THE ATIYAH CONJECTURE AND ARTINIAN RINGS

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Abstract. Let G be a group such that its finite subgroups have bounded order, let d denote the lowest common multiple of the orders of the finite subgroups of G, and let K be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Let $\mathcal{U}(G)$ denote the algebra of unbounded operators affiliated to the group von Neumann algebra $\mathcal{N}(G)$, and let $\mathcal{D}(KG,\mathcal{U}(G))$ denote the division closure of KG in $\mathcal{U}(G)$; thus $\mathcal{D}(KG,\mathcal{U}(G))$ is the smallest subring of $\mathcal{U}(G)$ containing KG that is closed under taking inverses. Suppose n is a positive integer, and $\alpha \in \mathrm{M}_n(KG)$. Then α induces a bounded linear map $\alpha\colon \ell^2(G)^n \to \ell^2(G)^n$, and $\ker\alpha$ has a well-defined von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker\alpha)$. This is a nonnegative real number, and one version of the Atiyah conjecture states that $d\dim_{\mathcal{N}(G)}(\ker\alpha) \in \mathbb{Z}$. Assuming this conjecture, we shall prove that if G has no nontrivial finite normal subgroup, then $\mathcal{D}(KG,\mathcal{U}(G))$ is a $d \times d$ matrix ring over a skew field. We shall also consider the case when G has a nontrivial finite normal subgroup, and other subrings of $\mathcal{U}(G)$ that contain KG.

1. Introduction

In this paper $\mathbb{N}$ will denote the positive integers $\{1, 2, \dots\}$, all rings will have a 1, subrings will have the same 1, and if $n \in \mathbb{N}$, then $\mathrm{M}_n(R)$ will indicate the $n \times n$ matrices over the ring R and $\mathrm{GL}_n(R)$ the invertible matrices in $\mathrm{M}_n(R)$. Let G be a group, let $\ell^2(G)$ denote the Hilbert space with orthonormal basis the elements of G, and let $\mathcal{B}(\ell^2(G))$ denote the bounded linear operators on $\ell^2(G)$. Thus we can write elements $a \in \ell^2(G)$ in the form $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ and $\sum_{g\in G} |a_g|^2 < \infty$. Then $\mathbb{C}G$ acts faithfully on the left of $\ell^2(G)$ as bounded linear operators via the left regular representation, so we may consider $\mathbb{C}G$ as a subalgebra of $\mathcal{B}(\ell^2(G))$. The weak closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$ is the group von Neumann algebra $\mathcal{N}(G)$ of G. Also if $n \in \mathbb{N}$, then $\mathrm{M}_n(\mathbb{C}G)$ acts as bounded linear operators on $\ell^2(G)^n$ and the weak closure of this ring in $\mathcal{B}(\ell^2(G)^n)$ is $\mathrm{M}_n(\mathcal{N}(G))$. Let 1 indicate the element of $\ell^2(G)$ which is 1 at the identity of G and zero elsewhere. Then the map $\theta \mapsto \theta 1\colon \mathcal{N}(G) \to \ell^2(G)$ is an injection, so we may regard $\mathcal{N}(G)$ as a subspace of $\ell^2(G)$. We can now define $\mathrm{tr}\colon \mathcal{N}(G) \to \mathbb{C}$ by $\mathrm{tr}(a) = a_1$. For $\alpha \in \mathrm{M}_n(\mathcal{N}(G))$, we can extend this definition by setting $\mathrm{tr}(\alpha) = \sum_{i=1}^n \mathrm{tr}(\alpha_{ii})$, where α_{ij} are the entries of α. A useful property is that if α is a positive operator, then $\mathrm{tr}(\alpha) \geq 0$. Also we can use tr to give any right $\mathcal{N}(G)$-module M a well defined dimension $\dim_{\mathcal{N}(G)} M$, which in general is a non-negative real number or ∞ [10, §6.1]. If e is a projection in $\mathrm{M}_n(\mathcal{N}(G))$, then $\dim_{\mathcal{N}(G)} e\,\mathrm{M}_n(\mathcal{N}(G)) = \mathrm{tr}(e)$. Furthermore if $\alpha \in \mathrm{M}_n(\mathcal{N}(G))$, so α is a Hilbert space map $\ell^2(G)^n \to \ell^2(G)^n$, then since $\ell^2(G)^n$ is a right $\mathcal{N}(G)$-module, $\dim_{\mathcal{N}(G)} \ker\alpha$ is well defined and is equal to

Date: Tue Nov 20 12:41:28 EST 2007.
2000 *Mathematics Subject Classification.* Primary: 16S34; Secondary: 20C07, 22D25, 46L99.
Key words and phrases. Atiyah conjecture, group von Neumann algebra.

$\dim_{\mathcal{N}(G)}\{\beta \in \mathcal{N}(G) \mid \alpha\beta = 0\}$. Finally $\mathcal{N}(G)$ has an involution which sends an operator to its adjoint; if $a = \sum_{g\in G} a_g g$, then $a^* = \sum_{g\in G} \overline{a}_g g^{-1}$, where the bar indicates complex conjugation.

A ring R is called regular, or sometimes von Neumann regular, if for every $x \in R$, there exists an idempotent $e \in R$ with $xR = eR$ [5, Theorem 1.1]. It is called finite, or directly finite, if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Finally a $*$-regular ring R is a regular ring with an involution * with the property that $x \in R$ and $x^*x = 0$ implies $x = 0$. In a $*$-regular ring, given $x \in R$, there is a unique projection e such that $xR = eR$; so $e = e^* = e^2$.

