

# Index Theory and the Baum-Connes conjecture

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These notes are based on lectures on index theory, topology, and operator algebras at the “School on High Dimensional Manifold Theory” at the ICTP in Trieste, and at the Seminari di Geometria 2002 in Bologna. We describe how techniques coming from the theory of operator algebras, in particular  $C^*$ -algebras, can be used to study manifolds. Operator algebras are extensively studied in their own right. We will focus on the basic definitions and properties, and on their relevance to the geometry and topology of manifolds. The link between topology and analysis is provided by index theorems. Starting with the classical Atiyah-Singer index theorem, we will explain several index theorems in detail.

Our point of view will be in particular, that an index lives in a canonical way in the  $K$ -theory of a certain  $C^*$ -algebra. The geometrical context will determine, which  $C^*$ -algebra to use.

A central pillar of work in the theory of  $C^*$ -algebras is the Baum-Connes conjecture. Nevertheless, it has important direct applications to the topology of manifolds, it implies e.g. the Novikov conjecture. We will explain the Baum-Connes conjecture and put it into our context.

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## 1 Index theory

The Atiyah-Singer index theorem is one of the great achievements of modern mathematics. It gives a formula for the index of a differential operator (the index is by definition the dimension of the space of its solutions minus the dimension of the solution space for its adjoint operator) in terms only of topological data associated to the operator and the underlying space. There are many good treatments of this subject available, apart from the original literature (most found in [1]). Much more detailed than the present notes can be, because of constraints of length and time, are e.g. [23, 5, 16].

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## 1.1 Elliptic operators and their index

We quickly review what type of operators we are looking at. This will also fix the notation.

**1.1 Definition.** Let  $M$  be a smooth manifold of dimension  $m$ ;  $E, F$  smooth (complex) vector bundles on  $M$ . A *differential operator* (of order  $d$ ) from  $E$  to  $F$  is a  $\mathbb{C}$ -linear map from the space of smooth sections  $C^\infty(E)$  of  $E$  to the space of smooth sections of  $F$ :

$$D: C^\infty(E) \rightarrow C^\infty(F),$$

such that in local coordinates and with local trivializations of the bundles it can be written in the form

$$D = \sum_{|\alpha| \leq d} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here  $A_\alpha(x)$  is a matrix of smooth complex valued functions,  $\alpha = (\alpha_1, \dots, \alpha_m)$  is an  $m$ -tuple of non-negative integers and  $|\alpha| = \alpha_1 + \dots + \alpha_m$ .  $\partial^{|\alpha|}/\partial x^\alpha$  is an abbreviation for  $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}$ . We require that  $A_\alpha(x) \neq 0$  for some  $\alpha$  with  $|\alpha| = d$  (else, the operator is of order strictly smaller than  $d$ ).

Let  $\pi: T^*M \rightarrow M$  be the bundle projection of the cotangent bundle of  $M$ . We get pull-backs  $\pi^*E$  and  $\pi^*F$  of the bundles  $E$  and  $F$ , respectively, to  $T^*M$ .

The *symbol*  $\sigma(D)$  of the differential operator  $D$  is the section of the bundle  $\text{Hom}(\pi^*E, \pi^*F)$  on  $T^*M$  defined as follows:

In the above local coordinates, using  $\xi = (\xi_1, \dots, \xi_m)$  as coordinate for the cotangent vectors in  $T^*M$ , in the fiber of  $(x, \xi)$ , the symbol  $\sigma(D)$  is given by multiplication with

$$\sum_{|\alpha|=m} A_\alpha(x) \xi^\alpha.$$

Here  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_m^{\alpha_m}$ .

The operator  $D$  is called *elliptic*, if  $\sigma(D)_{(x,\xi)}: \pi^*E_{(x,\xi)} \rightarrow \pi^*F_{(x,\xi)}$  is invertible outside the zero section of  $T^*M$ , i.e. in each fiber over  $(x, \xi) \in T^*M$  with  $\xi \neq 0$ . Observe that elliptic operators can only exist if the fiber dimensions of  $E$  and  $F$  coincide.

In other words, the symbol of an elliptic operator gives us two vector bundles over  $T^*M$ , namely  $\pi^*E$  and  $\pi^*F$ , together with a choice of an isomorphism of the fibers of these two bundles outside the zero section. If  $M$  is compact, this gives an element of the relative  $K$ -theory group  $K^0(DT^*M, ST^*M)$ , where  $DT^*M$  and  $ST^*M$  are the disc bundle and sphere bundle of  $T^*M$ , respectively (with respect to some arbitrary Riemannian metric).

Recall the following definition:

**1.2 Definition.** Let  $X$  be a compact topological space. We define the  $K$ -theory of  $X$ ,  $K^0(X)$ , to be the Grothendieck group of (isomorphism classes of) complex vector bundles over  $X$  (with finite fiber dimension). More precisely,  $K^0(X)$  consists of equivalence classes of pairs  $(E, F)$  of (isomorphism classes of) vector bundles over  $X$ , where  $(E, F) \sim (E', F')$  if and only if there exists another vector bundle  $G$  on  $X$  such that  $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ . One often writes  $[E] - [F]$  for the element of  $K^0(X)$  represented by  $(E, F)$ .

Let  $Y$  now be a closed subspace of  $X$ . The *relative K-theory*  $K^0(X, Y)$  is given by equivalence classes of triples  $(E, F, \phi)$ , where  $E$  and  $F$  are complex vector bundles over  $X$ , and  $\phi: E|_Y \rightarrow F|_Y$  is a given isomorphism between the restrictions of  $E$  and  $F$  to  $Y$ . Then  $(E, F, \phi)$  is isomorphic to  $(E', F', \phi')$  if we find isomorphisms  $\alpha: E \rightarrow E'$  and  $\beta: F \rightarrow F'$  such that the following diagram commutes.

$$\begin{array}{ccc} E|_Y & \xrightarrow{\phi} & F|_Y \\ \downarrow \alpha & & \downarrow \beta \\ E'|_Y & \xrightarrow{\phi'} & F'|_Y \end{array}$$

Two pairs  $(E, F, \phi)$  and  $(E', F', \phi')$  are equivalent, if there is a bundle  $G$  on  $X$  such that  $(E \oplus G, F \oplus G, \phi \oplus \text{id})$  is isomorphic to  $(E' \oplus G, F' \oplus G, \phi' \oplus \text{id})$ .

**1.3 Example.** The element of  $K^0(DT^*M, ST^*M)$  given by the symbol of an elliptic differential operator  $D$  mentioned above is represented by the restriction of the bundles  $\pi^*E$  and  $\pi^*F$  to the disc bundle  $DT^*M$ , together with the isomorphism  $\sigma(D)_{(x,\xi)}: E_{(x,\xi)} \rightarrow F_{(x,\xi)}$  for  $(x, \xi) \in ST^*M$ .

**1.4 Example.** Let  $M = \mathbb{R}^m$  and  $D = \sum_{i=1}^m (\partial/\partial_i)^2$  be the Laplace operator on functions. This is an elliptic differential operator, with symbol  $\sigma(D) = \sum_{i=1}^m \xi_i^2$ .

More generally, a second-order differential operator  $D: C^\infty(E) \rightarrow C^\infty(E)$  on a Riemannian manifold  $M$  is a *generalized Laplacian*, if  $\sigma(D)_{(x,\xi)} = |\xi|^2 \cdot \text{id}_{E_x}$  (the norm of the cotangent vector  $|\xi|$  is given by the Riemannian metric).

Notice that all generalized Laplacians are elliptic.

**1.5 Definition.** (*Adjoint operator*)

Assume that we have a differential operator  $D: C^\infty(E) \rightarrow C^\infty(F)$  between two Hermitian bundles  $E$  and  $F$  on a Riemannian manifold  $(M, g)$ . We define an  $L^2$ -inner product on  $C^\infty(E)$  by the formula

$$\langle f, g \rangle_{L^2(E)} := \int_M \langle f(x), g(x) \rangle_{E_x} d\mu(x) \quad \forall f, g \in C_0^\infty(E),$$

where  $\langle \cdot, \cdot \rangle_{E_x}$  is the fiber-wise inner product given by the Hermitian metric, and  $d\mu$  is the measure on  $M$  induced from the Riemannian metric. Here  $C_0^\infty$  is the space of smooth section with compact support. The Hilbert space completion of  $C_0^\infty(E)$  with respect to this inner product is called  $L^2(E)$ .

The *formal adjoint*  $D^*$  of  $D$  is then defined by

$$\langle Df, g \rangle_{L^2(F)} = \langle f, D^*g \rangle_{L^2(E)} \quad \forall f \in C_0^\infty(E), g \in C_0^\infty(F).$$

It turns out that exactly one operator with this property exists, which is another differential operator, and which is elliptic if and only if  $D$  is elliptic.

**1.6 Remark.** The class of differential operators is quite restricted. Many constructions one would like to carry out with differential operators automatically lead out of this class. Therefore, one often has to use *pseudodifferential operators*. Pseudodifferential operators are defined as a generalization of differential operators. There are many well written sources dealing with the theory of pseudodifferential operators. Since we will not discuss them in detail here, we omit even their precise definition and refer e.g. to [23] and [36]. What we have done

so far with elliptic operators can all be extended to pseudodifferential operators. In particular, they have a symbol, and the concept of ellipticity is defined for them. When studying elliptic differential operators, pseudodifferential operators naturally appear and play a very important role. An pseudodifferential operator  $P$  (which could e.g. be a differential operator) is elliptic if and only if a pseudodifferential operator  $Q$  exists such that  $PQ - \text{id}$  and  $QP - \text{id}$  are so called *smoothing* operators, a particularly nice class of pseudodifferential operators. For many purposes,  $Q$  can be considered to act like an inverse of  $P$ , and this kind of invertibility is frequently used in the theory of elliptic operators. However, if  $P$  happens to be an elliptic differential operator of positive order, then  $Q$  necessarily is not a differential operator, but only a pseudodifferential operator.

It should be noted that almost all of the results we present here for differential operators hold also for pseudodifferential operators, and often the proof is best given using them.

We now want to state several important properties of elliptic operators.

**1.7 Theorem.** *Let  $M$  be a smooth manifold,  $E$  and  $F$  smooth finite dimensional vector bundles over  $M$ . Let  $P: C^\infty(E) \rightarrow C^\infty(F)$  be an elliptic operator.*

*Then the following holds.*

(1) *Elliptic regularity:*

*If  $f \in L^2(E)$  is weakly in the null space of  $P$ , i.e.  $\langle f, P^*g \rangle_{L^2(E)} = 0$  for all  $g \in C_0^\infty(F)$ , then  $f \in C^\infty(E)$ .*

(2) *Decomposition into finite dimensional eigenspaces:*

*Assume  $M$  is compact and  $P = P^*$  (in particular,  $E = F$ ). Then the set  $s(P)$  of eigenvalues of  $P$  ( $P$  acting on  $C^\infty(E)$ ) is a discrete subset of  $\mathbb{R}$ , each eigenspace  $e_\lambda$  ( $\lambda \in s(P)$ ) is finite dimensional, and  $L^2(E) = \bigoplus_{\lambda \in s(P)} e_\lambda$  (here we use the completed direct sum in the sense of Hilbert spaces, which means by definition that the algebraic direct sum is dense in  $L^2(E)$ ).*

(3) *If  $M$  is compact, then  $\ker(P)$  and  $\ker(P^*)$  are finite dimensional, and then we define the index of  $P$*

$$\text{ind}(P) := \dim_{\mathbb{C}} \ker(P) - \dim_{\mathbb{C}} \ker(P^*).$$

*(Here, we could replace  $\ker(P^*)$  by  $\text{coker}(P)$ , because these two vector spaces are isomorphic).*

## 1.2 Characteristic classes

For explicit formulas for the index of a differential operator, we will have to use characteristic classes of certain bundles involved. Therefore, we quickly review the basics about the theory of characteristic classes.

**1.8 Theorem.** *Given a compact manifold  $M$  (or actually any finite CW-complex), there is a bijection between the isomorphism classes of  $n$ -dimensional complex vector bundles on  $M$ , and the set of homotopy classes of maps from  $M$  to  $BU(n)$ , the classifying space for  $n$ -dimensional vector bundles.  $BU(n)$  is*

by definition the space of  $n$ -dimensional subspaces of  $\mathbb{C}^\infty$  (with an appropriate limit topology).

The isomorphism is given as follows: On  $BU(n)$  there is the tautological  $n$ -plane bundle  $E(n)$ , the fiber at each point of  $BU(n)$  just being the subspace of  $\mathbb{C}^\infty$  which represents this point. Any map  $f: M \rightarrow BU(n)$  gives rise to the pull back bundle  $f^*E(n)$  on  $M$ . The theorem states that each bundle on  $M$  is isomorphic to such a pull back, and that two pull backs are isomorphic if and only the maps are homotopic.

**1.9 Definition.** A characteristic class  $c$  of vector bundles assigns to each vector bundle  $E$  over  $M$  an element  $c(E) \in H^*(M)$  which is natural, i.e. which satisfies

$$c(f^*E) = f^*c(E) \quad \forall f: M \rightarrow N, \quad E \text{ vector bundle over } N.$$

It follows that characteristic classes are given by cohomology classes of  $BU(n)$ .

**1.10 Theorem.** The integral cohomology ring  $H^*(BU(n))$  is a polynomial ring in generators  $c_0 \in H^0(BU(n))$ ,  $c_1 \in H^2(BU(n))$ ,  $\dots$ ,  $c_n \in H^{2n}(BU(n))$ . We call these generators the Chern classes of the tautological bundle  $E(n)$  of Theorem 1.8,  $c_i(E(n)) := c_i$ .

**1.11 Definition.** Write a complex vector bundle  $E$  over  $M$  as  $f^*E(n)$  for  $f: M \rightarrow BU(n)$  appropriate. Define  $c_i(fE) := f^*(c_i) \in H^{2i}(M; \mathbb{Z})$ , this is called the  $i$ -th Chern class of the bundle  $E = f^*E(n)$ .

