An note on a small part of the homology of $Emd(M, N)$

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Abstract

Let M and N be smooth manifolds and $Emd(M, N)$ the space of embeddings from M to N. If there exist small embeddings of M into N (with image contained in suitable coordinate neighborhoods), we show how elements in the homology of $Emd(M, N)$ can be detected in the homology of N. We consider examples which show that this way one can not get much information about "large" embeddings.

1 The space of small embeddings and its homology

1.1 Definition. Let N be a compact Riemannian manifold of dimension n, $r >$ smaller than the injectivity radius of N (i.e. the exponential map $\exp_x : T_xN \to N$ is a diffeomorphism restricted to the ball of radius r).

Let $Emb(M, N)$ be the space of all smooth embeddings from M to N. We equip it with the weak C^∞-topology (C^∞-convergence on compact subsets of M) (compare [2]). All statements are also valid for the weak C^r-topology, if $r \geq 1$.

Fix $p \in M$. Let $Emb_r(M, p; N)$ the subspace consisting of embeddings $i\colon M \to N$ with image contained in the r-neighborhood of $j(p)$.

Observe that $Emb_r(M, p; N)$ is nonempty if and only if M admits an embedding into $\mathbb{R}^n$. The case $(M, p) = (\mathbb{R}^n, 0)$ will be of particular importance.

1.2 Proposition. *The following observations are well known or immediately verified.*

(1) By definition, we have an embedding

$$Emb_r(M, N) \to Emb(M, N).$$

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(2) *Fix an embedding* $j\colon M_1 \to M_2$. *By pre-composition, this induces continuous maps*

$$Emb(M_2, N) \to Emb(M_1, N) \qquad Emb_r(M_2, N) \to Emb_r(M_1, N).$$

(3) *Fix* $q \in M$. *Evaluation defines continuous maps*

$$ev_q\colon Emb(M, N) \to N;\ i \mapsto i(q),$$

and similarly for $Emb_r(M, p; N)$. *In particular, the adjoint of the identity map gives a map*

$$ev\colon Emb(M, N) \times M \to N;\ (i, q) \mapsto i(q).$$

All of this maps are compatible with pre-composition. We get for an embedding $j\colon (M_1, p) \to (M_2, j(p))$ *e.g. a commutative diagram*

$$\begin{array}{ccc} Emb_r(M_2, j(p); N) & \longrightarrow & Emb_r(M_1, p; N) \\ \Big\downarrow{\scriptstyle ev_{j(q)}} & & \Big\downarrow{\scriptstyle ev_q} \\ N & = & N. \end{array}$$

(4) *Fix* $p \in M$. *Assume* $\dim(M) = m$. *Then we can consider the bundle* $V_m(TN) := Iso(T_pM, TN)$ *over* N, *with fiber over* $x \in N$ *the linear monomorphisms from* T_pM *to* T_xN. *Choosing a basis of* T_pM *induces between isomorphism of* $V_n(TN)$ *and the bundle of* m*-frames in* TN. *Clearly, this is a principal* $Gl_m(\mathbb{R})$*-bundle.*

The above evaluation maps can be lifted to maps

$$\overline{ev_p}\colon Emb(M, N) \to V_m(TN);\ i \mapsto (T_q i),$$

and similarly for $Emb_r(M, N)$.

(5) *Let* $Map(M, N)$ *be the space of all continuous maps from* M *to* N. *Clearly,* ev_p *and* ev *factor through* $Map(M, N)$*:*

$$Emb(M, N) \hookrightarrow Map(M, N) \xrightarrow{ev_p} N.$$

The aim of this note is to get some information about the (co)homology of $Emb(M, N)$ in terms of the (co)homology of N and of its frame bundle $V_m(TN)$. This way, we generalize the results of [4] and give a more algebraic topological proof of his result.

We will simply consider only the maps

$$(ev_p)_*\colon H_*(Emb(M, N)) \to H_*(N),$$
$$(ev_p)^*\colon H^*(N) \to H^*(Emb(M, N))$$

induced by evaluation at $p \in M$. In [4], more generally maps $H^*(N) \to H^*(Emb(M, N))$ was constructed for every continuous linear functional ϕ on $C^\infty(M)$ (using the de Rham point of view). This was done in the following way:

To a differential form $\omega \in \Omega^r(N)$ we assign the differential form ω_ϕ on $Emb(M, N)$. At tangent vectors $v_1, \cdots, v_r$ in $T_i Emb(M, N)$ (i.e. vector field of N along the embedding $i \colon M \to N$) $\omega_\phi(v_1, \dots, v_r)$ has the value

$$\phi(\omega(v_1, \dots, v_r) \circ i).$$

Note that $\omega(v_1, \dots, v_r)$ is a smooth function on $i(M)$.