Let $\mathcal{U}(G)$ denote the algebra of unbounded operators on $\ell^2(G)$ affiliated to $\mathcal{N}(G)$ [10, §8]. Then the involution on $\mathcal{N}(G)$ extends to an involution on $\mathcal{U}(G)$, and $\mathcal{U}(G)$ is a finite $*$-regular algebra. Also if M is a right $\mathcal{N}(G)$-module, then $\dim_{\mathcal{N}(G)} M = \dim_{\mathcal{N}(G)} M \otimes_{\mathcal{N}(G)} \mathcal{U}(G)$; in particular $\dim_{\mathcal{N}(G)} e\mathcal{U}(G) = \operatorname{tr}(e)$.

For any subring R of the ring S, we let $\mathcal{D}(R, S)$ denote the division closure of R in S; that is the smallest subring of S containing R that is closed under taking inverses. In the case G is a group and K is a subfield of $\mathbb{C}$, we shall set $\mathcal{D}(KG) = \mathcal{D}(KG, \mathcal{U}(G))$. For any group G, let $\operatorname{lcm}(G)$ indicate the least common multiple of the orders of the finite subgroups of G, and adopt the convention that $\operatorname{lcm}(G) = \infty$ if the orders of the finite subgroups of G are unbounded. One version of the strong Atiyah conjecture states that if G is a group with $\operatorname{lcm}(G) < \infty$, then the L^2-Betti numbers of every closed manifold with fundamental group G lie in the abelian group $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$. This is equivalent to the conjecture that if $n \in \mathbb{N}$, $A \in \mathrm{M}_n(\mathbb{Q}G)$ and $\alpha\colon \ell^2(G)^n \to \ell^2(G)^n$ is the map induced by left multiplication by A, then $\operatorname{lcm}(G) \dim_{\mathcal{N}(G)} \ker\alpha \in \mathbb{Z}$ [9, Lemma 2.2]. In this paper, we shall consider more generally the case when the coefficient ring is a subfield of $\mathbb{C}$.

Definition 1.1. Let G be a group with $\operatorname{lcm}(G) < \infty$, and let K be a subfield of $\mathbb{C}$. We say that the *strong Atiyah conjecture* holds for G over K if

$$\operatorname{lcm}(G) \dim_{\mathcal{N}(G)} \ker\alpha \in \mathbb{Z} \qquad \text{for all } \alpha \in \mathrm{M}_n(KG).$$

This is equivalent to the conjecture that if M is a finitely presented KG-module, then $\operatorname{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{N}(G) \in \mathbb{Z}$ [10, Lemma 10.7]. Obviously if G satisfies the strong Atiyah conjecture over $\mathbb{C}$, then G satisfies the strong Atiyah conjecture over K for all subfields K of $\mathbb{C}$. The strong Atiyah conjecture over $\mathbb{C}$ is known for large classes of groups; for example [7, Theorem 1.5] tells us that it is true if G has a normal free subgroup F such that G/F is an elementary amenable group. If K is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, it is known for even larger classes of groups, for example [4, Theorem 1.4] for groups which are residually torsion-free elementary amenable. The following result is well known; see for example [12, Lemma 3].

Proposition 1.2. *Let G be a torsion-free group (i.e. $\operatorname{lcm}(G) = 1$) and let K be a subfield of $\mathbb{C}$. Then G satisfies the strong Atiyah conjecture over K if and only if $\mathcal{D}(KG)$ is a skew field.*

The purpose of this paper is to generalize Proposition 1.2. We will denote the finite conjugate subgroup of the group G by $\Delta(G)$, and the torsion subgroup of $\Delta(G)$ by $\Delta^+(G)$ (this is a subgroup, compare [11, Lemma 19.3]. We shall prove

Theorem 1.3. *Let G be a group with $d := \operatorname{lcm}(G) < \infty$ and $\Delta^+(G) = 1$, and let K be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Then G satisfies the* *strong Atiyah conjecture over K if and only if $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field.*

It seems plausible that if K is a subfield of $\mathbb{C}$ which is closed under complex conjugation and G is a group with $\operatorname{lcm}(G) < \infty$ which satisfies the Atiyah conjecture over K, then $\mathcal{D}(KG)$ is a semisimple Artinian ring. However we cannot prove this, though we are able to prove a slightly weaker result, and to state this we require the following definition.

Definition 1.4. Let R be a subring of the ring S. The extended division closure, $\mathcal{E}(R,S)$, of R in S is the smallest subring of S containing R with the properties

(a) If $x \in \mathcal{E}(R,S)$ and $x^{-1} \in S$, then $x \in \mathcal{E}(R,S)$.

(b) If $x \in \mathcal{E}(R,S)$ and $xS = eS$ where e is a central idempotent of S, then $e \in \mathcal{E}(R,S)$.

