2.3.4 The Strategies

We have communication and playing phases, very similar to the construction of Renault. Given any L in the family of non-extendible sets, in the communication phase if a joint plan relative to L has been chosen, then Player One communicates not only this joint plan but also demonstrates to Player Two that the state of nature lies in the set L. The non-extendible property of L allows for this demonstration to be performed. From outside of L Player One will not be able to create the distribution on the signals that should be generated if indeed the state had been in L. Because of this, we do not care about the payoffs to states outside of L from the joint plans relative to the set L. Player One will be punished in the long run if she tries to communicate a joint plan corresponding to a non-extendible L when the chosen state of nature lies outside of L. What counts is how the joint plans equilibria from the various non-extendible sets fit together, and this is delivered by our two main theorems.

2.4 The Fourth Level of Difficulty

For the third level of difficulty we demonstrated the spanning property for the correspondence $\Gamma : \Delta(K) \to \mathbf{R}^K$ such that for any $p \in \Delta(K)$ $\Gamma(p)$ is the set of equilibrium payoffs of the game with initial probability p. We could do so with our two main theorems because we knew that $c\Gamma = \Gamma$.

In Simon, Spież, and Toruńczyk ("Topology in Some Games" 2002c), for the special family $\mathcal{L}$ of non-extendible subsets of K , we called a member L of $\mathcal{L}$ *singular* if for every $j \in J$ the image of $NR(L)$ in $\Delta(S)$ defined by the signaling function $\overline{\Lambda}^j$ is a singleton. Singularity is possible only for maximal members of $\mathcal{L}$. We proved equilibrium existence for the third level of difficulty with the weaker assumption that all maximal members of $\mathcal{L}$ were non-singular.

If $L \in \mathcal{L}$ is singular then the first player does not have the freedom to communicate one of any number of posterior probabilities in the interior of $\Delta(L)$. For every joint plan she is allowed to use at most only one point in the interior of $\Delta(L)$. This changes dramatically the mathematical background to equilibrium existence.

On the other hand the singularity of a set L implies an advantage to Player One when playing a strategy non-revealing with respect to L. Because Player

Two cannot distinguish between any two such non-revealing strategies, Player One is free to choose the strategy that maximizes her payoff for each state in L independently. This results in payoff correspondences that are convex valued and upper-hemi-continuous.

First, let us consider a topological conjecture whose confirmation implies the existence of an equilibrium in the case that K, the whole state space, is a singular member of the family $\mathcal{L}$.

Conjecture 2.4.1: Let $J = [a, b]$ be a non-trivial closed segment in $\mathbf{R}$, K a finite set and $\mathcal{L}$ a family of its non-void subsets such that $K \in \mathcal{L}$ and $\bigcup \mathcal{L} \backslash \{K\} = K$. Suppose for every $L \in \mathcal{L}$ there is given a saturated correspondence $F_L : \Delta(L) \to J^L$ with property $\mathcal{S}$ for $\Delta(L)$ and a closed convex subset U_L of J^K containing the point $(b, b, ..., b)$. Assume additionally that F_K is convex valued and non-empty. Set $\tilde{F}_L := \{(p, y) \in \Delta(L) \times J^K : p \in \Delta(L)$ and $y^L \in F_L(p)\}$ and assume that $\mathrm{im}(\tilde{F}_L) \subset U_L$ for every $L \in \mathcal{L}$. Define the correspondence $G : \Delta(K) \to J^K$ so that its graph is $\{(p, y) \mid y \in \tilde{F}_L(p)$ for some $L \in \mathcal{L}$ with $K \neq L \supset \mathrm{supp}(p),\ y \in \bigcap_{L \in \mathcal{L}} U_L\}$. Define the correspondence F so that its graph is $\{(p, y) \mid y \in F_K(p),\ y \in \cap_{L \in \mathcal{L}} U_L\}$. Consider the set $Y_1 = \text{image } (G)$. We consider a correspondence $\Gamma : \Delta(K) \to Y_1$, a variant of cG, defined by

$$\Gamma^{-1}(y) := \mathrm{co}(G^{-1}(y)) \cup \bigcup_{x \in F^{-1}(y)} \mathrm{co}(x, G^{-1}(y)).$$

Conclusion: the correspondence $F \cup \Gamma$ has the property $\mathcal{S}$ for $\Delta(K)$ (where $F \cup \Gamma$ is the correspondence defined by the union of the graphs of F and Γ).