If  $F$  is a real vector bundle over  $M$ , define the Pontryagin classes

$$p_i(F) := c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(M; \mathbb{Z}).$$

(The odd Chern classes of the complexification of a real vector bundle are two torsion and therefore are usually ignored).

### 1.2.1 Splitting principle

**1.12 Theorem.** Given a manifold  $M$  and a vector bundle  $E$  over  $M$ , there is another manifold  $N$  together with a map  $\phi: N \rightarrow M$ , which induces a monomorphism  $\phi^U: H^*(M; \mathbb{Z}) \rightarrow H^*(N; \mathbb{Z})$ , and such that  $\phi^*E = L^1 \oplus \dots \oplus L^n$  is a direct sum of line bundles.

Using Theorem 1.12, every question about characteristic classes of vector bundles can be reduced to the corresponding question for line bundles, and questions about the behavior under direct sums.

In particular, the following definitions makes sense:

**1.13 Definition.** The Chern character is an inhomogeneous characteristic class, assigning to each complex vector bundle  $E$  over a space  $M$  a cohomology class  $\text{ch}(E) \in H^*(M; \mathbb{Q})$ . It is characterized by the following properties:

- (1) Normalization: If  $L$  is a complex line bundle with first Chern class  $x$ , then

$$\text{ch}(L) = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in H^*(M; \mathbb{Q}).$$

Observe that in particular  $\text{ch}(\mathbb{C}) = 1$ .

(2) Additivity:  $L(E \oplus F) = L(E) + L(F)$ .

**1.14 Proposition.** *The Chern character is not only additive, but also multiplicative in the following sense: for two vector bundles  $E, F$  over  $M$  we have*

$$\text{ch}(E \otimes F) = \text{ch}(E) \cup \text{ch}(F).$$

**1.15 Definition.** The *Hirzebruch  $L$ -class* as normalized by Atiyah and Singer is an inhomogeneous characteristic class, assigning to each complex vector bundle  $E$  over a space  $M$  a cohomology class  $L(E) \in H^*(M; \mathbb{Q})$ . It is characterized by the following properties:

(1) Normalization: If  $L$  is a complex line bundle with first Chern class  $x$ , then

$$L(L) = \frac{x/2}{\tanh(x/2)} = 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots \in H^*(M; \mathbb{Q}).$$

Observe that in particular  $L(\mathbb{C}) = 1$ .

(2) Multiplicativity:  $L(E \oplus F) = L(E)L(F)$ .

**1.16 Definition.** The *Todd-class* is an inhomogeneous characteristic class, assigning to each complex vector bundle  $E$  over a space  $M$  a cohomology class  $\text{Td}(E) \in H^*(M; \mathbb{Q})$ . It is characterized by the following properties:

(1) Normalization: If  $L$  is a complex line bundle with first Chern class  $x$ , then

$$\text{Td}(L) = \frac{x}{1 - \exp(-x)} \in H^*(X; \mathbb{Q}).$$

Observe that in particular  $\text{Td}(\mathbb{C}) = 1$ .

(2) Multiplicativity:  $L(E \oplus F) = L(E)L(F)$ .

Note that  $\text{ch}$  as well as  $L$  and  $\text{Td}$  take values in the even dimensional part

$$H^{ev}(M; \mathbb{Q}) := \bigoplus_{k=0}^{\infty} H^{2k}(M; \mathbb{Q}).$$

## 1.2.2 Chern-Weyl theory

Chern-Weyl theory can be used to explicitly compute characteristic classes of finite dimensional vector spaces. For a short description compare [25]. To carry out the Chern-Weyl procedure, one has to choose a connection on the given vector bundle  $E$ . This connection has a curvature  $\Omega$ , which is a two form with values in the endomorphism bundle of the given vector bundle.

There are well defined homomorphisms

$$\sigma_r : \Omega^2(M; \text{End}(E)) \rightarrow \Omega^{2r}(M; \mathbb{C}),$$

which can be computed in local coordinates.

**1.17 Theorem.** *For any finite dimension vector bundle  $E$  (over a smooth manifold  $M$ ) with connection with curvature  $\Omega$ , for the image of the  $k$ -th Chern class  $c_k(E)$  in cohomology with complex coefficients, we have*

$$c_k(E) = \frac{1}{(2\pi i)^k} \sigma_k(\Omega) \in H^{2k}(M; \mathbb{C}).$$

*Since all other characteristic classes of complex vector bundles are given in terms of the Chern classes, this gives an explicit way to calculate arbitrary characteristic classes.*

### 1.2.3 Stable characteristic classes and K-theory

The elements of  $K^0(X)$  are represented by vector bundles. Therefore, it makes sense to ask whether a characteristic class of vector bundles can be used to define maps from  $K^0(X)$  to  $H^*(X)$ .

It turns out, that this is not always the case. The obstacle is, that two vector bundles  $E, F$  represent the *same* element in  $K^0(X)$  if (and only if) there is  $N \in \mathbb{N}$  such that  $E \oplus \mathbb{C}^N \cong F \oplus \mathbb{C}^N$ . Therefore, we have to make sure that  $c(E) = c(F)$  in this case. A characteristic class which satisfies this property is called *stable*, and evidently induces a map

$$c: K^0(X) \rightarrow H^*(X).$$

We deliberately did not specify the coefficients to be taken for cohomology, because most stable characteristic classes will take values in  $H^*(X; \mathbb{Q})$  instead of  $H^*(X; \mathbb{Z})$ .

The following proposition is an immediate consequence of the definition:

**1.18 Proposition.** *Assume a characteristic class  $c$  is multiplicative, i.e.  $c(E \oplus F) = c(E) \cup c(F) \in H^*(X)$ , and  $c(\mathbb{C}) = 1$ . Then  $c$  is a stable characteristic class.*

*Assume a characteristic class  $c$  is additive, i.e.  $c(E \oplus F) = c(E) + c(F)$ . Then  $c$  is a stable characteristic class.*

It follows in particular that the Chern character, as well as Hirzebruch's  $L$ -class are stable characteristic classes, i.e. they define maps from the K-theory  $K^0(X) \rightarrow H^*(X; \mathbb{Q})$ .

The relevance of the Chern character becomes apparent by the following theorem.

**1.19 Theorem.** *For a finite CW complex  $X$ ,*

$$\text{ch} \otimes \text{id}_{\mathbb{Q}}: K^0(X) \otimes \mathbb{Q} \rightarrow H^{ev}(X; \mathbb{Q}) \otimes \mathbb{Q} = H^{ev}(X; \mathbb{Q})$$

*is an isomorphism.*

We have constructed *relative* K-theory  $K^0(X, A)$  in terms of pairs of vector bundles on  $X$  with a given isomorphism of the restrictions to  $A$ . We can always find representatives such that one of the bundles is trivialized, and the other one  $E$  has in particular a trivialization  $E|_A = \mathbb{C}^n$  of its restriction to  $A$ . Such vector bundles correspond to homotopy classes  $[(X, A); (BU(n), pt)]$  of maps from  $X$  to  $BU(n)$  which map  $A$  to a fixed point  $pt$  in  $BU(n)$ .

For  $k > 0$ , we define relative Chern classes  $c_k(E, E|_A = \mathbb{C}^n) \in H^{2k}(X, A; \mathbb{Z})$  as pull back of  $c_k \in H^{2k}(BU(n), pt) \cong H^{2k}(BU(n))$ . The splitting principle also holds for such relative vector bundles, and therefore all the definitions we have made above go through in this relative situation. In particular, we can define a Chern character

$$\text{ch}: K^0(X, A) \rightarrow H^{ev}(X, A; \mathbb{Q}).$$

Given an elliptic differential operator  $D$ , we can apply this to our symbol element

$$\sigma(D) \in K^0(DT^*M, ST^*M),$$

to obtain  $\text{ch}(\sigma(D))$ .

**1.20 Proposition.** *Given a smooth manifold  $M$  of dimension  $m$ , there is a homomorphism*

$$\pi_! : H^{k+m}(DT^*M, ST^*M; \mathbb{R}) \rightarrow H^k(M),$$

called integration along the fiber. It is defined as follows: let  $\omega \in \Omega^{k+m}(DT^*M)$  be a closed differential form representing an element in  $H^{k+m}(DT^*M, ST^*M)$  (i.e. with vanishing restriction to the boundary). Locally, one can write  $\omega = \sum \alpha_i \cup \beta_i$ , where  $\beta_i$  are differential forms on  $M$  pulled back to  $DT^*M$  via the projection map  $\pi : DT^*M \rightarrow M$ , and  $\alpha_i$  are pulled back from the fiber in a local trivialization. Then  $\pi_!\omega$  is represented by

$$\sum_i \left( \int_{DT_x^*M} \alpha \right) \beta_i.$$

For more details about integration along the fiber, consult [8, Section 6]

### 1.3 Statement of the Atiyah-Singer index theorem

There are different variants of the Atiyah-Singer index theorem. We start with a cohomological formula for the index.

**1.21 Theorem.** *Let  $M$  be a compact oriented manifold of dimension  $m$ , and  $D : C^\infty(E) \rightarrow C^\infty(F)$  an elliptic operator with symbol  $\sigma(D)$ . Define the Todd character  $\text{Td}(M) := \text{Td}(TM \otimes \mathbb{C}) \in H^*(M; \mathbb{Q})$ . Then*

$$\text{ind}(D) = (-1)^{m(m+1)/2} \langle \pi_! \text{ch}(\sigma(D)) \cup \text{Td}(M), [M] \rangle.$$

The class  $[M] \in H_m(M; \mathbb{Q})$  is the fundamental class of the oriented manifold  $M$ , and  $\langle \cdot, \cdot \rangle$  is the usual pairing between homology and cohomology. For the characteristic classes, compare Subsection 1.2.

If we start with specific operators given by the geometry, explicit calculation usually give more familiar terms on the right hand side.

For example, for the signature operator we obtain Hirzebruch's signature formula expressing the signature in terms of the  $L$ -class, for the Euler characteristic operator we obtain the Gauss-Bonnet formula expressing the Euler characteristic in terms of the Pfaffian, and for the spin or spin<sup>c</sup> Dirac operator we obtain an  $\hat{A}$ -formula. For applications, these formulas prove to be particularly useful.

We give some more details about the signature operator, which we are going to use later again. To define the signature operator, fix a Riemannian metric  $g$  on  $M$ . Assume  $\dim M = 4k$  is divisible by four.

The signature operator maps from a certain subspace  $\Omega^+$  of the space of differential forms to another subspace  $\Omega^-$ . These subspaces are defined as follows. Define, on  $p$ -forms, the operator  $\tau := i^{p(p-1)+2k} *$ , where  $*$  is the Hodge- $*$  operator given by the Riemannian metric, and  $i^2 = -1$ . Since  $\dim M$  is divisible by 4, an easy calculation shows that  $\tau^2 = \text{id}$ . We then define  $\Omega^\pm$  to be the  $\pm 1$  eigenspaces of  $\tau$ .

The signature operator  $D_{sig}$  is now simply defined to be  $D_{sig} := d + d^*$ , where  $d$  is the exterior derivative on differential forms, and  $d^* = \pm * d *$  is its formal adjoint. We restrict this operator to  $\Omega^+$ , and another easy calculation shows that  $\Omega^+$  is mapped to  $\Omega^-$ .  $D_{sig}$  is elliptic, and a classical calculation

shows that its index is the signature of  $M$  given by the intersection form in middle homology.

The Atiyah-Singer index theorem now specializes to

$$\text{sign}(M) = \text{ind}(D_{\text{sig}}) = \langle 2^{2k} L(TM), [M] \rangle,$$

with  $\dim M = 4k$  as above.

*1.22 Remark.* One direction to generalize the Atiyah-Singer index theorem is to give an index formula for manifolds with boundary. Indeed, this is achieved in the Atiyah-Patodi-Singer index theorem. However, these results are much less topological than the results for manifolds without boundary. They are not discussed in these notes.

Next, we explain the K-theoretic version of the Atiyah-Singer index theorem. It starts with the element of  $K^0(DT^*M, ST^*M)$  given by the symbol of an elliptic operator. Given any compact manifold  $M$ , there is a well defined homomorphism

$$K^0(DT^*M, ST^*M) \rightarrow K^0(*) = \mathbb{Z},$$

constructed as follows. Embed  $M$  into high dimensional Euclidean space  $\mathbb{R}^N$ . This gives an embedding of  $T^*M$  into  $\mathbb{R}^{2N}$ , and further into its one point compactification  $S^{2N}$ , with normal bundle  $\nu$ . In this situation,  $\nu$  has a canonical complex structure. The embedding now defines a transfer map

$$K^0(DT^*M, ST^*M) \rightarrow K^0(S^{2N}, \infty),$$

by first using the Thom isomorphism to map to the (compactly supported) K-theory of the normal bundle, and then push forward to the K-theory of the sphere. The latter map is given by extending a vector bundle on the open subset  $\nu$  of  $S^{2N}$  which is trivialized outside a compact set (i.e. represents an element in compactly supported K-theory) trivially to all of  $S^{2N}$ .

Compose with the Bott periodicity isomorphism to map to  $K^0(pt) = \mathbb{Z}$ . The image of the symbol element under this homomorphism is denoted the *topological index*  $\text{ind}_t(D) \in K^0(*) = \mathbb{Z}$ . The reason for the terminology is that it is obtained from the symbol only, using purely topological constructions. The Atiyah-Singer index theorem states that analytical and topological index coincide:

**1.23 Theorem.**  $\text{ind}_t(D) = \text{ind}(D)$ .

## 1.4 The $G$ -index

### 1.4.1 The representation ring

Let  $G$  be a finite group, or more generally a compact Lie group. The representation ring  $RG$  of  $G$  is defined to be the Grothendieck group of all finite dimensional complex representations of  $G$ , i.e. an element of  $RG$  is a formal difference  $[V] - [W]$  of two finite dimensional  $G$ -representations  $V$  and  $W$ , and we have  $[V] - [W] = [X] - [Y]$  if and only if  $V \oplus Y \cong W \oplus X$  (strictly speaking, we have to pass to isomorphism classes of representations to avoid set theoretical problems). The direct sum of representations induces the structure of an abelian group on  $RG$ , and the tensor product makes it a commutative unital ring (the unit given by the trivial one-dimensional representation).