Obviously $\mathcal{E}(R,S) \supseteq \mathcal{D}(R,S)$. Note that if $\{R_i\}$ is a collection of subrings of S satisfying 1.4(a) and 1.4(b) above, then $\bigcap_i R_i$ is also a subring of S satisfying 1.4(a) and 1.4(b), consequently $\mathcal{E}(R,S)$ is a well defined subring of S containing R. Also if G is a group and K is a subfield of $\mathbb{C}$, then we write $\mathcal{E}(KG)$ for $\mathcal{E}(KG,\mathcal{U}(G))$. Observe that, if G is torsion-free and if the strong Atiyah conjecture holds for G over K, then $\mathcal{D}(KG)$ is a division ring, hence $x\mathcal{D}(KG) = \mathcal{D}(KG)$ for every $0 \neq x \in \mathcal{D}(KG)$ and consequently $\mathcal{E}(KG) = \mathcal{D}(KG)$ in this case. We are tempted to conjecture that this is always the case. We hope to show in a later paper that this should follow from a suitable version of the Atiyah conjecture.

We shall prove

Theorem 1.5. *Let G be a group with $\operatorname{lcm}(G) < \infty$, and let K be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Suppose that G satisfies the strong Atiyah conjecture over K. Then $\mathcal{E}(KG)$ is a semisimple Artinian ring.*

Thus in particular if K is a subfield of $\mathbb{C}$ that is closed under complex conjugation and G is a group with $\operatorname{lcm}(G) < \infty$ which satisfies the strong Atiyah conjecture over K, then KG can be embedded in a semisimple Artinian ring. Theorem 1.5 follows immediately from the more general Theorem 2.6 in Section 2.

In Section 3 we will show, somewhat unrelated to the rest of the paper, that KG can be embedded in a least subring of $\mathcal{U}(G)$ that is $*$-regular.

2. Proofs

Let R be a subring of the ring S and let $C = \{e \in S \mid e$ is a central idempotent of S and $eS = rS$ for some $r \in R\}$. Then we define

$$\mathcal{C}(R,S) = \sum_{e \in C} eR,$$

a subring of S. In the case $S = \mathcal{U}(G)$, we write $\mathcal{C}(R)$ for $\mathcal{C}(R,\mathcal{U}(G))$. For each ordinal α, define $\mathcal{E}_\alpha(R,S)$ as follows:

- $\mathcal{E}_0(R,S) = R$;
- $\mathcal{E}_{\alpha+1}(R,S) = \mathcal{D}(\mathcal{C}(\mathcal{E}_\alpha(R,S),S),S)$;
- $\mathcal{E}_\alpha(R,S) = \bigcup_{\beta<\alpha} \mathcal{E}_\beta(R,S)$ if α is a limit ordinal.

Then $\mathcal{E}(R,S) = \bigcup_\alpha \mathcal{E}_\alpha(R,S)$. Also in the case $R = KG$ where G is a group and K is a subfield of $\mathbb{C}$, we shall write $\mathcal{E}_\alpha(KG)$ for $\mathcal{E}_\alpha(KG,\mathcal{U}(G))$. If $A \subseteq \mathbb{R}$, then $\langle A \rangle$ will indicate the additive subgroup of $\mathbb{R}$ generated by A.

Lemma 2.1. *Let G be a group, let R be a subring of $\mathcal{U}(G)$, let $n \in \mathbb{N}$, and let $x \in R$. Suppose that $x\mathcal{U}(G) = e\mathcal{U}(G)$ where e is a central idempotent of $\mathcal{U}(G)$. Then $\langle \dim_{\mathcal{N}(G)} \beta\mathcal{U}(G)^n \mid \beta \in \mathrm{M}_n(R)\rangle = \langle \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G)^n \mid \alpha \in \mathrm{M}_n(R + eR)\rangle$.*

Proof. Set $E = eI_n$, the diagonal matrix in $\mathrm{M}_n(R + eR)$ that has e's on the main diagonal and zeros elsewhere. Then E is a central idempotent in $\mathrm{M}_n(\mathcal{U}(G))$. Obviously

$$\langle \dim_{\mathcal{N}(G)} \beta\mathcal{U}(G)^n \mid \beta \in \mathrm{M}_n(R)\rangle \subseteq \langle \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G)^n \mid \alpha \in \mathrm{M}_n(R + eR)\rangle,$$

so we need to prove the reverse inclusion. Let $\alpha \in \mathrm{M}_n(R+eR)$ and write $\alpha = \beta+E\gamma$ where $\beta, \gamma \in \mathrm{M}_n(R)$. Then we have

$$\dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G)^n = \dim_{\mathcal{N}(G)}(\beta + \gamma)E\mathcal{U}(G)^n + \dim_{\mathcal{N}(G)} \beta(1 - E)\mathcal{U}(G)^n.$$

Since $\dim_{\mathcal{N}(G)} \beta(1 - E)\mathcal{U}(G)^n = \dim_{\mathcal{N}(G)} \beta\mathcal{U}(G)^n - \dim_{\mathcal{N}(G)} \beta E\mathcal{U}(G)^n$, it suffices to prove that

$$\dim_{\mathcal{N}(G)} E\beta\mathcal{U}(G)^n \in \langle \dim_{\mathcal{N}(G)} \delta\mathcal{U}(G)^n \mid \delta \in \mathrm{M}_n(R)\rangle$$

for all $\beta \in \mathrm{M}_n(R)$. But $E\beta\mathcal{U}(G)^n = \beta(xI_n)\mathcal{U}(G)^n$ and the result follows. □

Lemma 2.1 immediately gives the following corollary.