Obviously, if ϕ is the δ-function at p, this map is the same as the one induced by ev_p. We now show that all these maps are essentially the same.

1.3 Lemma. *Let $\phi_1, \phi_2 \in C^\infty(M)'$ be two linear functional which represent the same element in de Rham homology (defined using currents). Then the two different maps $H^*(N) \to H^*(Emb(M, N))$ defined using ϕ_1 or ϕ_2 coincide.*

Proof. Composition of ev with the slant product gives a map

$$H^k(N) \times H_0(M) \xrightarrow{ev \times \mathrm{id}} H^K(Emb(M, N) \times M) \times H_0(M) \xrightarrow{H}{}^{k} (Emb(M, N)).$$

Using de Rham cohomology (with differential forms) and de Rham homology (with currents), the slant product with a zeroth homology class, represented by a functional ϕ on $C^\infty(M)$, is given exactly in the way described above. This already concludes the proof. □

1.4 Corollary. *If M is connected, then in the situation of Lemma 1.3 $\phi_1(1) = \phi_2(1)$ implies that the induced maps $H^*(N) \to H^*(Emb(M, N))$ coincide, where 1 is the function with constant value 1.*

Proof. If M is connected then $H_0(M; \mathbb{R}) \cong \mathbb{R} \cong H^0(M; \mathbb{R})$. Two cycles represent the same homology class, if they are equal on the generator 1 of the cohomology group. □

From now on, we will always use the functional given by the delta function at p. This way, we can avoid using the slant product, and simply look at the map induced by ev_p. Because of the Lemma 1.3 and its proof, this is no loss of generality, since the general case is a linear combination of such functionals.

The main result of [4] was, that under suitable conditions the map $H^1(N) \to H^1(Emb(M, N))$ is a monomorphism. We will reprove this result and show that its reason is, that the map $\overline{ev}$ has a section.

1.5 Theorem. *Set $M := \mathbb{R}^k$. Then there is a section*

$$f \colon V_k(TN) \to Emb_r(M, 0; N)$$

of $\overline{ev_0}$. In particular, the map induced by ev_0 in homology is split surjective, and in cohomology it is split injective (with arbitrary coefficients).

Proof. Recall that $V_k(TN)$ in this particular case is equal to k-frame bundle of TN, since a linear monomorphism of $T_0\mathbb{R}^k = \mathbb{R}^k$ to $T_x N$ is the same as a frame in $T_x N$. To such a frame we associate the embedding

$$(\mathbb{R}^k, 0) \cong (B_r(0) \subset \mathbb{R}^n, 0) \to N$$

given by the exponential map. Our choice of $r > 0$ implies that this is indeed an embedding. The exponential map depends smoothly on the basepoint, and is defined in such a way that this map is a splitting of ev_0. □

1.6 Remark. The statement of Theorem 1.5 remains true if N is not compact. In the proof, one simply has to replace the constant r by a smooth function $r(x) > 0$, such that $r(x)$ is smaller than the injectivity radius of N at x. Of course, the definition of $Emb_r(M, N)$ has to be modified accordingly.

Next, we want to get some information about the homology of $V_k(TN)$, and in particular compare it with the homology of N. This is done in a standard way, using the fiber bundle $V_k(\mathbb{R}^n) \to V_k(TN) \to N$, compare [6, 3]. Since $V_k(\mathbb{R}^n)$ is $n-k-1$-connected, the bundle projection $V_k(TN) \to N$ is $n-k$-connected. In particular, it induces an isomorphism in H_i and H^i for $i < n-k$, and an epimorphism in H_{n-k} as well as a monomorphism in H^{n-k}.

An additional refinement is possible if $k = n$ and N is an oriented manifold. In this case we can look at the subspace $V_n^o(TN)$ of $V_n(TN)$ of oriented frames (corresponding to the subspace $Emb_r^o(\mathbb{R}^n, N)$ of orientation preserving embeddings). The fiber of $V_n^o(TN)$ is isomorphic to $Gl_n^+(\mathbb{R})$, the group of orientation preserving linear isomorphisms of $\mathbb{R}^n$. This group is homotopy equivalent to $SO(n, \mathbb{R})$. In particular, it is connected. Consequently, the above argument implies that the map $V_n^o(TN) \to N$ induces an epimorphism in H_1 and a monomorphism in H^1.