Equivalently, we can consider  $RG$  as the free abelian group generated by all isomorphism classes of finite irreducible representations (since every representation decomposes uniquely as a direct sum of irreducible ones).

To get numerical information about representations, one uses characters: the character of a representation  $\rho: G \rightarrow Gl(V)$  is the complex valued function  $\chi_V$  on  $G$  with

$$\chi_V(g) = \text{tr}(\rho(g)).$$

Elements of the representation ring can be recovered from the corresponding characters. Therefore, equivariant index theorems are often formulated in terms of characters.

More about this representation ring can be found e.g. in [9].

### 1.4.2 Equivariant analytic index

Assume now that the manifold  $M$  is a compact smooth manifold with a smooth (left)  $G$ -action, and let  $E, F$  be complex  $G$ -vector bundles on  $M$ . This means that  $G$  acts on  $E$  and  $F$  by vector bundle automorphisms (i.e. carries fibers to fibers linearly), and the bundle projection maps  $\pi_E: E \rightarrow M$  and  $\pi_F: F \rightarrow M$  are  $G$ -equivariant, i.e. satisfy

$$\pi_E(g e) = g \pi_E(e) \quad \forall g \in G, e \in E.$$

We assume that the actions preserve a Riemannian metric on  $M$  and Hermitian metrics on  $E$  and  $F$ . Since  $G$  is compact, this can always be achieved by averaging an arbitrary given metric, using a Haar measure on  $G$ .

The action of  $G$  on  $E$  and  $M$  induces actions on the spaces  $C^\infty(M)$  and  $C^\infty(E)$  of smooth functions on  $M$ , and smooth sections of  $E$ , respectively. This is given by the formulas

$$\begin{aligned} g f(x) &= f(g^{-1} x); & f &\in C^\infty(M), g \in G \\ g s(x) &= g s(g^{-1} x); & s &\in C^\infty(E), g \in G. \end{aligned}$$

Let  $D: C^\infty(E) \rightarrow C^\infty(F)$  be a  $G$ -equivariant elliptic differential operator, i.e.

$$D(g s) = g D(s).$$

Because the action of  $D$  is assumed to be isometric, the adjoint operator  $D^*$  is  $G$ -equivariant, as well.

Since  $D$  is elliptic and  $M$  is compact, the kernel and cokernel of  $D$  are finite dimensional. If  $s \in \ker(D)$ ,  $D(g s) = g D(s) = 0$ , i.e.  $\ker(D)$  is a finite dimensional  $G$ -representation. The same is true for  $\text{coker}(D) = \ker(D^*)$ . We define the (analytic)  $G$ -index of  $D$  to be

$$\text{ind}^G(D) := [\ker(D)] - [\text{coker}(D)] \in RG.$$

If  $G$  is the trivial group then  $RG \cong \mathbb{Z}$ , given by the dimension, and then  $\text{ind}^G(D)$  evidently coincides with the usual index of  $D$ .

### 1.4.3 Equivariant K-theory

Generalizing the construction of the representation ring, assume that  $X$  is a compact Hausdorff space with a  $G$ -action.

Then the  $G$ -equivariant vector bundles over  $X$  form an abelian semigroup under direct sum, and we define the equivariant K-theory group  $K_G^0(X)$  as the Grothendieck group of this semigroup, consisting of formal differences as in the non-equivariant case. Similarly, if  $A$  is a closed subspace of  $X$  which is  $G$ -invariant (i.e.  $ga \in A$  for all  $a \in A$ ,  $g \in G$ ), then we can use the same recipe as in the non-equivariant case, but now with equivariant vector bundles (and equivariant maps) to define  $K_G^0(X, A)$ .

**1.24 Example.** If  $X = \{pt\}$  is the one point space with the (necessarily) trivial  $G$ -action, then an equivariant vector bundles is exactly a finite dimensional  $G$ -representation. It follows that

$$K_G^0(pt) = RG.$$

### 1.4.4 Equivariant topological index and equivariant index theorem

Now we define a *topological* equivariant index.

To do this, we proceed exactly in the same way as in the non-equivariant case, but now  $G$ -equivariantly.

First observe that the symbol of an equivariant differential operator

$$D: \Gamma(E) \rightarrow \Gamma(F)$$

is defined by two  $G$ -equivariant vector bundles, namely  $\pi^*E$  and  $\pi^*F$  over  $DT^*M$ , together with the isomorphism (given by the principal symbol of the operator) of the restrictions of  $\pi^*E$  and  $\pi^*F$  to  $ST^*M$ .  $G$ -equivariance of the operator implies immediately, that this isomorphism is  $G$ -equivariant as well. Consequently, in this situation the symbol can be considered to be an element

$$\sigma_G(D) \in K_G^0(DT^*M, ST^*M).$$

Next we choose an equivariant embedding of  $M$  into a suitable  $G$ -representation  $V$ . Here, we have to assume that  $G$  is compact to guarantee the existence of such an embedding. Now, an equivariant version of the Thom isomorphism (to the equivariant K-theory of the normal bundle) and push forward define a transfer homomorphism

$$K_G^0(DT^*M, ST^*M) \rightarrow K_G^0(V_+, \infty),$$

where  $V_+ = V \cup \{\infty\}$  is the one point compactification of the  $G$ -representation  $V$  to a sphere (where the  $G$ -action is extended to  $V_+$  by  $g\infty = \infty$  for all  $g \in G$ ).

Last, we compose with the  $G$ -equivariant Bott periodicity isomorphism

$$K_G^0(V_+, \infty) \xrightarrow{\cong} K_G^0(pt) = RG$$

to map to the representation ring.

The image of the equivariant symbol element of  $K_G^0(DT^*M, ST^*M)$  under this composition is the equivariant topological index  $\text{ind}_{top}^G(D)$ . Again, the Atiyah-Singer index theorem says

**1.25 Theorem.**  $\text{ind}^G(D) = \text{ind}_t^G(D) \in K_G^0(pt) = RG.$

### 1.5 Families of operators and their index

Another important generalization is given if we don't look at one operator on one manifold, but a family of operators on a family of manifolds. More precisely, let  $X$  be any compact topological space,  $\pi: Y \rightarrow X$  a locally trivial fiber bundle with fibers  $Y_x := \pi^{-1}(x) \cong M$  smooth compact manifolds ( $x \in X$ ), and structure group the diffeomorphisms of the typical fiber  $M$ . Let  $E, F$  be families of smooth vector bundles on  $Y$  (i.e. vector bundles which are smooth for each fiber of the fibration  $Y \rightarrow X$ ), and  $C^\infty(E), C^\infty(F)$  the continuous sections which are smooth along the fibers. More precisely,  $E$  and  $F$  are smooth fiber bundles over  $X$ , the typical fiber is a vector bundle over  $M$ , and the structure group consists of diffeomorphisms of this vector bundle which are fiberwise smooth.

Assume that  $D: C^\infty(E) \rightarrow C^\infty(F)$  is a family  $\{D_x\}$  of elliptic differential operator along the fiber  $Y_x \cong M$  ( $x \in X$ ), i.e., in local coordinates  $D$  becomes

$$\sum_{|\alpha| \leq m} A_\alpha(y, x) \frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

with  $y \in M$  and  $x \in X$  such that  $A_\alpha(y, x)$  depends continuously on  $x$ , and each  $D_x$  is an elliptic differential operator on  $Y_x$ .

If  $\dim_{\mathbb{C}} \ker(D_x)$  is independent of  $x \in X$ , then all of these vector spaces patch together to give a vector bundle called  $\ker(D)$  on  $X$ , and similarly for the (fiberwise) adjoint  $D^*$ . This then gives a  $K$ -theory element  $[\ker(D)] - [\ker(D^*)] \in K^0(X)$ .

Unfortunately, it does sometimes happen that these dimensions jump. However, using appropriate perturbations or stabilizations, one can always define the  $K$ -theory element

$$\text{ind}(D) := [\ker(D)] - [\ker(D^*)] \in K^0(X),$$

the analytic index of the family of elliptic operators  $D$ . For details on this and the following material, consult e.g. [23, Paragraph 15].

We define the symbol of  $D$  (or rather a family of symbols) exactly as in the non-parametrized case. This gives now rise to an element in  $K^0(DT_v^*Y, ST_v^*Y)$ , where  $T_v^*Y$  is the cotangent bundle along the fibers. Note that all relevant spaces here are fiber bundles over  $X$ , with typical fiber  $T^*M, DT^*M$  or  $ST^*M$ , respectively.

Now we proceed with a family version of the construction of the topological index, copying the construction in the non-family situation, and using

- a (fiberwise) embedding of  $Y$  into  $\mathbb{R}^N \times X$  (which is compatible with the projection maps to  $X$ )
- the Thom isomorphism for families of vector bundles
- the family version of Bott periodicity, namely

$$K^0(S^{2N} \times X, \{\infty\} \times X) \xrightarrow{\cong} K^0(X).$$

(Instead, one could also use the Künneth theorem together with ordinary Bott periodicity.)

This gives rise to  $\text{ind}_t(D) \in K^0(X)$ . The Atiyah-Singer index theorem for families states:

**1.26 Theorem.**  $\text{ind}(D) = \text{ind}_t(D) \in K^0(X)$ .

The upshot of the discussion of this and the last section (for the details the reader is referred to the literature) is that the natural receptacle for the index of differential operators in various situations are appropriate K-theory groups, and much of today's index theory deals with investigating these K-theory groups.

## 2 Survey on $C^*$ -algebras and their K-theory

More detailed references for this section are, among others, [43], [16], and [7].

### 2.1 $C^*$ -algebras

**2.1 Definition.** A *Banach algebra*  $A$  is a complex algebra which is a complete normed space, and such that  $|ab| \leq |a| |b|$  for each  $a, b \in A$ .

A *\*-algebra*  $A$  is a complex algebra with an anti-linear involution  $*$ :  $A \rightarrow A$  (i.e.  $(\lambda a)^* = \bar{\lambda} a^*$ ,  $(ab)^* = b^* a^*$ , and  $(a^*)^* = a$  for all  $a, b \in A$ ).

A *Banach \*-algebra*  $A$  is a Banach algebra which is a \*-algebra such that  $|a^*| = |a|$  for all  $a \in A$ .

A  *$C^*$ -algebra*  $A$  is a Banach \*-algebra which satisfies  $|a^* a| = |a|^2$  for all  $a \in A$ .

Alternatively, a  $C^*$ -algebra is a Banach \*-algebra which is isometrically \*-isomorphic to a norm-closed subalgebra of the algebra of bounded operators on some Hilbert space  $H$  (this is the Gelfand-Naimark representation theorem, compare e.g. [16, 1.6.2]).

A  $C^*$ -algebra  $A$  is called separable if there exists a countable dense subset of  $A$ .

**2.2 Example.** If  $X$  is a compact topological space, then  $C(X)$ , the algebra of complex valued continuous functions on  $X$ , is a commutative  $C^*$ -algebra (with unit). The adjoint is given by complex conjugation:  $f^*(x) = \overline{f(x)}$ , the norm is the supremum-norm.

Conversely, it is a theorem that every abelian unital  $C^*$ -algebra is isomorphic to  $C(X)$  for a suitable compact topological space  $X$  [16, Theorem 1.3.12].

Assume  $X$  is locally compact, and set

$$C_0(X) := \{f: X \rightarrow \mathbb{C} \mid f \text{ continuous, } f(x) \xrightarrow{x \rightarrow \infty} 0\}.$$

Here, we say  $f(x) \rightarrow 0$  for  $x \rightarrow \infty$ , or  $f$  *vanishes at infinity*, if for all  $\epsilon > 0$  there is a compact subset  $K$  of  $X$  with  $|f(x)| < \epsilon$  whenever  $x \in X - K$ . This is again a commutative  $C^*$ -algebra (we use the supremum norm on  $C_0(X)$ ), and it is unital if and only if  $X$  is compact (in this case,  $C_0(X) = C(X)$ ).

### 2.2 $K_0$ of a ring

Suppose  $R$  is an arbitrary ring with 1 (not necessarily commutative). A module  $M$  over  $R$  is called finitely generated projective, if there is another  $R$ -module  $N$  and a number  $n \geq 0$  such that

$$M \oplus N \cong R^n.$$

This is equivalent to the assertion that the matrix ring  $M_n(R) = \text{End}_R(R^n)$  contains an idempotent  $e$ , i.e. with  $e^2 = e$ , such that  $M$  is isomorphic to the image of  $e$ , i.e.  $M \cong eR^n$ .

**2.3 Example.** Description of projective modules.

- (1) If  $R$  is a field, the finitely generated projective  $R$ -modules are exactly the finite dimensional vector spaces. (In this case, every module is projective).
- (2) If  $R = \mathbb{Z}$ , the finitely generated projective modules are the free abelian groups of finite rank
- (3) Assume  $X$  is a compact topological space and  $A = C(X)$ . Then, by the Swan-Serre theorem [39],  $M$  is a finitely generated projective  $A$ -module if and only if  $M$  is isomorphic to the space  $\Gamma(E)$  of continuous sections of some complex vector bundle  $E$  over  $X$ .

**2.4 Definition.** Let  $R$  be any ring with unit.  $K_0(R)$  is defined to be the Grothendieck group of finitely generated projective modules over  $R$ , i.e. the group of equivalence classes  $[(M, N)]$  of pairs of (isomorphism classes of) finitely generated projective  $R$ -modules  $M, N$ , where  $(M, N) \equiv (M', N')$  if and only if there is an  $n \geq 0$  with

$$M \oplus N' \oplus R^n \cong M' \oplus N \oplus R^n.$$

The group composition is given by

$$[(M, N)] + [(M', N')] := [(M \oplus M', N \oplus N')].$$

We can think of  $(M, N)$  as the formal difference of modules  $M - N$ .

Any unital ring homomorphism  $f: R \rightarrow S$  induces a map

$$f_*: K_0(R) \rightarrow K_0(S): [M] \mapsto [S \otimes_R M],$$

where  $S$  becomes a right  $R$ -module via  $f$ . We obtain that  $K_0$  is a covariant functor from the category of unital rings to the category of abelian groups.