Corollary 2.2. *Let G be a group, let R be a subring of $\mathcal{U}(G)$, and let $n \in \mathbb{N}$. Then $\langle \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G) \mid \alpha \in \mathrm{M}_n(R)\rangle = \langle \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G) \mid \alpha \in \mathrm{M}_n(\mathcal{C}(R))\rangle$.*

Proof. Let $e_1, \dots, e_m$ be central idempotents of $\mathcal{U}(G)$ such that for each i, there exists $\alpha_i \in R$ with $e_i\mathcal{U}(G) = \alpha_i\mathcal{U}(G)$. Then by induction on m, Lemma 2.1 tells us that the result is true if $\alpha \in \mathrm{M}_n(R + e_1R + \cdots + e_mR)$. Since $\mathrm{M}_n(\mathcal{C}(R))$ is the union of $\mathrm{M}_n(R + e_1R + \cdots + e_mR)$, the result is proven. □

Lemma 2.3. *Let R be a subring of the ring S, let $n \in \mathbb{N}$, and let $A \in \mathrm{M}_n(\mathcal{D}(R, S))$. Then there exist $0 \le m \in \mathbb{Z}$ and $X, Y \in \mathrm{GL}_{m+n}(S)$ such that $X \operatorname{diag}(A, I_m)Y \in \mathrm{M}_{m+n}(R)$.*

Proof. This follows from [3, Proposition 7.1.3 and Exercise 7.1.4] and [8, Proposition 3.4]. □

Lemma 2.4. *Let G be a group and let K be a subfield of $\mathbb{C}$. Then $\langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G) \mid x \in \mathrm{M}_n(KG),\ n \in \mathbb{N}\rangle = \langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G) \mid x \in \mathrm{M}_n(\mathcal{E}(KG)),\ n \in \mathbb{N}\rangle$.*

Proof. Obviously

$$\langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G) \mid x \in \mathrm{M}_n(KG),\ n \in \mathbb{N}\rangle \subseteq \langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G) \mid x \in \mathrm{M}_n(\mathcal{E}(KG)),\ n \in \mathbb{N}\rangle.$$

We shall prove the reverse inclusion by transfinite induction. So let $n \in \mathbb{N}$ and $x \in \mathrm{M}_n(\mathcal{E}(KG))$. Then we may choose the least ordinal α such that $x \in \mathrm{M}_n(\mathcal{E}_\alpha(KG))$. Clearly α is not a limit ordinal, and the result is true if $\alpha = 0$, so we may write $\alpha = \beta+1$ for some ordinal β and assume that the result is true for all $y \in \mathrm{M}_n(\mathcal{E}_\beta(KG))$. By Corollary 2.2 the result is true for all $y \in \mathrm{M}_n(\mathcal{C}(\mathcal{E}_\beta(KG)))$ and now the result follows from Lemma 2.3. □

The following result from [6] will be crucial for our work here. Because of this, and because we use a slightly different formulation, we state it here.

Lemma 2.5. [6, Lemma 2] *Let G be a group, let $n \in \mathbb{N}$, and let $\alpha_1, \dots, \alpha_n \in \mathcal{U}(G)$. Then $(\sum_{j=1}^n \alpha_j\alpha_j^*)\mathcal{U}(G) \supseteq \alpha_1\mathcal{U}(G)$.*

Proof. By induction on n and [6, Lemma 2], we see that $(\sum_{j=1}^{n} \alpha_j \alpha_j^*)\mathcal{U}(G) \supseteq \alpha_1\alpha_1^*\mathcal{U}(G)$. The result now follows by applying [6, Lemma 2] in the case $\beta = 0$. □

Theorem 2.6. *Let G be a group and let K be a subfield of $\mathbb{C}$ which is closed under complex conjugation. Suppose there is an $\ell \in \mathbb{N}$ such that $\ell \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G)^n \in \mathbb{Z}$ for all $\alpha \in \mathrm{M}_n(KG)$ and for all $n \in \mathbb{N}$. Then $\mathcal{E}(KG)$ is a semisimple Artinian ring.*

Proof. First observe that Lemma 2.4 tells us that

$$\ell \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{E}(KG). \tag{2.7}$$

Next note that the hypothesis tells us that $\mathcal{E}(KG)$ has at most ℓ primitive central idempotents. Indeed if $e_1, \dots, e_{\ell+1}$ are (nonzero distinct) primitive central idempotents, then $e_ie_j = 0$ for $i \neq j$ and we see that the sum $\bigoplus_{i=1}^{\ell+1} e_i\mathcal{U}(G)$ is direct. But

$$\dim_{\mathcal{N}(G)} \bigoplus_{i=1}^{\ell+1} e_i\mathcal{U}(G) = \sum_{i=1}^{\ell+1} \dim_{\mathcal{N}(G)} e_i\mathcal{U}(G) \geq (\ell+1)/\ell > 1$$

by (2.7), and we have a contradiction. Thus $\mathcal{E}(KG)$ has n primitive central idempotents $e_1, \dots, e_n$ for some $n \in \mathbb{N}$, $n \leq l$. For each i, $1 \leq i \leq n$, choose $0 \neq \alpha_i \in e_i\mathcal{E}(KG)$ such that $\dim_{\mathcal{N}(G)} \alpha_i\mathcal{U}(G)$ is minimal.

