z} \subset G$. Then R is not flat over kG. If $k = \mathbb{C}$, this applies in particular to the algebra $\mathcal{U}G$ of unbounded operators affiliated to the group von Neumann algebra of G.

In this note, we give an alternative (and rather shorter and more elementary) proof of results proved in [1]. Moreover, the method gives slightly more general results. This proof is due to Warren Dicks. We feel that it should be published to put the methods of [1] in perspective.

Recall the following definition:

0.1 Definition. A ring R satisfies the *Ore condition* if for any $x, y \in R$ with x a non-zero divisor there are $s, t \in R$ with s a non-zero divisor such that $sy = tx$. Formally, this means that $s^{-1}t = yx^{-1}$, and the condition makes sure that a classical ring of quotients, inverting all non-zero divisors of R, can be constructed.

We study, which group rings satisfy the Ore condition. It is well known that this fails for a non-abelian free group.

On the other hand, abelian groups evidently satisfy the Ore condition. In this note we show that the lamplighter groups (and relatives) do not satisfy it. Note, however, that these groups are 2-step solvable, i.e. close relatives of abelian groups.

More precisely, we prove the following theorem:

0.2 Theorem. *Let* $G = \langle a, x \mid a^d = 1,\ [a, x^l a x^{-1}] = 1;\ l = 1, 2, \ldots \rangle$ *be the wreath product* $\mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}$ *(we use the commutator convention* $[x, y] = xyx^{-1}y^{-1}$*).*

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Let k be a commutative ring with unit. Then there is no non-zero divisor $s \in kG$ such that $s(a-1) = t(x-1)$ for any $t \in kG$. In particular, kG does not satisfy the Ore condition.

Proof. For the last statement note that the projection $G \twoheadrightarrow \mathbb{Z}$ which maps x to 1 shows that $(x-1)$ is not a zero divisor in kG since its homomorphic image in $k\mathbb{Z}$ isn't, either.

The main purpose of this very short note is to prove the first statement. Recall that any presentation $H = \langle S \mid R \rangle$ of a group H gives rise to an exact sequence of left kH-modules

$$\bigoplus_{r\in R} kH \xrightarrow{F} \bigoplus_{s\in S} kH \xrightarrow{\alpha} kH \xrightarrow{\epsilon} k \to 0. \tag{0.3}$$

Here, ϵ is the augmentation, α maps $u\overline{s} \in \bigoplus_{s\in S} kG$ (with $u \in kG$ and $\overline{s}$ the canonical basis element corresponding to the generator $s \in S$) to $u(s-1) \in kH$, and the map F is given by the Fox calculus, i.e. $u\overline{r}$ (where $u \in kH$ and $\overline{r}$ the canonical basis element corresponding to the relation $r \in R$) is mapped to

$$\sum_{s\in S} u\frac{\partial r}{\partial s}\overline{s}.$$

If $r = s_{i_1}^{\epsilon_1} \dots s_{i_n}^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$, then the Fox derivative is defined by

$$\frac{\partial r}{\partial s} := \sum_{k=1}^{n} s_{i_1}^{\epsilon_1} \dots s_{i_{k-1}}^{\epsilon_{k-1}} \frac{\partial s_{i_k}^{\epsilon_k}}{\partial s}.$$

Here $\partial s/\partial s = 1$, $\partial s^{-1}/\partial s = -s^{-1}$ and $\partial t/\partial s = 0$ if $s \neq t \in S$.

The above sequence can be considered as the cellular chain complex (with coefficients k) of the universal covering of the standard presentation CW-complex given by $\langle S \mid R \rangle$. Since this space is 2-connected, its first homology vanishes and its zeroth homology is isomorphic to k (by the augmentation), which implies that the sequence indeed is exact.

