Characteristic classes which don't change under addition of a complex bundle

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Abstract

We compute all characteristic classes for real bundles (with $\mathbb{Z}/2$-coefficients) which don't change when the underlying real bundle of a complex bundle is added.

Consider real vector bundles over a CW-complex X. In this note, we address the question whether there we can define characteristic classes of these bundles which are unchanged when we add a complex vector bundle (or rather the underlying real bundle of a complex vector bundle).

This corresponds to the following problem:

We have the fibration $U/O \to BO \to BU$, where the map $BO \to BU$ is induced from the inclusion and classifies the complexification of the canonical bundle over BO. Looping this fibration gives $\Omega U/O \to \Omega BO \to \Omega BU$. Applying $\Omega BG = G$ and using the part of real Bott periodicity which says $\Omega U/O = BO$, this becomes $BO \to O \to U$. An element in $KO^{-1}(X)$ is given by a map $X \to O$. This element becomes 0 under complexification $KO^{-1}(X) \to K^{-1}(X)$ if and only if the composition $X \to O \to U$ is null-homotopic. In the latter case, a lift $X \to BO$ exists, which is well defined only upto the action of ΩU on the fiber (i.e. for two lifts $f, g\colon X \to BO$ there is a map $H\colon X \to \Omega U$ such that $f(x) = H(x) \cdot g(x)$). By Bott periodicity, $\Omega U = BU$.

The action $BU \times BO \to BO$ is the classifying map of the direct sum of the underlying real bundle for the canonical bundle over BU with the canonical bundle over BO. (To be honest, this pretty much seems to follow from the long exact sequence connecting real and complex K-theory, giving $K^0(X) \to KO^0(X) \to KO^{-1}(X) \to K^{-1}(X)$ in the relevant region (where complex Bott periodicity is used at the very left to identify $K^{-2}(X)$ with $K^0(X)$). However, I haven't formally checked this assertion).

Any classes in the cohomology of BO which have the property that their pullback to $BU \times BO$ under the above "addition map" coincides with the pullback under the projection give rise to characteristic classes which are unchanged when a complex bundle is added. On the other hand, such classes define characteristic classes for elements in the kernel of $KO^{-1}(X) \to K^{-1}(X)$, since they are independent of the lift (this statement also follows directly from the abovementioned exact sequence, the map $K^0(X) \to KO^0(X)$ being given by taking the underlying real bundle of a complex bundle).

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1 Theorem. *The classes in $H^*(BO;\mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, w_3, \dots]$ (the polynomial ring generated by the universal Stiefel Whitney classes) which are unchanged if a complex bundle is added form a polynomial ring $\mathbb{Z}/2[q_1, q_3, q_r, \dots]$ generated by classes $q_{2k-1} \in H^{2k-1}(BO;\mathbb{Z}/2)$. Here $q_{2k-1} \equiv w_{2k-1}$ modulo decomposable elements. More precisely, we obtain q_{2k-1} in the following way:*

$$(q_1 + q_3 + \cdots) = (w_1 + w_3 + \cdots)(1 + w_2 + w_4 + \cdots)^{-1}.$$

In low degrees, we get

$$\begin{aligned} q_1 &= w_1 \\ q_3 &= w_3 + w_2 w_1 \\ q_5 &= w_5 + w_4 w_1 + w_3 w_2 + w_2^2 w_1 \\ q_7 &= w_7 + w_6 w_1 + w_5 w_2 + w_4 w_3 + w_2^2 w_3 \\ q_9 &= w_9 + w_8 w_1 + w_6 w_3 + w_4 w_5 + w_2 w_7 + w_2^2 w_5 \\ &\quad + w_4^2 w_1 + w_2^3 w_3 + w_2^2 w_4 w_1. \end{aligned}$$

Proof. Observe that the product of a class of even degree and a class of odd degree has odd degree, therefore $(q_1 + q_3 + \dots)$ indeed only contains classes of odd degree. Moreover, we invert

$$(1 + w_2 + w_4 + \cdots)^{-1} = (1 + (w_2 + w_4 + \cdots)^1 + (w_2 + w_4 + \cdots)^2 + \cdots),$$

such that in each degree the sum is finite, and module decomposable elements we see that $q_{2k-1} = w_{2k-1}$. The explicit formulas are easily derived.

Given a graded polynomial algebra, the generators can be changed by decomposable elements of the same degree to give a new set of polynomial generators (for the convenience of the reader, we give a proof in Lemma 2. In particular,

$$H^*(BO;\mathbb{Z}/2) = \mathbb{Z}/2[q_1, w_2, q_3, w_4, \dots]$$

and therefore the subalgebra generated by $q_1, q_3, \dots$ is a polynomial algebra on these generators.