Fix $m \in \{1, 2, \dots, n\}$. Since $\ell \dim_{\mathcal{N}(G)} \alpha\mathcal{U}(G) \in \mathbb{Z}$ for all $\alpha \in \mathcal{E}(KG)$ by (2.7), we may choose $g_1, \dots, g_r \in G$ with $\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$ maximal. Note that if $g_{r+1} \in G$, then

$$(\sum_{i=1}^{r+1} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G) \supseteq \sum_{i=1}^{r} g_i\alpha_m\mathcal{U}(G) \supseteq (\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$$

by Lemma 2.5, hence

$$\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r+1} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G) \geq \dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$$

and by maximality of $\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$, we see that

$$\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r+1} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G) = \dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G).$$

It follows that

$$(\sum_{i=1}^{r+1} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G) = (\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$$

and we deduce from Lemma 2.5 that $g\alpha_m\mathcal{U}(G) \subseteq (\sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1})\mathcal{U}(G)$ for all $g \in G$. Let $f \in \mathcal{U}(G)$ be the unique projection such that

$$f\mathcal{U}(G) = \sum_{i=1}^{r} g_i\alpha_m\alpha_m^* g_i^{-1}\mathcal{U}(G).$$

Then $gf\mathcal{U}(G) = \sum gg_i\alpha_m\alpha_m^* g_i^{-1}\mathcal{U}(G) \subseteq \sum gg_i\alpha_m\mathcal{U}(G) \subseteq f\mathcal{U}(G)$ for all $g \in G$, thus $gf\mathcal{U}(G) = f\mathcal{U}(G)$ and we deduce that $gfg^{-1}\mathcal{U}(G) = f\mathcal{U}(G)$ for all $g \in G$. Also gfg^{-1} is also a projection, thus $gfg^{-1} = f$ for all $g \in G$ and we conclude that f is a central projection in $\mathcal{E}(KG)$. Since $f \neq 0$, $f\mathcal{U}(G) \subseteq e_m\mathcal{U}(G)$ and e_m is primitive, we conclude that $f = e_m$ and consequently $\sum_{i=1}^r g_i\alpha_m\mathcal{U}(G) = e_m\mathcal{U}(G)$. By omitting some of the terms in this sum if necessary, we may assume that

$$\sum_{1\le i\le r,\ i\ne s} g_i\alpha_m\mathcal{U}(G) \ne e_m\mathcal{U}(G) \tag{2.8}$$

for all s such that $1 \le s \le r$. We make the following observation:

$$\text{If } 0 \ne x \in g_s\alpha_m\mathcal{E}(KG),\ \text{then } x\mathcal{U}(G) = g_s\alpha_m\mathcal{U}(G), \tag{2.9}$$

where $1 \le s \le r$. This is because $0 \ne x\mathcal{U}(G) \subseteq g_s\alpha_m\mathcal{U}(G)$ and by minimality of $\dim_{\mathcal{N}(G)} \alpha_m\mathcal{U}(G)$, we see that $\dim_{\mathcal{N}(G)} x\mathcal{U}(G) = \dim_{\mathcal{N}(G)} g_s\alpha_m\mathcal{U}(G)$ and consequently $x\mathcal{U}(G) = g_s\alpha_m\mathcal{U}(G)$.

We claim that $e_m\mathcal{E}(KG) = \bigoplus_{i=1}^r g_i\alpha_m\mathcal{E}(KG)$. Set $\sigma = (\sum_{i=1}^r g_i\alpha_m\alpha_m^* g_i^{-1})$. Since $\sigma\mathcal{U}(G) = e_m\mathcal{U}(G)$, we see that

$$(\sigma + (1-e_m))\mathcal{U}(G) \supseteq \sigma\mathcal{U}(G) + (1-e_m)\mathcal{U}(G) = e_m\mathcal{U}(G) + (1-e_m)\mathcal{U}(G) = \mathcal{U}(G).$$

Therefore, $\sigma + 1 - e_m$ is invertible in $\mathcal{U}(G)$ and hence $\sigma + 1 - e_m$ is invertible in $\mathcal{E}(KG)$. Thus

$$e_m\sigma\mathcal{E}(KG) = e_m(\sigma + 1 - e_m)\mathcal{E}(KG) = e_m\mathcal{E}(KG).$$

Moreover, $\sigma\mathcal{E}(KG) \subseteq e_m\mathcal{E}(KG)$ and therefore $e_m\sigma\mathcal{E}(KG) = \sigma\mathcal{E}(KG)$, hence

$$e_m\mathcal{E}(KG) = \sigma\mathcal{E}(KG) = \sum_{i=1}^r g_i\alpha_m\mathcal{E}(KG).$$

If this sum is not direct, then for some s with $1 \le s \le r$, we have $g_s\alpha_m\mathcal{E}(KG) \cap \sum_{i\ne s} g_i\alpha_m\mathcal{E}(KG) \ne 0$, and without loss of generality we may assume that $s = 1$.

So let $0 \ne x \in g_1\alpha_m\mathcal{E}(KG) \cap \sum_{i=2}^r g_i\alpha_m\mathcal{E}(KG)$. Then $0 \ne x\mathcal{U}(G) \subseteq g_1\alpha_m\mathcal{U}(G)$ and (2.9) shows that $x\mathcal{U}(G) = g_1\alpha_m\mathcal{U}(G)$. It follows that $g_1\alpha_m\mathcal{U}(G) \subseteq \sum_{i=2}^r g_i\alpha_m\mathcal{U}(G)$, consequently

$$\sum_{i=2}^r g_i\alpha_m\mathcal{U}(G) = e_m\mathcal{U}(G),$$

which contradicts (2.8) and our claim is established.