**2.5 Example.** Calculation of  $K_0$ .

- If  $R$  is a field, then  $K_0(R) \cong \mathbb{Z}$ , the isomorphism given by the dimension:  $\dim_R(M, N) := \dim_R(M) - \dim_R(N)$ .
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$ , given by the rank.
- If  $X$  is a compact topological space, then  $K_0(C(X)) \cong K^0(X)$ , the topological  $K$ -theory given in terms of complex vector bundles. To each vector bundle  $E$  one associates the  $C(X)$ -module  $\Gamma(E)$  of continuous sections of  $E$ .
- Let  $G$  be a discrete group. The group algebra  $\mathbb{C}G$  is a vector space with basis  $G$ , and with multiplication coming from the group structure, i.e. given by  $g \cdot h = (gh)$ .

If  $G$  is a finite group, then  $K_0(\mathbb{C}G)$  is the complex representation ring of  $G$ .

### 2.3 **K**-Theory of $C^*$ -algebras

**2.6 Definition.** Let  $A$  be a unital  $C^*$ -algebra. Then  $K_0(A)$  is defined as in Definition 2.4, i.e. by forgetting the topology of  $A$ .

#### 2.3.1 **K**-theory for non-unital $C^*$ -algebras

When studying (the  $K$ -theory of)  $C^*$ -algebras, one has to understand morphisms  $f: A \rightarrow B$ . This necessarily involves studying the kernel of  $f$ , which is a closed ideal of  $A$ , and hence a *non-unital*  $C^*$ -algebra. Therefore, we proceed by defining the  $K$ -theory of  $C^*$ -algebras without unit.

**2.7 Definition.** To any  $C^*$ -algebra  $A$ , with or without unit, we assign in a functorial way a new, unital  $C^*$ -algebra  $A_+$  as follows. As  $\mathbb{C}$ -vector space,  $A_+ := A \oplus \mathbb{C}$ , with product

$$(a, \lambda)(b, \mu) := (ab + \lambda a + \mu b, \lambda\mu) \quad \text{for } (a, \lambda), (b, \mu) \in A \oplus \mathbb{C}.$$

The unit is given by  $(0, 1)$ . The star-operation is defined as  $(a, \lambda)^* := (a^*, \bar{\lambda})$ , and the new norm is given by

$$|(a, \lambda)| = \sup\{|ax + \lambda x| \mid x \in A \text{ with } |x| = 1\}$$

*2.8 Remark.*  $A$  is a closed ideal of  $A_+$ , the kernel of the canonical projection  $A_+ \rightarrow \mathbb{C}$  onto the second factor. If  $A$  itself is unital, the unit of  $A$  is of course different from the unit of  $A_+$ .

**2.9 Example.** Assume  $X$  is a locally compact space, and let  $X_+ := X \cup \{\infty\}$  be the one-point compactification of  $X$ . Then

$$C_0(X)_+ \cong C(X_+).$$

The ideal  $C_0(X)$  of  $C_0(X)_+$  is identified with the ideal of those functions  $f \in C(X_+)$  such that  $f(\infty) = 0$ .

**2.10 Definition.** For an arbitrary  $C^*$ -algebra  $A$  (not necessarily unital) define

$$K_0(A) := \ker(K_0(A_+) \rightarrow K_0(\mathbb{C})).$$

Any  $C^*$ -algebra homomorphism  $f: A \rightarrow B$  (not necessarily unital) induces a unital homomorphism  $f_+: A_+ \rightarrow B_+$ . The induced map

$$(f_+)_*: K_0(A_+) \rightarrow K_0(B_+)$$

maps the kernel of the map  $K_0(A_+) \rightarrow K_0(\mathbb{C})$  to the kernel of  $K_0(B_+) \rightarrow K_0(\mathbb{C})$ . This means it restricts to a map  $f_*: K_0(A) \rightarrow K_0(B)$ . We obtain a covariant functor from the category of (not necessarily unital)  $C^*$ -algebras to abelian groups.

Of course, we need the following result.

**2.11 Proposition.** *If  $A$  is a unital  $C^*$ -algebra, the new and the old definition of  $K_0(A)$  are canonically isomorphic.*

### 2.3.2 Higher topological $K$ -groups

We also want to define higher topological  $K$ -theory groups. We have an ad hoc definition using suspensions (this is similar to the corresponding idea in topological  $K$ -theory of spaces). For this we need the following.

**2.12 Definition.** Let  $A$  be a  $C^*$ -algebra. We define the cone  $CA$  and the suspension  $SA$  as follows.

$$\begin{aligned} CA &:= \{f: [0, 1] \rightarrow A \mid f(0) = 0\} \\ SA &:= \{f: [0, 1] \rightarrow A \mid f(0) = 0 = f(1)\}. \end{aligned}$$

These are again  $C^*$ -algebras, using pointwise operations and the supremum norm.

Inductively, we define

$$S^0 A := A \quad S^n A := S(S^{n-1} A) \quad \text{for } n \geq 1.$$

**2.13 Definition.** Assume  $A$  is a  $C^*$ -algebra. For  $n \geq 0$ , define

$$K_n(A) := K_0(S^n A).$$

These are the *topological  $K$ -theory groups of  $A$* . For each  $n \geq 0$ , we obtain a functor from the category of  $C^*$ -algebras to the category of abelian groups.

For unital  $C^*$ -algebras, we can also give a more direct definition of higher  $K$ -groups (in particular useful for  $K_1$ , which is then defined in terms of (classes of) invertible matrices). This is done as follows:

**2.14 Definition.** Let  $A$  be a unital  $C^*$ -algebra. Then  $Gl_n(A)$  becomes a topological group, and we have continuous embeddings

$$Gl_n(A) \hookrightarrow Gl_{n+1}(A): X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We set  $Gl_\infty(A) := \lim_{n \rightarrow \infty} Gl_n(A)$ , and we equip  $Gl_\infty(A)$  with the direct limit topology.

**2.15 Proposition.** Let  $A$  be a unital  $C^*$ -algebra. If  $k \geq 1$ , then

$$K_k(A) = \pi_{k-1}(Gl_\infty(A)) (\cong \pi_k(BGl_\infty(A))).$$

*Observe that any unital morphism  $f: A \rightarrow B$  of unital  $C^*$ -algebras induces a map  $Gl_n(A) \rightarrow Gl_n(B)$  and therefore also between  $\pi_k(Gl_\infty(A))$  and  $\pi_k(Gl_\infty(B))$ . This map coincides with the previously defined induced map in topological  $K$ -theory.*

**2.16 Remark.** Note that the topology of the  $C^*$ -algebra enters the definition of the higher topological  $K$ -theory of  $A$ , and in general the topological  $K$ -theory of  $A$  will be vastly different from the algebraic  $K$ -theory of the algebra underlying  $A$ . For connections in special cases, compare [38].

**2.17 Example.** It is well known that  $Gl_n(\mathbb{C})$  is connected for each  $n \in \mathbb{N}$ . Therefore

$$K_1(\mathbb{C}) = \pi_0(Gl_\infty(\mathbb{C})) = 0.$$

A very important result about  $K$ -theory of  $C^*$ -algebras is the following long exact sequence. A proof can be found e.g. in [16, Proposition 4.5.9].

**2.18 Theorem.** *Assume  $I$  is a closed ideal of a  $C^*$ -algebra  $A$ . Then, we get a short exact sequence of  $C^*$ -algebras  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ , which induces a long exact sequence in  $K$ -theory*

$$\rightarrow K_n(I) \rightarrow K_n(A) \rightarrow K_n(A/I) \rightarrow K_{n-1}(I) \rightarrow \cdots \rightarrow K_0(A/I).$$

## 2.4 Bott periodicity and the cyclic exact sequence

One of the most important and remarkable results about the  $K$ -theory of  $C^*$ -algebras is Bott periodicity, which can be stated as follows.

**2.19 Theorem.** *Assume  $A$  is a  $C^*$ -algebra. There is a natural isomorphism, called the Bott map*

$$K_0(A) \rightarrow K_0(S^2 A),$$

which implies immediately that there are natural isomorphism

$$K_n(A) \cong K_{n+2}(A) \quad \forall n \geq 0.$$

*2.20 Remark.* Bott periodicity allows us to define  $K_n(A)$  for each  $n \in \mathbb{Z}$ , or to regard the  $K$ -theory of  $C^*$ -algebras as a  $\mathbb{Z}/2$ -graded theory, i.e. to talk of  $K_n(A)$  with  $n \in \mathbb{Z}/2$ . This way, the long exact sequence of Theorem 2.18 becomes a (six-term) cyclic exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \mu_* \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

The connecting homomorphism  $\mu_*$  is the composition of the Bott periodicity isomorphism and the connecting homomorphism of Theorem 2.18.

**2.21 Example.** Assume  $A = C(X)$  is the space of continuous functions on a compact Hausdorff space. Particularly interesting is the case where  $X$  is the one point space.

Then

$$S^{2N} A = \{f: S^{2N} \times A \rightarrow \mathbb{C} \mid f(*, a) = 0 \quad \forall a \in A\},$$

where  $*$  is a base point in the two sphere  $S^{2N}$ .

To see this, use that  $C(X, C(Y)) = C(X \times Y)$ ; and the fact that

$$\{f: [0, 1]^{2N} \rightarrow A \mid f|_{\partial[0,1]^{2N}} = 0\} = \{f: S^{2N} \rightarrow A \mid f(*) = 0\}.$$

It follows that

$$K^0(S^{2N} A) \cong K^0(S^{2N} \times A, \{*\} \times A)$$

is the relative topological  $K$ -theory of the pair of spaces  $(S^{2N} \times A, \{*\} \times A)$ .

In particular, we recover the Bott periodicity isomorphism

$$K^0(S^{2N}, \infty) \rightarrow K^0(pt); \quad K^0(S^{2N} \times X, \{\infty\} \times X) \rightarrow K^0(X)$$

used in the definition of the topological index and the topological family index, respectively.

## 2.5 The $C^*$ -algebra of a group

Let  $\Gamma$  be a discrete group. Define  $l^2(\Gamma)$  to be the Hilbert space of square summable complex valued functions on  $\Gamma$ . We can write an element  $f \in l^2(\Gamma)$  as a sum  $\sum_{g \in \Gamma} \lambda_g g$  with  $\lambda_g \in \mathbb{C}$  and  $\sum_{g \in \Gamma} |\lambda_g|^2 < \infty$ .

We defined the *complex group algebra* (often also called the *complex group ring*)  $\mathbb{C}\Gamma$  to be the complex vector space with basis the elements of  $\Gamma$  (this can also be considered as the space of complex valued functions on  $\Gamma$  with finite support, and as such is a subspace of  $l^2(\Gamma)$ ). The product in  $\mathbb{C}\Gamma$  is induced by the multiplication in  $\Gamma$ , namely, if  $f = \sum_{g \in \Gamma} \lambda_g g$ ,  $u = \sum_{g \in \Gamma} \mu_g g \in \mathbb{C}\Gamma$ , then

$$\left( \sum_{g \in \Gamma} \lambda_g g \right) \left( \sum_{g \in \Gamma} \mu_g g \right) := \sum_{g, h \in \Gamma} \lambda_g \mu_h (gh) = \sum_{g \in \Gamma} \left( \sum_{h \in \Gamma} \lambda_h \mu_{h^{-1}g} \right) g.$$

This is a convolution product.

We have the *left regular representation*  $\lambda_\Gamma$  of  $\Gamma$  on  $l^2(\Gamma)$ , given by

$$\lambda_\Gamma(g) \cdot \left( \sum_{h \in \Gamma} \lambda_h h \right) := \sum_{h \in \Gamma} \lambda_h gh$$

for  $g \in \Gamma$  and  $\sum_{h \in \Gamma} \lambda_h h \in l^2(\Gamma)$ .

This unitary representation extends linearly to  $\mathbb{C}\Gamma$ .

The *reduced  $C^*$ -algebra*  $C_r^*\Gamma$  of  $\Gamma$  is defined to be the norm closure of the image  $\lambda_\Gamma(\mathbb{C}\Gamma)$  in the  $C^*$ -algebra of bounded operators on  $l^2(\Gamma)$ .

*2.22 Remark.* It's no surprise that there is also a *maximal  $C^*$ -algebra*  $C_{max}^*\Gamma$  of a group  $\Gamma$ . It is defined using not only the left regular representation of  $\Gamma$ , but simultaneously all of its representations. We will not make use of  $C_{max}^*\Gamma$  in these notes, and therefore will not define it here.

Given a topological group  $G$ , one can define  $C^*$ -algebras  $C_r^*G$  and  $C_{max}^*G$  which take the topology of  $G$  into account. They actually play an important role in the study of the Baum-Connes conjecture, which can be defined for (almost arbitrary) topological groups, but again we will not cover this subject here. Instead, we will throughout stick to discrete groups.

**2.23 Example.** If  $\Gamma$  is finite, then  $C_r^*\Gamma = \mathbb{C}\Gamma$  is the complex group ring of  $\Gamma$ .

In particular, in this case  $K_0(C_r^*\Gamma) \cong R\Gamma$  coincides with the (additive group of) the complex representation ring of  $\Gamma$ .

## 3 The Baum-Connes conjecture

The Baum-Connes conjecture relates an object from algebraic topology, namely the K-homology of the classifying space of a given group  $\Gamma$ , to representation theory and the world of  $C^*$ -algebras, namely to the K-theory of the reduced  $C^*$ -algebra of  $\Gamma$ .

Unfortunately, the material is very technical. Because of lack of space and time we can not go into the details (even of some of the definitions). We recommend the sources [41], [42], [16], [2], [28] and [7].