Now we specialize to the lamplighter group G. Assume that $u, v \in kG$ with $u(a-1) = v(x-1)$. In other words, $u\overline{a} - v\overline{x}$ are mapped to zero under the boundary map α. Exactness implies that there are $(z_r)_{r\in R}$ such that $F(\sum_{r\in R} z_r\overline{r}) = u\overline{a} - v\overline{x}$. We want to prove that u is a zero divisor. Therefore we are only concerned with the $\overline{a}$ component of $F(\sum_{r\in R} z_r\overline{r})$. This means we first must compute $\partial r/\partial a$ for all the relators in our presentation of G. This is easily done:

$$\frac{\partial a^d}{a} = 1 + a + \dots + a^{d-1} \tag{0.4}$$

$$\frac{\partial[a, x^l a x^{-1}]}{a} = \frac{\partial(ax^l a x^{-l} a^{-1} x^l a^{-1} x^{-l})}{a} \tag{0.5}$$

$$= 1 + ax^l - ax^l a x^{-l} a^{-1} - ax^l a x^{-l} a^{-1} x^l a^{-1}, \tag{0.6}$$

the latter for $l > 0$. Using the fact that $x^l a x^{-l}$ commutes with a for each l, we can simplify

$$\frac{\partial[a, x^l a x^{-1}]}{a} = 1 + ax^l - x^l a x^{-l} - x^l = x^l(x^{-l} a x^l - 1) - (x^l a x^{-l} - 1).$$

Note that a and each of its conjugates $x^l a x^{-l}$ ($l \neq 0$) are of order d, therefore $(1 + a + \cdots a^{d-1})(1 - a) = 0$ and $(x^l a x^{-l} - 1)x^l(1 + a + \cdots + a^{d-1})x^{-l} = 0$. It suffices to show that each finite left kG-linear combination x of $1 + a + \cdots + a^{d-1}$ and of $x^l a x^{-1} - 1$ ($l \neq 0$) is a (right) zero divisor. By finiteness, we can assume that $|l| \leq L$ for each non-zero multiple of $x^l a x^{-1} - 1$ ($l \neq 0$) in x. Define $y := (1 - a) \prod_{-L \leq l \leq -1,\ 1 \leq l \leq L} (1 + x^l a x^{-l} + \cdots + x^l a^{d-1} x^{-l})$. The factors in y all commute with each other. Consequently $xy = 0$. On the other hand, y is in the subring $k[\oplus_{k \in \mathbb{Z}} \mathbb{Z}/d]$ of kG, where the different summands of the subgroup $\oplus_{k \in \mathbb{Z}} \mathbb{Z}/d$ of the wreath product G are generated by the conjugates $x^l a x^{-l}$. This implies that $y \neq 0$, and concludes the proof. (If the characteristic of k is different from d, we can use the map induced from the group homomorphism from $\oplus_{k \in \mathbb{Z}} \mathbb{Z}/d$ to $\mathbb{Z}/d$ which maps $t^l a t^{-l}$ to the trivial element if $l \neq 0$ and a to the generator of $\mathbb{Z}/d$. The image of y is $(1 - a)d^L \neq 0 \in k[\mathbb{Z}/d]$. The general case to check that y is non-trivial is not much harder.) □

0.7 Corollary. *Assume kG of Theorem 0.2 embeds into a ring R such that $(1 - x)$ becomes invertible in R. Then R is not flat over kG.*

If $k \subset \mathbb{C}$, this applies in particular to the division closure DG of kG in the ring $\mathcal{U}G$ of unbounded operators affiliated to the group von Neumann algebra $\mathcal{N}G$, and to $\mathcal{U}G$ itself.

Proof. Tensor the exact sequence (0.3) over kG with R. Then $(a-1)(x-1)^{-1}\overline{x} - \overline{a}$ is in the kernel of $\mathrm{id}_R \otimes \alpha$, but the proof of Theorem 0.2 implies that if $u\overline{x} + v\overline{a}$ is in the image of $\mathrm{id}_R \otimes F$, then v is a zero divisor, in particular $v \neq -1$, therefore the tensored sequence is not exact. □

References

[1] Peter Linnell, Wolfgang Lück, and Thomas Schick. The ore condition, affiliated operator, and the lamplighter group. Preprint 2002, arXiv http://front.math.ucdavis.edu/math.RA/0202027. (document)