Now we show that $g_1\alpha_m\mathcal{E}(KG)$ is an irreducible $\mathcal{E}(KG)$-module. Suppose $0 \ne x \in g_1\alpha_m\mathcal{E}(KG)$. Then $x\mathcal{U}(G) = g_1\alpha_m\mathcal{U}(G)$ by (2.9) and using Lemma 2.5, we see as before that $xx^* + \sum_{i=2}^r g_i\alpha_i\alpha_i^* g_i^{-1} + 1 - e_m$ is a unit in $\mathcal{U}(G)$ and hence is also a unit in $\mathcal{E}(KG)$. This proves that $x\mathcal{E}(KG) = g_1\alpha_m\mathcal{E}(KG)$ and we deduce that $\mathcal{E}(KG)$ is a finite direct sum of irreducible $\mathcal{E}(KG)$-modules. It follows that $\mathcal{E}(KG)$ is a semisimple Artinian ring. □

Proposition 2.10. *Let G be a group with $\Delta(G)$ finite and let K be a subfield of $\mathbb{C}$ with $K = \overline{K}$ which contains all $|\Delta(G)|$-th roots of unity, e.g. $K = \mathbb{C}$ or K is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Then $\mathcal{E}(KG) = \mathcal{D}(KG)$.*

Proof. If e is a central idempotent in $\mathcal{U}(G)$, then $e \in \mathcal{N}(\Delta(G))$, in particular $e \in \mathbb{C}G$, and by our assumption on K even $e \in KG$. The result follows. □

The following result is well known, but we include a proof.

Lemma 2.11. *Let G be a group, let e be a projection in $\mathcal{N}(G)$, and let $\alpha \in \mathcal{N}(G)$. Then $\operatorname{tr}(e\alpha\alpha^*e) \le \operatorname{tr}(\alpha\alpha^*)$.*

Proof. Since $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for all $x, y \in \mathcal{N}(G)$, we see that $\operatorname{tr}(e\alpha\alpha^*(1-e)) = \operatorname{tr}((1-e)\alpha\alpha^* e) = 0$. Therefore $\operatorname{tr}(\alpha\alpha^*) = \operatorname{tr}(e\alpha\alpha^* e) + \operatorname{tr}((1-e)\alpha\alpha^*(1-e))$. Since $\operatorname{tr}((1-e)\alpha\alpha^*(1-e)) \geq 0$, the result follows. □

Lemma 2.12. *Let G be a group, and let (α_n) be a sequence in $\mathcal{N}(G)$ converging strongly to α. Suppose that $\ker\alpha = 0$. Then $\dim_{\mathcal{N}(G)}(\ker\alpha_n)$ converges to 0.*

Proof. By the principle of uniform boundedness, $\|\alpha_n\|$ is bounded. Also by multiplying everything by a unitary operator if necessary, we may assume that α is positive. Then $\alpha_n - \alpha$ converges strongly to 0 and $(\alpha_n - \alpha)^*$ is bounded, hence $(\alpha_n - \alpha)^*(\alpha_n - \alpha)$ converges strongly to 0 and in particular $\lim_{n\to\infty} \operatorname{tr}((\alpha_n - \alpha)^*(\alpha_n - \alpha)) = 0$. Let $e_n \in \mathcal{N}(G)$ denote the projection of $\ell^2(G)$ onto $\ker\alpha_n$. Then $e_n\alpha_n^* = \alpha_n e_n = 0$ and using Lemma 2.11, we obtain

$$\begin{aligned}\operatorname{tr}((\alpha_n - \alpha)^*(\alpha_n - \alpha)) &\geq \operatorname{tr}(e_n(\alpha_n - \alpha)^*(\alpha_n - \alpha)e_n)\\ &= \operatorname{tr}(e_n\alpha^*\alpha e_n) \geq 0.\end{aligned}$$

Thus $\lim_{n\to\infty} \operatorname{tr}(e_n\alpha^*\alpha e_n) = 0$. Suppose by way of contradiction that $\lim_{n\to\infty} \dim_{\mathcal{N}(G)}(\ker\alpha_n) \neq 0$. Then by taking a subsequence if necessary, we may assume that $\dim_{\mathcal{N}(G)}(\ker\alpha_n) > \epsilon$ for some $\epsilon > 0$, for all $n \in \mathbb{N}$. By considering the spectral family associated to $\alpha^*\alpha$ [10, Definition 1.68], there is a closed $\alpha^*\alpha$-invariant $\mathcal{N}(G)$-submodule X of $\ell^2(G)$ and a $\delta > 0$ such that $\dim_{\mathcal{N}(G)}(X) > 1 - \epsilon/2$ and $\alpha^*\alpha > \delta$ on X. Because $\dim_{\mathcal{N}(G)}(X) > 1 - \epsilon/2$ and $\dim_{\mathcal{N}(G)}(\ker\alpha_n) > \epsilon$, we find that $\dim_{\mathcal{N}(G)}(X \cap \ker\alpha_n) > \epsilon/2$ (use [10, Theorem 6.7]). Let f_n denote the projection of $\ell^2(G)$ onto $X \cap \ker\alpha_n$, so $\operatorname{tr} f_n > \epsilon/2$. Since $\alpha^*\alpha > \delta$ on $X \cap \ker\alpha_n$, $f_n\alpha^*\alpha f_n \geq \delta f_n$, and because of positivity of tr we see that $\operatorname{tr}(f_n\alpha^*\alpha f_n) \geq \operatorname{tr}(\delta f_n) > \delta\epsilon/2$. Therefore $\operatorname{tr}(e_n\alpha^*\alpha e_n) > \epsilon\delta/2$ by Lemma 2.11, which shows that $\operatorname{tr}(e_n\alpha^*\alpha e_n)$ does not converge to 0, and the result follows. □