### 3.1 The Baum-Connes conjecture for torsion-free groups

**3.1 Definition.** Let  $X$  be any CW-complex.  $K_*(X)$  is the K-homology of  $X$ , where K-homology is the homology theory dual to topological K-theory. If  $BU$  is the spectrum of topological K-theory, and  $X_+$  is  $X$  with a disjoint basepoint added, then

$$K_n(X) := \pi_n(X_+ \wedge BU).$$

**3.2 Definition.** Let  $\Gamma$  be a discrete group. A classifying space  $B\Gamma$  for  $\Gamma$  is a CW-complex with the property that  $\pi_1(B\Gamma) \cong \Gamma$ , and  $\pi_k(B\Gamma) = 0$  if  $k \neq 1$ . A classifying space always exists, and is unique up to homotopy equivalence. Its universal covering  $E\Gamma$  is a contractible CW-complex with a free cellular  $\Gamma$ -action, the so called *universal space for  $\Gamma$ -actions*.

*3.3 Remark.* In the literature about the Baum-Connes conjecture, one will often find the definition

$$RK_n(X) := \varinjlim K_n(Y),$$

where the limit is taken over all finite subcomplexes  $Y$  of  $X$ . Note, however, that K-homology (like any homology theory in algebraic topology) is compatible with direct limits, which implies  $RK_n(X) = K_n(X)$  as defined above. The confusion comes from the fact that operator algebraists often use Kasparov's bivariant KK-theory to define  $K_*(X)$ , and this coincides with the homotopy theoretic definition only if  $X$  is compact.

Recall that a group  $\Gamma$  is called torsion-free, if  $g^n = 1$  for  $g \in \Gamma$  and  $n > 0$  implies that  $g = 1$ .

We can now formulate the Baum-Connes conjecture for torsion-free discrete groups.

**3.4 Conjecture.** *Assume  $\Gamma$  is a torsion-free discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\bar{\mu}_* : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma) \tag{3.5}$$

(which will be defined later). The Baum-Connes conjecture says that this map is an isomorphism.

**3.6 Example.** The map  $\bar{\mu}_*$  of Equation (3.5) is also defined if  $\Gamma$  is not torsion-free. However, in this situation it will in general not be an isomorphism. This can already be seen if  $\Gamma = \mathbb{Z}/2$ . Then  $C_r^*\Gamma = \mathbb{C}\Gamma \cong \mathbb{C} \oplus \mathbb{C}$  as a  $\mathbb{C}$ -algebra. Consequently,

$$K_0(C_r^*\Gamma) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{3.7}$$

On the other hand, using the homological Chern character,

$$K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{n=0}^{\infty} H_{2n}(B\Gamma; \mathbb{Q}) \cong \mathbb{Q}. \tag{3.8}$$

(Here we use the fact that the rational homology of every finite group is zero in positive degrees, which follows from the fact that the transfer homomorphism  $H_k(B\Gamma; \mathbb{Q}) \rightarrow H_k(\{1\}; \mathbb{Q})$  is (with rational coefficients) up to a factor  $|\Gamma|$  a left inverse to the map induced from the inclusion, and therefore is injective.)

The calculations (3.7) and (3.8) prevent  $\mu_0$  of (3.5) from being an isomorphism.

### 3.2 The Baum-Connes conjecture in general

To account for the problem visible in Example 3.6 if we are dealing with groups with torsion, one replaces the left hand side by a more complicated gadget, the equivariant K-homology of a certain  $\Gamma$ -space  $E(\Gamma, fin)$ , the classifying space for proper actions. We will define all of this later. Then, the Baum-Connes conjecture says the following.

**3.9 Conjecture.** *Assume  $\Gamma$  is a discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\mu_* : K_*^\Gamma(E(\Gamma, fin)) \rightarrow K_*(C_r^*\Gamma) \quad (3.10)$$

(we will define it later). The conjecture says that this map is an isomorphism.

*3.11 Remark.* If  $\Gamma$  is torsion-free, then  $K_*(B\Gamma) = K_*^\Gamma(E(\Gamma, fin))$ , and the assembly maps  $\bar{\mu}$  of Conjectures 3.4 and  $\mu$  of 3.9 coincide (see Proposition 3.30).

Last, we want to mention that there is also a *real version* of the Baum-Connes conjecture, where on the left hand side the K-homology is replaced by KO-homology, i.e. the homology dual to the K-theory of real vector bundles (or an equivariant version hereof), and on the right hand side  $C_r^*\Gamma$  is replaced by the real reduced  $C^*$ -algebra  $C_{r,\mathbb{R}}^*\Gamma$ .

### 3.3 Consequences of the Baum-Connes conjecture

#### 3.3.1 Idempotents in $C_r^*\Gamma$

The connection between the Baum-Connes conjecture and idempotents is best shown via Atiyah's  $L^2$ -index theorem, which we discuss first.

Given a closed manifold  $M$  with an elliptic differential operator  $D: C^\infty(E) \rightarrow C^\infty(F)$  between two bundles on  $M$ , and a normal covering  $\tilde{M} \rightarrow M$  (with deck transformation group  $\Gamma$ , normal means that  $M = \tilde{M}/\Gamma$ ), we can lift  $E$ ,  $F$  and  $D$  to  $\tilde{M}$ , and get an elliptic  $\Gamma$ -equivariant differential operator  $\tilde{D}: C^\infty(\tilde{E}) \rightarrow C^\infty(\tilde{F})$ . If  $\Gamma$  is not finite, we can not use the equivariant index of Section 1.4. However, because the action is free, it is possible to define an equivariant analytic index

$$\text{ind}_\Gamma(\tilde{D}) \in K_{\dim M}(C_r^*\Gamma).$$

This is described in Example 3.39.

Atiyah used a certain real valued homomorphism, the  $\Gamma$ -dimension

$$\text{dim}_\Gamma: K_0(C_r^*\Gamma) \rightarrow \mathbb{R},$$

to define the  $L^2$ -index of  $\tilde{D}$  (on an even dimensional manifold):

$$L^2\text{-ind}(\tilde{D}) := \text{dim}_\Gamma(\text{ind}_\Gamma(\tilde{D})).$$

The  $L^2$ -index theorem says

$$L^2\text{-ind}(\tilde{D}) = \text{ind}(D),$$

in particular, it follows that the  $L^2$ -index is an integer.

**3.12 Definition.** The  $\Gamma$ -dimension used above can be defined as follows: an element  $x$  of  $K_0(C_r^*\Gamma)$  is given by a (formal difference of) finitely generated projective modules over  $C_r^*\Gamma$ . Such a module is the image of a projection  $p \in M_n(C_r^*\Gamma)$ , i.e. a matrix  $(p_{ij})$  with entries in  $C_r^*\Gamma$  and such that  $p^2 = p = p^*$ .  $C_r^*\Gamma$  is by definition a certain algebra of bounded operators on  $l^2\Gamma$ . On this algebra, we can define a trace  $\mathrm{tr}_\Gamma$  by  $\mathrm{tr}_\Gamma(a) = \langle a(e), e \rangle_{l^2\Gamma}$ , where  $e$  is the function in  $l^2\Gamma$  which has value one at the unit element, and zero everywhere else.

We then define

$$\dim_\Gamma(x = [\mathrm{im}(p) - \mathrm{im}(q)]) = \sum_{i=1}^n \mathrm{tr}_\Gamma(p_{ii}) - \mathrm{tr}_\Gamma(q_{ii}).$$

An alternative description of the left hand side of (3.5) and (3.10) shows that, as long as  $\Gamma$  is torsion-free, the image of  $\mu_0$  coincides with the subset of  $K_0(C_r^*\Gamma)$  consisting of  $\mathrm{ind}_\Gamma(\tilde{D})$ , where  $\tilde{D}$  is as above. In particular, if  $\mu_0$  is surjective (and  $\Gamma$  is torsion-free), for each  $x \in K_0(C_r^*\Gamma)$  we find a differential operator  $D$  such that  $x = \mathrm{ind}_\Gamma(\tilde{D})$ . As a consequence,  $\dim_\Gamma(x) \in \mathbb{Z}$ , i.e. the range of  $\dim_\Gamma$  is contained in  $\mathbb{Z}$ . This is the statement of the so called *trace conjecture*.

**3.13 Conjecture.** *Assume  $\Gamma$  is a torsion-free discrete group. Then*

$$\dim_\Gamma(K_0(C_r^*\Gamma)) \subset \mathbb{Z}.$$

On the other hand, if  $x \in K_0(C_r^*\Gamma)$  is represented by a projection  $p = p^2 \in C_r^*\Gamma$ , then elementary properties of  $\dim_\Gamma$  (monotonicity and faithfulness) imply that  $0 \leq \dim_\Gamma(p) \leq 1$ , and  $\dim_\Gamma(p) \notin \{0, 1\}$  if  $p \neq 0, 1$ .

Therefore, we have the following consequence of the Baum-Connes conjecture. If  $\Gamma$  is torsion-free and the Baum-Connes map  $\mu_0$  is surjective, then  $C_r^*\Gamma$  does not contain any projection different from 0 or 1.

This is the assertion of the Kadison-Kaplansky conjecture:

**3.14 Conjecture.** *Assume  $\Gamma$  is torsion-free. Then  $C_r^*\Gamma$  does not contain any non-trivial projections.*

The following consequence of the Kadison-Kaplansky conjecture deserves to be mentioned:

**3.15 Proposition.** *If the Kadison-Kaplansky conjecture is true for a group  $\Gamma$ , then the spectrum  $s(x)$  of every self adjoint element  $x \in C_r^*\Gamma$  is connected. Recall that the spectrum is defined in the following way:*

$$s(x) := \{\lambda \in \mathbb{C} \mid (x - \lambda \cdot 1) \text{ not invertible}\}.$$

If  $\Gamma$  is not torsion-free, it is easy to construct non-trivial projections, and it is clear that the range of  $\mathrm{ind}_\Gamma$  is not contained in  $\mathbb{Z}$ . Baum and Connes originally conjectured that it is contained in the abelian subgroup  $\mathrm{Fin}^{-1}(\Gamma)$  of  $\mathbb{Q}$  generated by  $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$ . This conjecture is not correct, as is shown by an example of Roy [33]. In [24], Lück proves that the Baum-Connes conjecture implies that the range of  $\dim_\Gamma$  is contained in the *subring* of  $\mathbb{Q}$  generated by  $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$ .

**3.16 Remark.** An alternative, topological proof of the fact that the Baum-Connes implies the Kadison-Kaplansky conjecture is given by Mislin in [6]. Their proof does not use Atiyah's  $L^2$ -index theorem.

### 3.3.2 Obstructions to positive scalar curvature

The Baum-Connes conjecture implies the so called “stable Gromov-Lawson-Rosenberg” conjecture. This implication is a theorem due to Stephan Stolz. The details of this is discussed in the lectures of Stephan Stolz [37], therefore we can be very brief. We just mention the result.

**3.17 Theorem.** *Fix a group  $\Gamma$ . Assume that  $\mu$  in the real version of (3.10) discussed in Section 4 is injective (which follows e.g. if  $\mu$  in (3.10) is an isomorphism), and assume that  $M$  is a closed spin manifold with  $\pi_1(M) = \Gamma$ . Assume that a certain (index theoretic) invariant  $\alpha(M) \in K_{\dim M}(C_{\mathbb{R},r}^*\Gamma)$  vanishes. Then there is an  $n \geq 0$  such that  $M \times B^n$  admits a metric with positive scalar curvature.*

Here,  $B$  is any simply connected 8-dimensional spin manifold with  $\hat{A}(M) = 1$ . Such a manifold is called a *Bott manifold*.

The converse of Theorem 3.17, i.e. positive scalar curvature implies vanishing of  $\alpha(M)$ , is true for arbitrary groups and without knowing anything about the Baum-Connes conjecture.

### 3.3.3 The Novikov conjecture about higher signatures

**Direct approach** The original form of the Novikov conjecture states that higher signatures are homotopy invariant.

More precisely, let  $M$  be an (even dimensional) closed oriented manifold with fundamental group  $\Gamma$ . Let  $B\Gamma$  be a classifying space for  $\Gamma$ . There is a unique (up to homotopy) *classifying map*  $u: M \rightarrow B\Gamma$  which is defined by the property that it induces an isomorphism on  $\pi_1$ . Equivalently,  $u$  classifies a universal covering of  $M$ .

Let  $L(M) \in H^*(M; \mathbb{Q})$  be the Hirzebruch L-class (as normalized by Atiyah and Singer). Given any cohomology class  $a \in H^*(B\Gamma, \mathbb{Q})$ , we define the higher signature

$$\sigma_a(M) := \langle L(M) \cup u^*a, [M] \rangle \in \mathbb{Q}.$$

Here  $[M] \in H_{\dim M}(M; \mathbb{Q})$  is the fundamental class of the oriented manifold  $M$ , and  $\langle \cdot, \cdot \rangle$  is the usual pairing between cohomology and homology.

Recall that the Hirzebruch signature theorem states that  $\sigma_1(M)$  is the signature of  $M$ , which evidently is an oriented homotopy invariant.

The Novikov conjecture generalizes this as follows.

**3.18 Conjecture.** *Assume  $f: M \rightarrow M'$  is an oriented homotopy equivalence between two even dimensional closed oriented manifolds, with (common) fundamental group  $\pi$ . “Oriented” means that  $f_*[M] = [M']$ . Then all higher signatures of  $M$  and  $M'$  are equal, i.e.*

$$\sigma_a(M) = \sigma_a(M') \quad \forall a \in H^*(B\Gamma, \mathbb{Q}).$$

There is an equivalent reformulation of this conjecture in terms of K-homology. To see this, let  $D$  be the signature operator of  $M$ . (We assume here that  $M$  is smooth, and we choose a Riemannian metric on  $M$  to define this operator. It is an elliptic differential operator on  $M$ .) The operator  $D$  defines an element in the K-homology of  $M$ ,  $[D] \in K_{\dim M}(M)$ . Using the map  $u$ , we can push  $[D]$  to

$K_{\dim M}(B\Gamma)$ . We define the higher signature  $\sigma(M) := u_*[D] \in K_{\dim M}(B\Gamma) \otimes \mathbb{Q}$ . It turns out that

$$2^{\dim M/2} \sigma_a(M) = \langle a, ch(\sigma(M)) \rangle \quad \forall a \in H^*(B\Gamma; \mathbb{Q}),$$

where  $ch: K_*(B\Gamma) \otimes \mathbb{Q} \rightarrow H_*(B\Gamma, \mathbb{Q})$  is the homological Chern character (an isomorphism).