Combining this information with the information we get from the split of $\overline{ev_0}\colon Emb_r(\mathbb{R}^k, 0; N) \to V_k(TN)$, we see that $ev_0\colon Emb_r(\mathbb{R}^k, 0; N) \to N$ induces a epimorphism on H_i, and a monomorphism on H^i, for $i \leq n-k$.

In the special case $k = n$ and N orientable, the statement is true for $i \leq 1$.

The latter result follows from the commutative diagram

$$\begin{array}{ccccc}
Emb_r^o(\mathbb{R}^n, 0; N) & \xrightarrow{\overline{ev_0}} & V_n^o(TN) & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow = \\
Emb_r(\mathbb{R}^n, 0; N) & \xrightarrow{\overline{ev_0}} & V_n(TN) & \longrightarrow & N.
\end{array}$$

Assume $V_k(TN) \to N$ has a split, i.e. TN has a trivial k-dimensional subbundle. Then, the above considerations imply that $ev_0\colon Emb_r(\mathbb{R}^k, 0; N) \to N$ induces a split epimorphism on H^i, and a monomorphism on H^i, for all $i \in \mathbb{N}$. In particular, this is true for $k = n$ if N is parallelizable.

Assume M is an m-dimensional manifold which admits an embedding $j\colon M \hookrightarrow \mathbb{R}^k$. Fix a basepoint $p \in M$ and assume $j(p) = 0$ (this is no loss of generality). This induces a particular monomorphism $T_pj\colon T_pM \to T_0\mathbb{R}^k = \mathbb{R}^k$, which induces by pre-composition a map $V_k(TN) \to V_m(TN)$ (here it is most convenient, to define V_k using embeddings of $\mathbb{R}^k$, and V_m using embeddings of T_pM). This gives a commutative diagram

$$\begin{array}{ccccc}
Emb_r(\mathbb{R}^k, 0; N) & \xrightarrow{\overline{ev_0}} & V_k(TN) & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow = \\
Emb_r(M, p; N) & \xrightarrow{\overline{ev_p}} & V_m(TN) & \longrightarrow & N.
\end{array}$$

Note that the splitting of $\overline{ev_0}$ does not provide us with a splitting of $\overline{ev_p}$. However, the map immediately implies that $ev_p\colon Emb_r(M, p; N) \to N$ induces an epimorphism on H_i in the same range where $ev_0\colon Emb_r(\mathbb{R}^k, p; N)$ does (and similarly for the dual statement in cohomology).

Last, observe that ev_p factors through $Emb(M, N)$. Therefore, we have proved the following theorem.

1.7 Theorem. *Assume* M *admits an embedding into* $\mathbb{R}^k$. *Then we have an epimorphism*

$$H_i(Emb(M, N)) \to H_i(N) \qquad \text{for } i \leq n - k.$$

If $k = n$ *and* N *is orientable, we get an epimorphism*

$$H_1(Emb(M, N)) \to H_1(N).$$

The dual maps in cohomology are monomorphisms in the same range.

If TN *has a* k*-dimensional trivial sub-bundle, the above result holds for all* $i \in \mathbb{N}$. *In particular, this is the case if* $k = n$ *and* N *is parallelizable.*

In the same way, we get further information (in the range $i \leq l$) if the restriction of TN splits of a k-dimensional trivial sub-bundle. This condition can be replaced by assuming that the restriction of TN to any l-dimensional immersed submanifold splits of a k-dimensional trivial bundle, as long as we work with rational or real coefficients.

1.1 Reverse implications for TN

We have just seen that N being parallelizable implies that $ev_p : Emb_r(M, p; N) \to N$ induces a split surjection in homology, and a split injection in cohomology.

On the other hand, note that the pull back of TN to $V_m(TN)$ canonically splits off a trivial m-dimensional sub-bundle $V_M(TN) \times T_pM$, and the same follows then for $(ev_p)^*TN$ on $Emb(M, p; N)$. Consequently, if ev_p induces an injection in homology (in a given range of degrees) the characteristic cohomology classes of TN look (in the given range) like the characteristic classes of a bundle splitting off a trivial m-dimensional sub-bundle. In particular, the Pontryagin classes $p_i(TN)$ vanish for $4i > n-m$ (in the given range), and, if N is orientable, the Euler class $e(TN)$ vanishes if $m > 0$ and n is in the given range.

1.8 Remark. Note that we information about $H_*(Emb(M, N))$ all comes from the subspace of small embeddings $Emb_r(M, N)$. In particular, there might be many components of $Emb(M, N)$ we don't get any information about. We will discuss this a little bit in the next section.

1.9 Remark. Let $Emb^0(M, N)$ be the space of embeddings with the C^0-topology. The identity map is continuous from $Emb(M, N)$ with the C^∞-topology to $Emb^0(M, N)$ and gives a factorization of the evaluation map. Therefore, the statements of Theorem 1.5 remain true with $Emb(M, N)$ replaced by $Emb^0(M, N)$.