Proposition 2.13. *Let G be a group with $\Delta^+(G) = 1$ and let K be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Assume that $\operatorname{lcm}(G) = d \in \mathbb{N}$ and that G satisfies the strong Atiyah conjecture over K. Then $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field.*

Let p be a prime, let q be the largest power of p that divides d, and let $H \leq G$ with $|H| = q$ (so H is a "Sylow" p-subgroup of G). Set $e = \frac{1}{q}\sum_{h\in H} h$, a projection in $\mathbb{Q}H$. We shall use the center valued von Neumann dimension $\dim^u$, as defined in [10, Definition 9.12]. Since $\Delta^+(G) = 1$, we see that $\dim^u(e\mathcal{U}(G)) = 1/q$ and $\dim^u((1-e)\mathcal{U}(G)) = (q-1)/q$. Therefore by [10, Theorem 9.13(1)],

$$(1-e)\mathcal{U}(G) \cong e\mathcal{U}(G)^{q-1}$$

and we deduce that there exist orthogonal projections $e = e_1, e_2, \ldots, e_q \in \mathcal{U}(G)$ (so $e_ie_j = 0$ for $i \neq j$) such that $\sum_{i=1}^q e_i = 1$ and $e_i\mathcal{U}(G) \cong e\mathcal{U}(G)$ for all i. By [2, Exercise 13.15A, p. 76], there exist similarities (that is self adjoint unitaries) $u_i \in \mathcal{U}(G)$ with $u_1 = 1$ such that $e_i = u_i e u_i$. There is a countable subgroup F of G such that $u_i \in \mathcal{N}(F)$ for all i. By the Kaplansky density theorem [1, Corollary, p. 8] for each i ($1 \leq i \leq q$) there exists a sequence $u_{ij} \in KF$ such that $u_{ij} \to u_i$ as $j \to \infty$ in the strong operator topology in $\mathcal{N}(F)$ with $u_{1j} = 1$ for all j. Set $v_j = \sum_{i=1}^q u_{ij}eu_{ij}$. Then $v_j \to \sum_{i=1}^q e_i = 1$ strongly, hence for $1 \leq i \leq q$,

$$\lim_{j\to\infty} \dim_{\mathcal{N}(F)}(v_j\mathcal{U}(F)) = \lim_{j\to\infty} \dim_{\mathcal{N}(F)}(u_{ij}\mathcal{U}(F)) = 1$$

by Lemma 2.12. Now $\dim_{\mathcal{N}(F)}(x\mathcal{U}(F)) = \dim_{\mathcal{N}(G)}(x\mathcal{U}(G))$ for all $x \in \mathcal{U}(F)$, consequently

$$\lim_{j\to\infty} \dim_{\mathcal{N}(G)} v_j\mathcal{U}(G) = \lim_{j\to\infty} \dim_{\mathcal{N}(G)}(u_{ij}\mathcal{U}(G)) = 1 \qquad \text{for } 1 \le i \le q,$$

and since by assumption G satisfies the strong Atiyah conjecture over K, there exists $n \in \mathbb{N}$ such that $\dim_{\mathcal{N}(G)} v_j\mathcal{U}(G) = \dim_{\mathcal{N}(G)}(u_{ij}\mathcal{U}(G)) = 1$ for $1 \le i \le q$ for all $j \ge n$, in particular $\dim_{\mathcal{N}(G)}(v_n\mathcal{U}(G)) = \dim_{\mathcal{N}(G)}(u_{in}\mathcal{U}(G)) = 1$ and we conclude that v_n and u_{in} $(1 \le i \le q)$ are units in $\mathcal{U}(G)$. Therefore v_n and u_{in} $(1 \le i \le q)$ are units in $\mathcal{D}(KG)$ and we deduce that $\sum_{i=1}^q u_{in}e\mathcal{D}(KG) = \mathcal{D}(KG)$, because

$$D(KG) = v_nD(KG) = \sum_{i=1}^q (u_{in}eu_{in})D(KG) \subseteq \sum_{i=1}^q u_{in}eD(KG) \subseteq D(KG).$$

Since $\dim_{\mathcal{N}(G)} e\mathcal{U}(G) = 1/q$, we see that $\bigoplus_{i=1}^q u_{in}e\mathcal{U}(G) = \mathcal{U}(G)$, a direct sum, and we deduce that

$$\bigoplus_{i=1}^q u_{in}e\mathcal{D}(KG) = \mathcal{D}(KG), \tag{2.14}$$

also a direct sum.