Therefore, the Novikov conjecture translates to the statement that  $\sigma(M) = \sigma(M')$  if  $M$  and  $M'$  are oriented homotopy equivalent.

Now one can show *directly* that

$$\bar{\mu}(\sigma(M)) = \bar{\mu}(\sigma(M')) \in K_*(C_r^*\Gamma),$$

if  $M$  and  $M'$  are oriented homotopy equivalent. Consequently, rational injectivity of the Baum-Connes map  $\bar{\mu}$  immediately implies the Novikov conjecture. If  $\Gamma$  is torsion-free, this is part of the assertion of the Baum-Connes conjecture. Because of this relation, injectivity of the Baum-Connes map  $\mu$  is often called the ‘‘analytic Novikov conjecture’’.

**Groups with torsion** For an arbitrary group  $\Gamma$ , we have a factorization of  $\bar{\mu}$  as follows:

$$K_*(B\Gamma) \xrightarrow{f} K_*^\Gamma(E(\Gamma, fin)) \xrightarrow{\mu} K_*(C_r^*\Gamma).$$

One can show that  $f$  is rationally injective, so that rational injectivity of the Baum-Connes map  $\mu$  implies the Novikov conjecture also in general.

### 3.4 The universal space for proper actions

**3.19 Definition.** Let  $\Gamma$  be a discrete group and  $X$  a Hausdorff space with an action of  $\Gamma$ . We say that the action is *proper*, if for all  $x, y \in X$  there are open neighborhood  $U_x \ni x$  and  $U_y \ni y$  such that  $gU_x \cap U_y$  is non-empty only for finitely many  $g \in \Gamma$  (the number depending on  $x$  and  $y$ ).

The action is said to be *cocompact*, if  $X/\Gamma$  is compact.

**3.20 Lemma.** *If the action of  $\Gamma$  on  $X$  is proper, then for each  $x \in X$  the isotropy group  $\Gamma_x := \{g \in \Gamma \mid gx = x\}$  is finite.*

**3.21 Definition.** Let  $\Gamma$  be a discrete group. A CW-complex  $X$  is a  $\Gamma$ -CW-complex, if  $X$  is a CW-complex with a cellular action of  $\Gamma$  with the additional property that, whenever  $g(D) \subset D$  for a cell  $D$  of  $X$  and some  $g \in \Gamma$ , then  $g|_D = \text{id}_D$ , i.e.  $g$  doesn't move  $D$  at all.

**3.22 Remark.** There exists also the notion of  $G$ -CW-complex for topological groups  $G$  (taking the topology of  $G$  into account). These have to be defined in a different way, namely by gluing together  $G$ -equivariant cells  $D^n \times G/H$ . In general, such a  $G$ -CW-complex is not an ordinary CW-complex.

**3.23 Lemma.** *The action of a discrete group  $\Gamma$  on a  $\Gamma$ -CW-complex is proper if and only if every isotropy group is finite.*

**3.24 Definition.** A proper  $\Gamma$ -CW-complex  $X$  is called *universal*, or more precisely *universal for proper actions*, if for every proper  $\Gamma$ -CW-complex  $Y$  there is a  $\Gamma$ -equivariant map  $f: Y \rightarrow X$  which is unique up to  $\Gamma$ -equivariant homotopy. Any such space is denoted  $E(\Gamma, fin)$  or  $\underline{E}\Gamma$ .

**3.25 Proposition.** *A  $\Gamma$ -CW-complex  $X$  is universal for proper actions if and only if the fixed point set*

$$X^H := \{x \in X \mid hx = x \quad \forall h \in H\}$$

*is empty whenever  $H$  is an infinite subgroup of  $\Gamma$ , and is contractible (and in particular non-empty) if  $H$  is a finite subgroup of  $\Gamma$ .*

**3.26 Proposition.** *If  $\Gamma$  is a discrete group, then  $E(\Gamma, fin)$  exists and is unique up to  $\Gamma$ -homotopy equivalence.*

*3.27 Remark.* The general context for this discussion are actions of a group  $\Gamma$  where the isotropy belongs to a fixed family of subgroups of  $\Gamma$  (in our case, the family of all finite subgroups). For more information, compare [40].

**3.28 Example.**

- If  $\Gamma$  is torsion-free, then  $E(\Gamma, fin) = E\Gamma$ , the universal covering of the classifying space  $B\Gamma$ . Indeed,  $\Gamma$  acts freely on  $E\Gamma$ , and  $E\Gamma$  is contractible.
- If  $\Gamma$  is finite, then  $E(\Gamma, fin) = \{*\}$ .
- If  $G$  is a connected Lie group with maximal compact subgroup  $K$ , and  $\Gamma$  is a discrete subgroup of  $G$ , then  $E(\Gamma, fin) = G/K$  [2, Section 2].

*3.29 Remark.* In the literature (in particular, in [2]), also a slightly different notion of universal spaces is discussed. One allows  $X$  to be any proper metrizable  $\Gamma$ -space, and requires the universal property for all proper metrizable  $\Gamma$ -spaces  $Y$ . For discrete groups (which are the only groups we are discussing here), a universal space in the sense of Definition 3.24 is universal in this sense.

However, for some of the proofs of the Baum-Connes conjecture (for special groups) it is useful to use certain models of  $E(\Gamma, fin)$  (in the broader sense) coming from the geometry of the group, which are not  $\Gamma$ -CW-complexes.

## 3.5 Equivariant K-homology

Let  $\Gamma$  be a discrete group. We have seen that, if  $\Gamma$  is not torsion-free, the assembly map (3.5) is not an isomorphism. To account for that, we replace  $K_*(B\Gamma)$  by the equivariant K-theory of  $E(\Gamma, fin)$ . Let  $X$  be any proper  $\Gamma$ -CW complex. The original definition of equivariant K-homology is due to Kasparov, making ideas of Atiyah precise. In this definition, elements of  $K_*^\Gamma(X)$  are equivalence classes of generalized elliptic operators. In [11], a more homotopy theoretic definition of  $K_*^\Gamma(X)$  is given, which puts the Baum-Connes conjecture in the context of other isomorphism conjectures.

### 3.5.1 Homotopy theoretic definition of equivariant K-homology

The details of this definition are quite technical, using spaces and spectra over the orbit category of the discrete group  $\Gamma$ . The objects of the orbit category are the orbits  $\Gamma/H$ ,  $H$  any subgroup of  $\Gamma$ . The morphisms from  $\Gamma/H$  to  $\Gamma/K$  are simply the  $\Gamma$ -equivariant maps.

In this section, spectra are used in the sense of homotopy theory, they are a generalization of topological spaces, in particular of CW-complexes. For a basic

introduction to this theory, one may consult e.g. [21, Chapter 3] or [10]. For the (more intricate) constructions mentioned in here, only the original literature [11] is available.

In this setting, any spectrum over the orbit category gives rise to an equivariant homology theory. The decisive step is then the construction of a (periodic) topological K-theory spectrum  $\mathbf{K}^\Gamma$  over the orbit category of  $\Gamma$ . This gives us then a functor from the category of (arbitrary)  $\Gamma$ -CW-complexes to the category of (graded) abelian groups, the *equivariant K-homology*  $K_*^\Gamma(X)$  ( $X$  any  $\Gamma$ -CW-complex).

The important property (which justifies the name “topological K-theory spectrum”) is that

$$K_k^\Gamma(\Gamma/H) = \pi_k(\mathbf{K}^\Gamma(\Gamma/H)) \cong K_k(C_r^*H)$$

for every subgroup  $H$  of  $\Gamma$ . In particular,

$$K_k^\Gamma(\{*\}) \cong K_k(C_r^*\Gamma).$$

Moreover, we have the following properties:

**3.30 Proposition.** (1) *Assume  $\Gamma$  is the trivial group. Then*

$$K_*^\Gamma(X) = K_*(X),$$

*i.e. we get back the ordinary K-homology introduced above.*

(2) *If  $H \leq \Gamma$  and  $X$  is an  $H$ -CW-complex, then there is a natural isomorphism*

$$K_*^H(X) \cong K_*^\Gamma(\Gamma \times_H X).$$

*Here  $\Gamma \times_H X = \Gamma \times H / \sim$ , where we divide out the equivalence relation generated by  $(gh, x) \sim (g, hx)$  for  $g \in \Gamma$ ,  $h \in H$  and  $x \in X$ . This is in the obvious way a left  $\Gamma$ -space.*

(3) *Assume  $X$  is a free  $\Gamma$ -CW-complex. Then there is a natural isomorphism*

$$K_*(\Gamma \backslash X) \rightarrow K_*^\Gamma(X).$$

*In particular, using the canonical  $\Gamma$ -equivariant map  $E\Gamma \rightarrow E(\Gamma, \text{fin})$ , we get a natural homomorphism*

$$K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(E\Gamma) \rightarrow K_*^\Gamma(E(\Gamma, \text{fin})).$$

### 3.5.2 Analytic definition of equivariant K-homology

Here we will give the original definition, which embeds into the powerful framework of equivariant KK-theory, and which is used for almost all proofs of special cases of the Baum-Connes conjecture. However, to derive some of the consequences of the Baum-Connes conjecture, most notably about the positive scalar curvature question —this is discussed in one of the lectures of Stephan Stolz [37]— the homotopy theoretic definition is used.

**3.31 Definition.** A Hilbert space  $H$  is called  $(\mathbb{Z}/2)$ -graded, if  $H$  comes with an orthogonal sum decomposition  $H = H_0 \oplus H_1$ . Equivalently, a unitary operator  $\epsilon$  with  $\epsilon^2 = 1$  is given on  $H$ . The subspaces  $H_0$  and  $H_1$  can be recovered as the  $+1$  and  $-1$  eigenspaces of  $\epsilon$ , respectively.

A bounded operator  $T: H \rightarrow H$  is called *even* (with respect to the given grading), if  $T$  commutes with  $\epsilon$ , and *odd*, if  $\epsilon$  and  $T$  anti-commute, i.e. if  $T\epsilon = -\epsilon T$ . An even operator decomposes as  $T = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ , an odd one as  $T = \begin{pmatrix} 0 & T_0 \\ T_1 & 0 \end{pmatrix}$  in the given decomposition  $H = H_0 \oplus H_1$ .

**3.32 Definition.** A *generalized elliptic  $\Gamma$ -operator on  $X$* , or a *cycle for  $\Gamma$ -K-homology of the  $\Gamma$ -space  $X$* , simply a *cycle* for short, is a triple  $(H, \pi, F)$ , where

- $H = H_0 \oplus H_1$  is a  $\mathbb{Z}/2$ -graded  $\Gamma$ -Hilbert space (i.e. the direct sum of two Hilbert spaces with unitary  $\Gamma$ -action)
- $\pi$  is a  $\Gamma$ -equivariant  $*$ -representation of  $C_0(X)$  on even bounded operators of  $H$  (equivariant means that  $\pi(fg^{-1}) = g\pi(f)g^{-1}$  for all  $f \in C_0(X)$  and all  $g \in \Gamma$ ).
- $F: H \rightarrow H$  is a bounded,  $\Gamma$ -equivariant, self adjoint operator such that  $\pi(f)(F^2 - 1)$  and  $[\pi(f), F] := \pi(f)F - F\pi(f)$  are compact operators for all  $f \in C_0(X)$ . Moreover, we require that  $F$  is odd, i.e.  $F = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$  in the decomposition  $H = H_0 \oplus H_1$ .

*3.33 Remark.* There are many different definitions of cycles, slightly weakening or strengthening some of the conditions. Of course, this does not effect the equivariant K-homology groups which are eventually defined using them.

**3.34 Definition.** We define the direct sum of two cycles in the obvious way.

**3.35 Definition.** Assume  $\alpha = (H, \pi, F)$  and  $\alpha' = (H', \pi', F')$  are two cycles.

- (1) They are called (isometrically) *isomorphic*, if there is a  $\Gamma$ -equivariant grading preserving isometry  $\Psi: H \rightarrow H'$  such that  $\Psi \circ \pi(f) = \pi'(f) \circ \Psi$  for all  $f \in C_0(X)$  and  $\Psi \circ F = F' \circ \Psi$ .
- (2) They are called *homotopic* (or *operator homotopic*) if  $H = H'$ ,  $\pi = \pi'$ , and there is a norm continuous path  $(F_t)_{t \in [0,1]}$  of operators with  $F_0 = F$  and  $F_1 = F'$  and such that  $(H, \pi, F_t)$  is a cycle for each  $t \in [0, 1]$ .
- (3)  $(H, \pi, F)$  is called *degenerate*, if  $[\pi(f), F] = 0$  and  $\pi(f)(F^2 - 1) = 0$  for each  $f \in C_0(X)$ .
- (4) The two cycles are called *equivalent* if there are degenerate cycles  $\beta$  and  $\beta'$  such that  $\alpha \oplus \beta$  is operator homotopic to a cycle isometrically isomorphic to  $\alpha' \oplus \beta'$ .

The set of equivalence classes of cycles is denoted  $KK_0^\Gamma(X)$ . (Caution, this is slightly unusual, mostly one will find the notation  $K^\Gamma(X)$  instead of  $KK^\Gamma(X)$ ).

**3.36 Proposition.** *Direct sum induces the structure of an abelian group on  $KK_0^\Gamma(X)$ .*

**3.37 Proposition.** *Any proper  $\Gamma$ -equivariant map  $\phi: X \rightarrow Y$  between two proper  $\Gamma$ -CW-complexes induces a homomorphism*

$$KK_0^\Gamma(X) \rightarrow KK_0^\Gamma(Y)$$

by  $(H, \pi, F) \mapsto (H, \pi \circ \phi^*, F)$ , where  $\phi^*: C_0(Y) \rightarrow C_0(X): f \mapsto f \circ \phi$  is defined since  $\phi$  is a proper map (else  $f \circ \phi$  does not necessarily vanish at infinity).

Recall that a continuous map  $\phi: X \rightarrow Y$  is called *proper* if the inverse image of every compact subset of  $Y$  is compact.