Given an arbitrary complex vector bundle E over BO, the odd Stiefel Whitney classes of the underlying real bundle (also denoted E) vanish: $w_{2k-1}(E) = 0$. Let ξ be the canonical real bundle over BO with total Stiefel-Whitney class $w(\xi) = 1 + w_1 + w_2 + \dots$. Then $w(\xi + E) = w(\xi)w(X)$. Consider the total even Stiefel-Whitney classes $w^{ev}(\xi) = 1 + w_2 + w_4 + \dots$, $w^{ev}(E) = w(E)$. Since the odd Stiefel-Whitney classes of E vanish,

$$w^{ev}(\xi + E) = w^{ev}(\xi)w^{ev}(E) = w^{ev}(\xi)w(E).$$

Consequently,

$$w(\xi + E)w^{ev}(\xi + E) = w(\xi)w(E)w^{ev}(\xi)w(E) = w(\xi)w^{ev}(\xi)^{-1},$$

i.e. $w(\xi)w^{ev}(\xi)^{-1}$ is unchanged if a complex bundle is added. Lastly,

$$w \cdot (w^{ev})^{-1} = w^{odd}(w^{ev})^{-1} + w^{ev}(w^{ev})^{-1} = w^{odd}(w^{ev})^{-1} + 1.$$

This implies that our class $(w_1 + w_3 + \cdots)(1 + w_2 + w_4 + \cdots)^{-1}$ has the required property.

This translates to the assertion about the pullback under the classifying map for the direct sum bundle over $BU \times BO$, since

$$H^*(BU \times BO; \mathbb{Z}/2) = H^*(BU; \mathbb{Z}/2) \otimes H^*(BO; \mathbb{Z}/2),$$

and the former is the polynomial algebra

$$H^*(BU; \mathbb{Z}/2) = \mathbb{Z}/2[w_2(E), w_4(E), \dots],$$

where E is the underlying real bundle of the canonical complexs bundle over BU.

The Künneth formula for $BU \times BO$ has the given simple form since we are using coefficients in a field.

It remains to check that all classes with the required invariance property are polynomials in $q_1, q_3, \dots$. To do this, we write

$$H^*(BO; \mathbb{Z}/2) = \mathbb{Z}/2[q_1, w_2, q_3, w_4, \dots]$$

as introduced above. We have to consider two algebra homomorphism

$$\begin{aligned} &\mathbb{Z}/2[q_1, w_2, q_3, w_4, \dots] = H^*(BO; \mathbb{Z}/2) \\ &\quad \to H^*(BO; \mathbb{Z}/2) \otimes H^*(BU; \mathbb{Z}/2) = \mathbb{Z}/2[q_1, w_2, w_2(E), q_3, w_4, w_4(E), \dots], \end{aligned}$$

the first one sending q_{2k-1} to q_{2k-1} and w_{2k} to w_{2k}, the second one sending q_{2k-1} to q_{2k-1}, and w_{2k} to the summand of degree $2k$ in

$$(1 + w_1 + w_2 + \dots)(1 + w_2(E) + w_4(E) + \dots)$$

(note that we already checked that q_{2k-1} is send to itself under this homomorphism). We have to find the kernel of the difference of these two homomorphism.

We have to check that this kernel is exactly the polynomial ring generated by the q_{2k-1}. To see this, compose both maps with the projection

$$\mathbb{Z}/2[q_1, w_2, w_2(E), \dots] \to \mathbb{Z}/2[q_1, w_2(E), q_3, w_e(E), \dots]$$

sending q_{2k-1} to q_{2k-1}, w_{2k} to 0 and $w_{2k}(E)$ to $w_{2k}(E)$.

Then the first map sends q_{2k-1} to itself and w_{2k} to zero, whereas the second one sends q_{2k-1} to itself and w_{2k} to $w_{2k}(E)$, because we have to apply our projection to $(1 + w_1 + w_2 + \dots)(1 + w_2(E) + w_4(E) + \dots)$, giving

$$(1 + w_3 + w_5 + \dots)(1 + w_2(E) + w_4(E) + \dots) \equiv 1 + w_2(E) + w_4(E) + \dots$$

modulo elements of odd order.

Any monomial containing at least one w_{2k} is send to zero under the first composition, whereas the second composition is an isomorphism. The kernel of the difference will therefore only contain polynomials in $q_1, q_3, \dots$. This concludes the proof. □

We used the following lemma:

2 Lemma. *Given a graded polynomial algebra $A = K[x_1, x_2, x_3, \dots]$ with x_i of degree $\phi(i)$. If $v_1, v_2, \dots$ are decomposable elements, and degree v_i is $\phi(i)$, then A is a polynomial algebra on $x_i + v_i$.*

Proof. The map $\alpha\colon A \to A$ sending x_i to x_i+v_i is an isomorphism. We construct an inverse $\beta\colon A \to A$ by induction on the degree of the generators x_i. If x_i is of minimal degree, it is send to x_i (in this case, $v_i = 0$, since there are no elements of lower degree which could give rise to non-trivial products).

If $\beta(x_j)$ is defined for all generators x_j with $deg(x_j) < deg(x_i)$, define $\beta(x_i) := x_i - v_i + \beta(v_i)$, $\beta(v_i)$ being already defined since it is of lower order. Then $\beta \circ \alpha = \mathrm{id}_A$.

It follows that α is injective. But α is also surjective: we prove that all elements x_i are contained in the image, by induction on the degree of x_i. For generators of minimal degree, $x_i + v_i = x_i$, because there are no decomposable elements of this degree, so they are in the image. In the induction step, v_i is decomposable, therefore is a product of elements of lower degree, therefore lies in the image. Since the same is true for $x_i + v_i$, also x_i lies in the image. □