It is conceivable that some of the statements remain true with the space of smooth embeddings replaced by the space of continuous embeddings. However, the proof given here relies on the smooth structure.

1.10 Remark. We can deal with homology as well as cohomology, and that not only real, but integer coefficients are allowed, in contrast to the treatment of [4].

1.2 Fibrations

1.11 Theorem. *The evaluation map*

$$ev_p : Emb_r(M, p; N) \to N$$

is a locally trivial fiber bundle with typical fiber the space $Emb(M, \mathbb{R}^n)$ *of embeddings of* $i\colon M \to \mathbb{R}^n$ *with* $i(p) = 0$.

Proof. Fix $x_0 \in N$ and $\epsilon > 0$ small enough. Choose a Riemannian trivialization τ of $TN|_{B_\epsilon(x_0)}$ (it could e.g. be given by geodesic coordinates at x_0).

Now we define a smooth family of diffeomorphisms $\phi_x\colon B_r(x) \to B_r(x_0)$, where ϕ_x is the composition of $\exp_x^{-1}$ with the map between T_xN and $T_{x_0}N$ given by the trivialization and then with $\exp_{x_0}$.

Set $Emb_r(M, p; N, x_0) := \{i \in Emb_r(M, p; N) \mid i(p) = x_0\}$. We now get a trivialization

$$\begin{aligned} Emb_r(M, p; N)|_{B_r(x_0)} &\to B_r(x_0) \times Emb_r(M, p; N, x_0) \\ i &\mapsto (i(p), \phi_{i(p)} \circ i). \end{aligned}$$

□

1.12 Question and Remark. Is it true that also $\overline{ev_p}\colon Emb_r(M, p; N) \to V_m(N)$ is a locally trivial fiber bundle?

The method of Theorem 1.11 applies to prove this result if we restrict to the subspaces of "metric" embeddings $i\colon M \to B_r(i(p))$ such that $Ti\colon T_pM \to T_{i(p)}N$ is an isometric embedding (for a fixed, but arbitrary Riemannian metric on T_pM).

A slight modification implies that $Emb_r(M, p; N) \to V_m(N)$ is a local fibration (i.e. has locally the homotopy lifting property): we have to lift a homotopy $X \times [0, 1] \to U \subset V_m(N)$ to $Emb_r(M, p; N)$. When we try to trivialize $Emb_r(M, p; N) \to V_m(N)$ over a neighborhood U of an m-frame α_0 as in the proof of Theorem 1.11, we have to use an additional (intermediate) linear isomorphism $B_\alpha\colon T_{x_0}N \to T_{x_0}N$ which maps the (image of the) m-frame $\alpha \in U$ we start with to the m-frame α_0. If (the image of) α is close to α_0 we define this linear map by mapping the l-th vector of α to the l-th vector of α_0, and doing nothing on the complement of the span of α (if α is close enough to α_0, this is an isomorphism).

The problem is that this map need not be an isometry, and consequently that we don't create a diffeomorphism $B_r(x) \to B_r(x_0)$.

Let

$$v(\alpha, \beta) := \sup\{\left|B_\alpha^{-1} B_\beta v\right| \mid 0 \neq v \in T_{x_0}N\}$$

be the maximal distortion for the corresponding "linear part" of the map from the fiber over $\beta \in U$ to the fiber over $\alpha \in U$.

To lift the homotopy $h\colon X \times [0, 1] \to U$, we use the "trivializations", but compose them with (nonlinear) "contractions" of $T_{x_0}N$ which are the identity near zero, but counteract the distortion $v(h(x), x)$ which would prevent our lift from living in $Emb_r(M, p; N)$.

Since N and $V_m(N)$ are paracompact and Hausdorff, any local fibration, in particular any locally trivial fiber bundle, are fibrations, and we can apply the long exact homotopy sequence of the fibration. Consequently, we get long exact sequences

$$\cdots \to \pi_k(Emb_r(M, p; \mathbb{R}^n, 0)) \to \pi_k(Emb_r(M, p; N)) \to \pi_k(N) \to \cdots$$
$$\cdots \to \pi_k(Emb_r^f(M, p; \mathbb{R}^n, 0)) \to \pi_k(Emb_r(M, p; N)) \to \pi_k(V_m(TN)) \to \cdots$$

where $Emb_r^f(M, p; \mathbb{R}^n, 0)$ is the space of embeddings with prescribed differential at p which we encountered earlier. Observe that the $\overline{ev_p}\colon Emb_r(M, p; N) \to V_m(N)$ has a split, such that the second long exact sequence splits into short exact sequences

$$0 \to \pi_k(Emb_r^f(M, p; \mathbb{R}^n, 0)) \to \pi_k(Emb_r(M, p; N)) \to \pi_k(V_m(TN)) \to 0.$$

Note, however, that the space of embeddings into $\mathbb{R}^n$ is a very complicated space. But we can see at last that we can split off the homotopy groups of $V_m(TN)$, which are closely related to the homotopy groups of N.