It turns out that the analytic definition of equivariant K-homology is quite flexible. It is designed to make it easy to construct elements of these groups—in many geometric situations they automatically show up. We give one of the most typical examples of such a situation, which we will have used in Section 3.3.1..

We need the following definition:

**3.38 Definition.** Let  $\overline{M}$  be a (not necessarily compact) Riemannian manifold without boundary, which is complete as a metric space. Define

$$L^2\Omega^p(\overline{M}) := \{\omega \text{ measurable } p\text{-form on } M \mid \int_{\overline{M}} |\omega(x)|_x^2 d\mu(x) < \infty\}.$$

Here,  $|\omega(x)|_x$  is the pointwise norm (at  $x \in \overline{M}$ ) of  $\omega(x)$ , which is given by the Riemannian metric, and  $d\mu(x)$  is the measure induced by the Riemannian metric.

$L^2\Omega^p(\overline{M})$  can be considered as the Hilbert space completion of the space of compactly supported  $p$ -forms on  $\overline{M}$ . The inner product is given by integrating the pointwise inner product, i.e.

$$\langle \omega, \eta \rangle_{L^2} := \int_{\overline{M}} \langle \omega(x), \eta(x) \rangle_x d\mu(x).$$

**3.39 Example.** Assume that  $M$  is a compact even dimensional Riemannian manifold. Let  $X = \overline{M}$  be a normal covering of  $M$  with deck transformation group  $\Gamma$  (normal means that  $X/\Gamma = M$ ). Of course, the action is free, in particular, proper. Let  $E = E_0 \oplus E_1$  be a graded Hermitian vector bundle on  $M$ , and

$$D: C^\infty(E) \rightarrow C^\infty(E)$$

an odd elliptic self adjoint differential operator (odd means that  $D$  maps the subspace  $C^\infty(E_0)$  to  $C^\infty(E_1)$ , and vice versa). If  $M$  is oriented, the signature operator on  $M$  is such an operator, if  $M$  is a spin-manifold, the same is true for its Dirac operator.

Now we can pull back  $E$  to a bundle  $\overline{E}$  on  $\overline{M}$ , and lift  $D$  to an operator  $\overline{D}$  on  $\overline{E}$ . The assumptions imply that  $\overline{D}$  extends to an unbounded self adjoint operator on  $L^2(\overline{E})$ , the space of square integrable sections of  $\overline{E}$ . This space is the completion of  $C_c^\infty(\overline{E})$  with respect to the canonical inner product (compare Definition 3.38). (The subscript c denotes sections with compact support). Using the functional calculus, we can replace  $\overline{D}$  by

$$F := (\overline{D}^2 + 1)^{-1/2} \overline{D}: L^2(\overline{E}) \rightarrow L^2(\overline{E}).$$

Observe that

$$L^2(\overline{E}) = L^2(\overline{E}_0) \oplus L^2(\overline{E}_1)$$

is a  $\mathbb{Z}/2$ -graded Hilbert space with a unitary  $\Gamma$ -action, which admits an (equivariant) action  $\pi$  of  $C_0(\overline{M}) = C_0(X)$  by fiber-wise multiplication. This action preserves the grading. Moreover,  $\overline{D}$  as well as  $F$  are odd,  $\Gamma$ -equivariant, self adjoint operators on  $L^2(\overline{E})$  and  $F$  is a bounded operator. From ellipticity it follows that

$$\pi(f)(F^2 - 1) = -\pi(f)(\overline{D}^2 + 1)^{-1}$$

is compact for each  $f \in C_0(\overline{M})$  (observe that this is not true for  $(\overline{D}^2 + 1)^{-1}$  itself, if  $\overline{M}$  is not compact). Consequently,  $(L^2(E), \pi, F)$  defines an (even) cycle for  $\Gamma$ -K-homology, i.e. it represents an element in  $KK_0^\Gamma(X)$ .

One can slightly reformulate the construction as follows:  $\overline{M}$  is a principal  $\Gamma$ -bundle over  $M$ , and  $l^2(\Gamma)$  has a (unitary) left  $\Gamma$ -action. We therefore can construct the associated flat bundle

$$L := l^2(\Gamma) \times_\Gamma \overline{M}$$

on  $M$  with fiber  $l^2(\Gamma)$ . Now we can twist  $D$  with this bundle  $L$ , i.e. define

$$\overline{D} := \nabla_L \otimes \text{id} + \text{id} \otimes D: C^\infty(L \otimes E) \rightarrow C^\infty(L \otimes E),$$

using the given flat connection  $\nabla_L$  on  $L$ . Again, we can complete to  $L^2(L \otimes E)$  and define

$$F := (\overline{D}^2 + 1)^{-1/2} \overline{D}.$$

The left action of  $\Gamma$  on  $l^2\Gamma$  induces an action of  $\Gamma$  on  $L$  and then a unitary action on  $L^2(L \otimes E)$ . Since  $\nabla_L$  preserves the  $\Gamma$ -action,  $\overline{D}$  is  $\Gamma$ -equivariant. There is a canonical  $\Gamma$ -isometry between  $L^2(L \otimes E)$  and  $L^2(\overline{E})$  which identifies the two versions of  $\overline{D}$  and  $F$ . The action of  $C_0(\overline{M})$  on  $L^2(L \otimes E)$  can be described by identifying  $C_0(\overline{M})$  with the continuous sections of  $M$  on the associated bundle

$$C_0(\Gamma) \times_\Gamma \overline{M},$$

where  $C_0(\Gamma)$  is the  $C^*$ -algebra of functions on  $\Gamma$  vanishing at infinity, and then using the obvious action of  $C_0(\Gamma)$  on  $l^2(\Gamma)$ .

It is easy to see how this examples generalizes to  $\Gamma$ -equivariant elliptic differential operators on manifolds with a proper, but not necessarily free,  $\Gamma$ -action (with the exception of the last part, of course).

Work in progress of Baum, Higson and Schick [3] suggests the (somewhat surprising) fact that, given any proper  $\Gamma$ -CW-complex  $Y$ , we can, for each element  $y \in KK_0^\Gamma(Y)$ , find such a proper  $\Gamma$ -manifold  $X$ , together with a  $\Gamma$ -equivariant map  $f: X \rightarrow Y$  and an elliptic differential operator on  $X$  giving an element  $x \in KK_0^\Gamma(X)$  as in the example, such that  $y = f_*(x)$ .

Analytic K-homology is homotopy invariant, a proof can be found in [7].

**3.40 Theorem.** *If  $\phi_1, \phi_2: X \rightarrow Y$  are proper  $\Gamma$ -equivariant maps which are homotopic through proper  $\Gamma$ -equivariant maps, then*

$$(\phi_1)_* = (\phi_2)_*: KK_*^\Gamma(X) \rightarrow KK_*^\Gamma(Y).$$

**3.41 Theorem.** *If  $\Gamma$  acts freely on  $X$ , then*

$$KK_*^\Gamma(X) \cong K_*(\Gamma \backslash X),$$

where the right hand side is the ordinary  $K$ -homology of  $\Gamma \backslash X$ .

**3.42 Definition.** Assume  $Y$  is an arbitrary proper  $\Gamma$ -CW-complex. Set

$$RK_*^\Gamma(Y) := \varinjlim KK_*^\Gamma(X),$$

where we take the direct limit over the direct system of  $\Gamma$ -invariant subcomplexes of  $Y$  with compact quotient (by the action of  $\Gamma$ ).

**3.43 Definition.** To define higher (analytic) equivariant  $K$ -homology, there are two ways. The short one only works for complex  $K$ -homology. One considers cycles and an equivalence relation exactly as above — with the notable exception that one does not require any grading! This way, one defines  $KK_1^\Gamma(X)$ . Because of Bott periodicity (which has period 2), this is enough to define all  $K$ -homology groups ( $KK_n^\Gamma(X) = KK_{n+2k}^\Gamma(X)$  for any  $k \in \mathbb{Z}$ ).

A perhaps more conceptual approach is the following. Here, one generalizes the notion of a graded Hilbert space by the notion of a  $p$ -multigraded Hilbert space ( $p \geq 0$ ). This means that the graded Hilbert space comes with  $p$  unitary operators  $\epsilon_1, \dots, \epsilon_p$  which are odd with respect to the grading, which satisfy  $\epsilon_i^2 = -1$  and  $\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 0$  for all  $i$  and  $j$  with  $i \neq j$ . An operator  $T: H \rightarrow H$  on a  $p$ -multigraded Hilbert space is called *multigraded* if it commutes with  $\epsilon_1, \dots, \epsilon_p$ . Such operators can (in addition) be even or odd.

This definition can be reformulated as saying that a multigraded Hilbert space is a (right) module over the Clifford algebra  $Cl_p$ , and a multigraded operator is a module map.

We now define  $KK_p^\Gamma(X)$  using cycles as above, with the additional assumption that the Hilbert space is  $p$ -graded, that the representation  $\pi$  takes values in  $\pi$ -multigraded even operators, and that the operator  $F$  is an odd  $p$ -multigraded operator. Isomorphism and equivalence of these multigraded cycles is defined as above, requiring that the multigradings are preserved throughout.

This definition gives an equivariant homology theory if we restrict to *proper* maps. Moreover, it satisfies Bott periodicity. The period is two for the (complex)  $K$ -homology we have considered so far. All results mentioned in this section generalize to higher equivariant  $K$ -homology.

If  $X$  is a proper  $\Gamma$ -CW-complex, the analytically defined representable equivariant  $K$ -homology groups  $RK_p^\Gamma(X)$  are canonically isomorphic to the equivariant  $K$ -homology groups  $K_p^\Gamma(X)$  defined by Davis and Lück in [11] as described in Section 3.5.1.

## 3.6 The assembly map

Here, we will use the homotopy theoretic description of equivariant  $K$ -homology due to Davis and Lück [11] described in Section 3.5.1. The assembly map then becomes particularly convenient to describe. From the present point of view, the main virtue is that they define a functor from *arbitrary*, not necessarily proper,  $\Gamma$ -CW-complexes to abelian groups.

The Baum-Connes assembly map is now simply defined using the equivariant collapse  $E(\Gamma, fin) \rightarrow *$ :

$$\mu: K_k^\Gamma(E(\Gamma, fin)) \rightarrow K_k^\Gamma(*) = K_k(C_r^*\Gamma). \quad (3.44)$$

If  $\Gamma$  is torsion-free, then  $E\Gamma = E(\Gamma, fin)$ , and the assembly map of (3.5) is defined as the composition of (3.44) with the appropriate isomorphism in Proposition 3.30.

### 3.7 Survey of KK-theory

The analytic definition of  $\Gamma$ -equivariant K-homology can be extended to a bivariate functor on  $\Gamma$ - $C^*$ -algebras. Here, a  $\Gamma$ - $C^*$ -algebra is a  $C^*$ -algebra  $A$  with an action (by  $C^*$ -algebra automorphisms) of  $\Gamma$ . If  $X$  is a proper  $\Gamma$ -space,  $C_0(X)$  is such a  $\Gamma$ - $C^*$ -algebra.

Given two  $\Gamma$ - $C^*$ -algebras  $A$  and  $B$ , Kasparov defines the bivariate KK-groups  $KK_*^\Gamma(A, B)$ . The most important property of this bivariate KK-theory is that it comes with a (composition) product, the *Kasparov product*. This can be stated most conveniently as follows:

Given a discrete group  $\Gamma$ , we have a category  $KK^\Gamma$  whose objects are  $\Gamma$ - $C^*$ -algebras (we restrict here to separable  $C^*$ -algebras). The morphisms in this category between two  $\Gamma$ - $C^*$ -algebras  $A$  and  $B$  are called  $KK_*^\Gamma(A, B)$ . They are  $\mathbb{Z}/2$ -graded abelian groups, and the composition preserves the grading, i.e. if  $\phi \in KK_i^\Gamma(A, B)$  and  $\psi \in KK_j^\Gamma(B, C)$  then  $\psi\phi \in KK_{i+j}^\Gamma(A, C)$ .

There is a functor from the category of separable  $\Gamma$ - $C^*$ -algebras (where morphisms are  $\Gamma$ -equivariant  $*$ -homomorphisms) to the category  $KK_*^\Gamma$  which maps an object  $A$  to  $A$ , and such that the image of a morphism  $\phi: A \rightarrow B$  is contained in  $KK_0^\Gamma(A, B)$ .

If  $X$  is a proper cocompact  $\Gamma$ -CW-complex then (by definition)

$$KK_p^\Gamma(C_0(X), \mathbb{C}) = KK_{-p}^\Gamma(X).$$

Here,  $\mathbb{C}$  has the trivial  $\Gamma$ -action.

On the other hand, for any  $C^*$ -algebra  $A$  without a group action (i.e. with trivial action of the trivial group  $\{1\}$ ),  $KK_*^{\{1\}}(\mathbb{C}, A) = K_*(A)$ .

There is a functor from  $KK^\Gamma$  to  $KK^{\{1\}}$ , called *descent*, which assigns to every  $\Gamma$ - $C^*$ -algebra  $A$  the *reduced crossed product*  $C_r^*(\Gamma, A)$ . The crossed product has the property that  $C_r^*(\Gamma, \mathbb{C}) = C_r^*\Gamma$ .

### 3.8 KK assembly

We now want to give an account of the analytic definition of the assembly map, which was the original definition. The basic idea is that the assembly map is given by taking an index. To start with, assume that we have an even generalized elliptic  $\Gamma$ -operator  $(H, \pi, F)$ , representing an element in  $K_0^\Gamma(X)$ , where  $X$  is a proper  $\Gamma$ -space such that  $\Gamma \backslash X$  is compact. The index of this operator should give us an element in  $K_0(C_r^*\Gamma)$ . Since the cycle is even,  $H$  split as  $H = H_0 \oplus H_1$ , and  $F = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}$  with respect to this splitting. Indeed, now, the kernel and cokernel of  $P$  are modules over  $\mathbb{C}\Gamma$ , and should, in most cases, give modules over  $C_r^*\Gamma$ .