The key property in the definition of Γ is that for every $y \in Y_1$ we convexify sets in $F^{-1}(y) \cup G^{-1}(y)$ that have at most only member p with the property that p is in $\Delta(K) \backslash (\cup_{L \in \mathcal{L} \backslash \{K\}} \Delta(L))$. The correspondence Γ will have star-shaped inverse images, (which implies also that they are acyclic). We believe that the confirmation of Conjecture 2.4.1 will be proven with a better understanding of how one can map Y_1 to neighborhoods of the graph of Γ, and the proof will be similar to that of Theorem 2.3.1, the gluing theorem.

With a plurality of singular members of $\mathcal{L}$, equilibrium existence for the fourth level of difficulty would be proven with confirmations of both Conjecture 2.4.1 and the following conjecture.

Conjecture 2.4.2: Let $J = [a, b]$ be a non-trivial closed segment in $\mathbf{R}$, K a finite set and $\mathcal{L}$ a non-comparable family of its non-void subsets such that $\bigcup \mathcal{L} = K$. (Non-comparable means $L, N \in \mathcal{L}, L \subset N \Rightarrow L = N$.) Suppose for every $L \in \mathcal{L}$ that there is a saturated correspondence $F_L : \Delta(L) \to J^L$ with property $\mathcal{S}$ for $\Delta(L)$. Set $\tilde{F}_L := \{(p, y) \in \Delta(L) \times J^K : \ p \in \Delta(L) \ \text{ and } \ y^L \in F_L(p)\}$. Define the correspondence $\Gamma : \Delta(K) \to J^K$ so that for every $y \in J^K$

$$\Gamma^{-1}(y) := \bigcup_{x_{L_i} \in \tilde{F}_{L_i}^{-1}(y), \ \forall i \neq j \ \ L_i \neq L_j} \mathrm{co}(x_{L_1}, \ldots, x_{L_m}).$$

($\Gamma^{-1}(y)$ will be a union of simplexes, and therefore in general Γ^{-1} will not be a-cyclic valued.)
Conclusion: the correspondence Γ is non-empty.

Because Conjecture 2.4.2 does not involve convex inverse images, it is closer to the original Borsuk-Ulam Theorem. It suggests the following related conjecture.

Conjecture 2.4.3: Let C be a compact n-dimensional PL manifold with boundary in $\mathbf{R}^n$ and let $F : C \to Y$ be a correspondence into a compact manifold Y of dimension strictly less than n. We assume that F has the spanning property for C. Define the correspondence $G : C \to Y$ by $G(x) := \{y \mid \ \text{there exits a subset } V \subset \partial C \text{ with } |V| \leq 2,\ y \in \cap_{v \in V} F(v) \text{ and } x \in \text{co } (V)\}$.
Conclusion: G also has the spanning property $\mathcal{S}$ for C.

Conjecture 2.4.1 is probably not difficult to solve, however it's solution would not mean much without a reasonable hope of solving either Conjecture 2.4.2 or Conjecture 2.4.3. Probably it would be best to attempt to solve Conjecture 2.4.3 first, as we suspect that its solution would give many hints how to solve Conjecture 2.4.2. And it would be best to attempt to solve Conjecture 2.4.3 first with the additional assumption that F is a continuous function and both C and Y are disks (the context of the Borsuk-Ulam Theorem).

2.5 Equilibrium Stability

Our original concern with games of the first level of difficulty was the existence of equilibria. Later we became interested in whether these equilibria can give to the players strictly more than what they would receive from being punished for deviation. Perhaps there are games for which all equilibria