2 Non-small embeddings

Here, we want to show that the evaluation map does not give much information about the homology of components of $Emb(M, N)$ where the embeddings are not contained in small balls.

We use the simple observation that $ev_p\colon Emb(M, N) \to N$ factors through the space $Map(M, N)$ of all continuous maps from M to N. Consequently, if $(ev_p)_*\colon H_*(Map_i(M, N)) \to H_*(M)$ is not surjective, neither can the map from $H_*(Emb_i(M, N))$ be surjective, where $Emb_i(M, N)$ is the component of $i\colon M \hookrightarrow N$ in $Emb(M, N)$, and similarly for $Map_i(M, N)$.

Note that $ev_p\colon Map(M, N) \to N$ is always a fibration (as long as the inclusion $p \hookrightarrow M$ is a cofibration, which is the case for manifolds), compare [5, 2.8.2].

Now, we specialize to $M = S^1$ and N a connected orientable surface of genus > 1. We claim that $Map(S^1, N)$ has many components which contain embeddings $i\colon S^1 \to N$, such that $(ev_p)_*\colon H_1(Map_i(S^1, N)) \to H_1(N)$ is not surjective.

To see this, we examine the long exact sequence in homotopy of the fibration $\Omega_{x_0}N \to Map(S^1, N) \to N$, where $\Omega_{x_0}N$ is the space of loops in N based at x_0. Recall that N is aspherical, which implies that $\pi_k(\Omega_{x_0}N) = 0$ for $k > 0$, and $\pi_0(\Omega_{x_0}N) = \pi_1(N, x_0)$ (the isomorphism is canonical, mapping $[\gamma] \in \pi_1(N, x_0)$ to $[\gamma] \in \pi_0(\Omega_{x_0}N)$).

As the relevant part of the long exact sequence of this fibration we obtain therefore for any basepoint $i \in \Omega_{x_0}N$

$$0 \to \pi_1(Map(S^1, N), i) \xrightarrow{(ev_p)_*} \pi_1(N, x_0) \xrightarrow{\delta} \pi_1(N, x_0) \to \pi_0(Map(S^1, N)) \to \{*\}$$

($\pi_0(N)$ consists of a single point since N is connected).

Usually, this sequence is considered when i is the constant loop. However, it is valid for any loop i, and we want to investigate it when i represents some interesting elements in $\pi_1(N, x_0)$.

Obviously, to understand the map $(ev_p)_*$, we have to understand the boundary map δ. In this situation, δ maps $\alpha \in \pi_1(N, x_0)$ to $\alpha^{-1}[i]\alpha \in \pi_1(N, x_0)$ (this follows easily from the definition, where δ is induced from the effect of fiber transport to the basepoint i).

This sequence being exact at the points involving π_0 (which is only a set, not a group) simply means that the image of $(ev_p)_*$ is exactly the subgroup which is mapped to the basepoint $[i]$ under the map δ. Our calculation shows that this is exactly the centralizer of $[i]$.

It is known [1, Section 1 and Theorem 3] that the centralizer of each non-trivial element of $\pi_1(N)$ is cyclic. If $[i] \in \pi_1(N)$ is primitive (i.e. not a proper power) it therefore is the cyclic subgroup generated by $[i]$.

In particular, the map induced by ev_p is far from being surjective, here.

Passing to the abelianization, we get the map

$$(ev_p)_* \colon H_1(Map_i(S^1, N)) \to H_1(N).$$

Note that we obtain H_1 of the component of $Map(S^1, N)$ containing i. Our considerations show that the image is contained in the image of a cyclic group in the abelianization. Therefore, $(ev_p)_*$ is not surjective.

2.1 Question. When is $ev_p : Emb(M, N) \to N$ a fibration?

References

[1] Gilbert Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[2] Morris W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[3] John W. Milnor and James D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[4] Olaf Müller. A note on $H^1(Emb(M, N))$. arXiv: http://de.arXiv.org/abs/math.AT/0202049, 2002.

[5] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York, 1981. Corrected reprint.

[6] Norman Steenrod. *The topology of fibre bundles.* Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.