If  $\Gamma$  is finite, the latter is indeed the case (since  $C_r^*\Gamma = \mathbb{C}\Gamma$ ). Moreover, since  $\Gamma \backslash X$  is compact and  $\Gamma$  is finite,  $X$  is compact, which implies that  $C_0(X)$  is unital.

We may then assume that  $\pi$  is unital (switching to an equivalent cycle with Hilbert space  $\pi(1)H$ , if necessary). But then the axioms for a cycle imply that  $F^2 - 1$  is compact, i.e. that  $F$  is invertible modulo compact operators, or that  $F$  is Fredholm, which means that  $\ker(P)$  and  $\ker(P^*)$  are finite dimensional. Since  $\Gamma$  acts on them,  $[\ker(P)] - [\ker(P^*)]$  defines an element of the representation ring  $R\Gamma = K_0(C_r^*\Gamma)$  for the finite group  $\Gamma$ . It remains to show that this map respects the equivalence relation defining  $K_0^\Gamma(X)$ .

However, if  $\Gamma$  is not finite, the modules  $\ker(P)$  and  $\ker(P^*)$ , even if they are  $C_r^*\Gamma$ -modules, are in general not finitely generated projective.

To grasp the difficulty, consider Example 3.39. Using the description where  $F$  acts on a bundle over the base space  $M$  with infinite dimensional fiber  $L \otimes E$ , we see that loosely speaking, the null space of  $F$  should rather “contain” certain copies of  $l^2\Gamma$  than copies of  $C_r^*\Gamma$  (for finite groups, “accidentally” these two are the same!). However, in general  $l^2\Gamma$  is not projective over  $C_r^*\Gamma$  (although it is a module over this algebra). To be specific, assume that  $M$  is a point,  $E_0 = \mathbb{C}$  and  $E_1 = 0$ , and  $D = 0$ . Here we obtain,  $L^2(E_0) = l^2\Gamma$ ,  $L^2(E_1) = 0$ ,  $F = 0$ , and indeed,  $\ker(P) = l^2\Gamma$ .

In the situation of our example, there is a way around this problem: Instead of twisting the operator  $D$  with the flat bundle  $l^2(\Gamma) \times_\Gamma \overline{M}$ , we twist with  $C_r^*(\Gamma) \times_\Gamma \overline{M}$ , to obtain an operator  $D'$  acting on a bundle with fiber  $C_r^*\Gamma \otimes \mathbb{C}^{\dim E}$ . This way, we replace  $l^2\Gamma$  by  $C_r^*\Gamma$  throughout. Still, it is not true in general that the kernels we get in this way are finitely generated projective modules over  $C_r^*\Gamma$ . However, it is a fact that one can always add to the new  $F'$  an appropriate compact operator such that this is the case. Then the obvious definition gives an element

$$\text{ind}(D') \in K_0(C_r^*\Gamma).$$

This is the Mishchenko-Fomenko index of  $D'$  which does not depend on the chosen compact perturbation. Mishchenko and Fomenko give a formula for this index extending the Atiyah-Singer index formula, compare e.g. [27, Section 1] or [31, Section 1] and [32, Section 3].

One way to get around the difficulty in the general situation (not necessarily studying a lifted differential operator) is to deform  $(H, \pi, F)$  to an equivalent  $(H, \pi, F')$  which is better behaved (reminiscent to the compact perturbation above). This allows to proceed with a rather elaborate generalization of the Mishchenko-Fomenko example we just considered, essentially replacing  $l^2(\Gamma)$  by  $C_r^*\Gamma$  again. In this way, one defines an index as an element of  $K_*(C_r^*\Gamma)$ .

This gives a homomorphism  $\mu^\Gamma: KK_*^\Gamma(C_0(X)) \rightarrow K_*(C_r^*\Gamma)$  for each proper  $\Gamma$ -CW-complex  $X$  where  $\Gamma \backslash X$  is compact. This passes to direct limits and defines, in particular,

$$\mu_*: RK_*^\Gamma(E(\Gamma, \text{fin})) \rightarrow K_*(C_r^*\Gamma).$$

Next, we proceed with an alternative definition of the Baum-Connes map using KK-theory and the Kasparov product. The basic observation here is that, given any proper  $\Gamma$ -CW-space  $X$ , there is a specific projection  $p \in C_r^*(\Gamma, C_0(X))$  (unique up to an appropriate type of homotopy) which gives rise to a canonical element  $[L_X] \in K_0(C_r^*(\Gamma, C_0(X))) = KK_0(\mathbb{C}, C_r^*(\Gamma, C_0(X)))$ . This defines by

composition the homomorphism

$$KK_*^\Gamma(X) = KK_*^\Gamma(C_0(X), \mathbb{C}) \xrightarrow{\text{descent}} KK_*(C_r^*(\Gamma, C_0(X)), C_r^*\Gamma) \\ \xrightarrow{[L_x]^\circ} KK_*(\mathbb{C}, C_r^*\Gamma) = K_*(C_r^*\Gamma).$$

Again, this passes to direct limits and defines as a special case the Baum-Connes assembly map

$$\mu: RK_*^\Gamma(E(\Gamma, \text{fin})) \rightarrow K_*(C_r^*\Gamma).$$

*3.45 Remark.* It is a non-trivial fact (due to Hambleton and Pedersen [12]) that this assembly map coincides with the map  $\mu$  of (3.10).

Almost all positive results about the Baum-Connes have been obtained using the powerful methods of KK-theory, in particular the so called Dirac-dual Dirac method, compare e.g. [41].

### 3.9 The status of the conjecture

The Baum-Connes conjecture is known to be true for the following classes of groups.

- (1) discrete subgroups of  $SO(n, 1)$  and  $SU(n, 1)$  [17]
- (2) Groups with the *Haagerup property*, sometimes called *a-T-menable groups*, i.e. which admit an isometric action on some affine Hilbert  $H$  space which is proper, i.e. such that  $g_n v \xrightarrow{n \rightarrow \infty} \infty$  for every  $v \in H$  whenever  $g_n \xrightarrow{n \rightarrow \infty} \infty$  in  $G$  [13]. Examples of groups with the Haagerup property are amenable groups, Coxeter groups, groups acting properly on trees, and groups acting properly on simply connected CAT(0) cubical complexes
- (3) One-relator groups, i.e. groups with a presentation  $G = \langle g_1, \dots, g_n \mid r \rangle$  with only one defining relation  $r$  [4].
- (4) Cocompact lattices in  $Sl_3(\mathbb{R})$ ,  $Sl_3(\mathbb{C})$  and  $Sl_3(\mathbb{Q}_p)$  ( $\mathbb{Q}_p$  denotes the  $p$ -adic numbers) [22]
- (5) Word hyperbolic groups in the sense of Gromov [26].
- (6) Artin's full braid groups  $B_n$  [34].

Since we will encounter amenability later on, we recall the definition here.

**3.46 Definition.** A finitely generated discrete group  $\Gamma$  is called amenable, if for any given finite set of generators  $S$  (where we require  $1 \in S$  and require that  $s \in S$  implies  $s^{-1} \in S$ ) there exists a sequence of finite subsets  $X_k$  of  $\Gamma$  such that

$$\frac{|SX_k| := \{sx \mid s \in S, x \in X_k\}}{|X_k|} \xrightarrow{k \rightarrow \infty} 1.$$

$|Y|$  denotes the number of elements of the set  $Y$ .

An arbitrary discrete group is called amenable, if each finitely generated subgroup is amenable.

Examples of amenable groups are all finite groups, all abelian, nilpotent and solvable groups. Moreover, the class of amenable groups is closed under taking subgroups, quotients, extensions, and directed unions.

The free group on two generators is not amenable. “Most” examples of non-amenable groups do contain a non-abelian free group.

There is a certain stronger variant of the Baum-Connes conjecture, the *Baum-Connes conjecture with coefficients*. It has the following stability properties:

- (1) If a group  $\Gamma$  acts on a tree such that the stabilizer of every edge and every vertex satisfies the Baum-Connes conjecture with coefficients, the same is true for  $\Gamma$  [29].
- (2) If a group  $\Gamma$  satisfies the Baum-Connes conjecture with coefficients, then so does every subgroup of  $\Gamma$  [29].
- (3) If we have an extension  $1 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3 \rightarrow 1$ ,  $\Gamma_3$  is torsion-free and  $\Gamma_1$  as well as  $\Gamma_3$  satisfy the Baum-Connes conjecture with coefficients, then so does  $\Gamma_2$ .

It should be remarked that in the above list, all groups except for word hyperbolic groups, and cocompact subgroups of  $Sl_3$  actually satisfy the Baum-Connes conjecture with coefficients.

The Baum-Connes assembly map  $\mu$  of (3.10) is known to be rationally injective for considerably larger classes of groups, in particular the following.

- (1) Discrete subgroups of connected Lie groups [18].
- (2) Discrete subgroups of  $p$ -adic groups [19].
- (3) Bolic groups (a certain generalization of word hyperbolic groups) [20].
- (4) Groups which admit an amenable action on some compact space [15].

Last, it should be mentioned that recent constructions of Gromov show that certain variants of the Baum-Connes conjecture, among them the Baum-Connes conjecture with coefficients, and an extension called the *Baum-Connes conjecture for groupoids*, are false [14]. At the moment, no counterexample to the Baum-Connes conjecture 3.9 seems to be known. However, there are many experts in the field who think that such a counterexample eventually will be constructed [14].

## 4 Real $C^*$ -algebras and K-theory

### 4.1 Real $C^*$ -algebras

The applications of the theory of  $C^*$ -algebras to geometry and topology we present here require at some point that we work with real  $C^*$ -algebras. Most of the theory is parallel to the theory of complex  $C^*$ -algebras. For more details on real  $C^*$ -algebras and their K-theory, including the role this plays in index theory, compare [35].

**4.1 Definition.** A unital real  $C^*$ -algebra is a Banach-algebra  $A$  with unit over the real numbers, with an isometric involution  $*$ :  $A \rightarrow A$ , such that

$$|x|^2 = |x^*x| \quad \text{and } 1 + x^*x \text{ is invertible } \quad \forall x \in A.$$

It turns out that this is equivalent to the existence of a  $*$ -isometric embedding of  $A$  as a closed subalgebra into  $\mathcal{B}H_{\mathbb{R}}$ , the bounded operators on a suitable real Hilbert space (compare [30]).

**4.2 Example.** If  $X$  is a compact topological space, then  $C(X; \mathbb{R})$ , the algebra of real valued continuous function on  $X$ , is a real  $C^*$ -algebra with unit (and with trivial involution).

More generally, if  $X$  comes with an involution  $\tau: X \rightarrow X$  (i.e.  $\tau^2 = \text{id}_X$ ), then  $C_{\tau}(X) := \{f: X \rightarrow \mathbb{C} \mid f(\tau x) = \overline{f(x)}\}$  is a real  $C^*$ -algebra with involution  $f^*(x) = \overline{f(\tau x)}$ .

Conversely, every commutative unital real  $C^*$ -algebra is isomorphic to some  $C_{\tau}(X)$ .

If  $X$  is only locally compact, we can produce examples of non-unital real  $C^*$ -algebras as in Example 2.2.

Essentially everything we have done for (complex)  $C^*$ -algebras carries over to real  $C^*$ -algebras, substituting  $\mathbb{R}$  for  $\mathbb{C}$  throughout. In particular, the definition of the K-theory of real  $C^*$ -algebras is literally the same as for complex  $C^*$ -algebras (actually, the definitions make sense for even more general topological algebras), and a short exact sequence of real  $C^*$ -algebras gives rise to a long exact K-theory sequence.

The notable exception is Bott periodicity. We don't get the period 2, but the period 8.

**4.3 Theorem.** *Assume that  $A$  is a real  $C^*$ -algebra. Then we have a Bott periodicity isomorphism*

$$K_0(A) \cong K_0(S^8 A).$$

*This implies*

$$K_n(A) \cong K_{n+8}(A) \quad \text{for } n \geq 0.$$

**4.4 Remark.** Again, we can use Bott periodicity to define  $K_n(A)$  for arbitrary  $n \in \mathbb{Z}$ , or we may view  $K_n(A)$  as an 8-periodic theory, i.e. with  $n \in \mathbb{Z}/8$ .

The long exact sequence of Theorem 2.18 becomes a 24-term cyclic exact sequence.

The *real reduced  $C^*$ -algebra* of a group  $\Gamma$ , denoted  $C_{\mathbb{R},r}^* \Gamma$ , is the norm closure of  $\mathbb{R}\Gamma$  in the bounded operators on  $l^2\Gamma$ .

## 4.2 Real K-homology and Baum-Connes

More details about the contents of this subsection can be found in [32, Section 2].

A variant of the cohomology theory given by complex vector bundles is KO-theory, which is given by real vector bundles. The homology theory dual to this is KO-homology. If  $KO$  is the spectrum of topological KO-theory, then  $KO_n(X) = \pi_n(X_+ \wedge KO)$ .

The homotopy theoretic definition of equivariant K-homology can be varied easily to define equivariant KO-homology. The analytic definition can also be adapted easily, replacing  $\mathbb{C}$  by  $\mathbb{R}$  throughout, using in particular real Hilbert spaces. However, we have to stick to  $n$ -multigraded cycles to define  $KK_n^{\Gamma}(X)$ , it is not sufficient to consider only even and odd cycles.

All the constructions and properties translate appropriately from the complex to the real situation, again with the notable exception that Bott periodicity does not give the period 2, but 8. The upshot of all of this is that we get a real version of the Baum-Connes conjecture, namely

**4.5 Conjecture.** *The real Baum-Connes assembly map*

$$\mu_n: KO_n^\Gamma(E(\Gamma, fin)) \rightarrow KO_n(C_{\mathbb{R},r}^*\Gamma),$$

*is an isomorphism.*

It should be remarked that all known results about injectivity or surjectivity of the Baum-Connes map can be proved for the real version as well as for the complex version, since each proof translates without too much difficulty. Moreover, it is known that the complex version of the Baum-Connes conjecture for a group  $\Gamma$  implies the real version (for this abstract result, the isomorphism is needed as input, since this is based on the use of the five-lemma at a certain point).

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