Now suppose that ε is a central idempotent in $\mathcal{C}(\mathcal{D}(KG))$. We want to prove that $\varepsilon = 0$ or 1, so assume otherwise. Now $\varepsilon u_{in}e\mathcal{U}(G) \cong \varepsilon e\mathcal{U}(G)$ for all i, which implies that $\dim_{\mathcal{N}(G)}(\varepsilon\mathcal{U}(G)) = q\dim_{\mathcal{N}(G)}(\varepsilon e\mathcal{U}(G))$. Moreover, because of the Atiyah conjecture, $d\dim_{\mathcal{N}(G)}(\varepsilon e\mathcal{U}(G)) \in \mathbb{Z}$. These two observations together imply that $d\dim_{\mathcal{N}(G)}(\varepsilon\mathcal{U}(G)) \in q\mathbb{Z}$. Since this is true for all primes p, it follows that $\dim_{\mathcal{N}(G)} \varepsilon\mathcal{U}(G) \in \mathbb{Z}$, so 0 and 1 are the only central idempotents of $\mathcal{C}(\mathcal{D}(KG))$.

Summing up, we have shown that $\mathcal{C}(\mathcal{D}(KG))$ contains no nontrivial central idempotents. Using Theorem 2.6, we see that $\mathcal{D}(KG)$ is a semisimple Artinian ring with no nontrivial central idempotents. Thus $\mathcal{D}(KG)$ is an $l \times l$ matrix ring over a division ring for some $l \in \mathbb{N}$. In particular, $\mathcal{D}(KG)$ is the direct sum of l mutually isomorphic $\mathcal{D}(KG)$-submodules, so if f is a primitive idempotent in $\mathcal{D}(KG)$, we see that $\dim_{\mathcal{N}(G)}(f\mathcal{U}(G)) = 1/l$. Furthermore Lemma 2.3 (or Lemma 2.4) show that $l|d$. On the other hand (2.14) shows that $q|l$, for all primes p, so $d|e$ and the result follows.

Proof of Theorem 1.3. If G satisfies the strong Atiyah conjecture over K, then $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field by Proposition 2.13. Conversely suppose $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field F. We need to show that if M is a finitely presented KG-module, then $\operatorname{lcm}(G)\dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{U}(G) \in \mathbb{Z}$. However

$$M \otimes_{KG} \mathcal{U}(G) \cong M \otimes_{KG} \mathrm{M}_d(F) \otimes_{\mathrm{M}_d(F)} \otimes\mathcal{U}(G),$$

consequently $(M \otimes_{KG} \mathcal{U}(G))^d$ is a finitely generated free $\mathcal{U}(G)$-module and we conclude that $\operatorname{lcm}(G)\dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{U}(G) \in \mathbb{Z}$ as required. □

3. Embeddings in ∗-regular rings

There are other closures of group rings KG in $\mathcal{U}(G)$ which may be useful, especially when $\operatorname{lcm}(G) = \infty$. In general the intersection of regular subrings of a von Neumann regular ring is not regular [5, Example 1.10], however we do have the following result.

Proposition 3.1. *Let G be a group and let $\{R_i \mid i \in \mathcal{I}\}$ be a collection of $*$-regular subrings of $\mathcal{U}(G)$. Then $\bigcap_{i\in\mathcal{I}} R_i$ is also a $*$-regular subring of $\mathcal{U}(G)$.*

Proof. Set $S = \bigcap_{i\in\mathcal{I}} R_i$. Obviously S is a $*$-subring of $\mathcal{U}(G)$; we need to show that S is $*$-regular, that is given $s \in S$, there is a projection $e \in S$ such that $sS = eS$. We note that $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ for all i. Indeed if $x \in R_i$ and x is invertible in $\mathcal{U}(G)$, then $xR_i = eR_i$ where e is a projection in R_i, consequently $x\mathcal{U}(G) = e\mathcal{U}(G)$ and since x is invertible in $\mathcal{U}(G)$, we must have $e = 1$ and we deduce that $xR_i = R_i$. Similarly $R_ix = R_i$ and thus x is invertible in R_i, so $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ as asserted. Since R_i is $*$-regular, for each $i \in \mathcal{I}$, there is a projection $e_i \in R_i$ such that $e_iR_i = sR_i$. We now have $e_i\mathcal{U}(G) = e_j\mathcal{U}(G)$ for all i, j and we deduce that $e_i = e_j$ for all $i, j \in \mathcal{I}$, so there exists $f \in S$ such that $f = e_i$ for all i. Since $f\mathcal{U}(G) = s\mathcal{U}(G)$, we see that $fs = s$, so $s \in fS$ and hence $sS \subseteq fS$. Thus the result will be proven if we can show that $ss^*S \supseteq fS$. By Lemma 2.5,

$$(ss^* + (1 - f))\mathcal{U}(G) \supseteq (1 - f)\mathcal{U}(G) + s\mathcal{U}(G) = (1 - f)\mathcal{U}(G) + f\mathcal{U}(G) = \mathcal{U}(G)$$

and we see that $ss^* + 1 - f$ is a unit in $\mathcal{U}(G)$. Let $t \in \mathcal{U}(G)$ be the inverse of $ss^* + 1 - f$, so

$$(ss^* + 1 - f)t = 1. \tag{3.2}$$

Since $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ for all i, we deduce that $t \in R_i$ for all i and hence $t \in S$. Moreover $fs = s$ and $f(1 - f) = 0$, so if we multiply (3.2) on the left by f, we obtain $ss^*t = f$ and the result is proven. □

Thus if K is a subfield of $\mathbb{C}$ that is closed under complex conjugation and G is any group, then there is a least subring of $\mathcal{U}(G)$ containing KG that is $*$-regular.

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