Modern index Theory — lectures held at CIRM rencontré "Theorie d'indice", Mar 2006

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Abstract

Every elliptic (pseudo)-differential operator D on a closed manifold gives rise to a Fredholm operator acting on L^2-sections of the bundle in question. It therefore has an index $\mathrm{ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D))$. This index depends only on the *symbol* of D. The Atiyah-Singer index theorem expresses this index by means of a *topological expression* in terms of this symbol.

Using a Chern character and applied to special operators coming from geometry, there is a very explicit cohomological formula for this index.

It turns out that, in more general contexts, the suitable definition of index is not given in terms of the difference of kernel and cokernel, but more precisely as an element of a K-theory group (where this group depends on the geometric situation in question). One instance where this can be observed is the index theorem for families, where the index is an element in the (topological) K-theory of the parameter space. Again, the index theorem gives a topological expression for this index.

In the version we are studying, the index takes values in the K-theory of a C^*-algebra associated to the index problem. In this context, one can make use of the fact that *positivity* implies invertibility to conclude that the index of a positive operator vanishes.

Geometric conditions sometimes imply positivity; the most prominent example is the Dirac operator on a spin manifold, which by the Lichnerowicz formula is positive if the manifold has a metric with positive scalar curvature. Consequently, the index of the Dirac operator is an obstruction to the existence of a metric with positive scalar curvature

(as long as the index is independent of the chosen metric; which follows e.g. from the index theorems).

A convenient modern way to describe the K-theory of C^*-algebras and the index element of a pseudo-differential operator is given by Kasparov's bivariant KK-theory. This theory also allows connections between the index of these operators, and topological invariants constructed from this operators and living in the K-homology of the underlying spaces. In this context, we will also introduce the Mishchenko-Fomenko index. This index can be used to define the Baum-Connes assembly map.

The Baum-Connes conjecture expresses the K-theory of C^*-algebras in terms of the K-homology of topological spaces; in those cases where it holds (and so far, no counterexamples are known) this provides new views on the index obstructions to positive scalar curvature; it allows for instance to prove the stable Gromov-Lawson-Rosenberg conjecture which gives a precise description of those spin manifolds which admit a metric with positive scalar curvature -upto a certain stabilization procedure.

A special invariant which can be read off from the Mishchenko-Fomenko index is the L^2-index of Atiyah; which also has an interpretation as a difference of (regularized) dimensions of kernel and cokernel, but now of an operator acting on a (typically non-compact) covering space of the manifold one started with. Atiyah proves that this index coincides with the classical finite dimensional index of the underlying operator. This allows the use of secondary L^2-invariants to classify e.g. different metrics of positive scalar curvature on a given manifold. For this, one uses related L^2-invariants, namely L^2-rho invariants (eta- and rho-invariants are introduced in the series of lectures of Paolo Piazza).

Further related L^2-invariants are the L^2-Betti numbers, which measure the dimension of the kernel of the Laplacian (on forms) of the universal covering. These have interesting properties; one of the most remarkable is that these numbers (a priori arbitrary positive reals) are always integers for large classes of torsion-free fundamental groups; this is related to the existence of zero divisors in the complex group ring. The Atiyah conjecture predicts that this integrality result always holds.

Some (optional) more refined invariants will round off the presentation, compare the list below.

1 Organisation of the talks

(1) First lecture: classical constructions

- Ellipticity of (pseudo)differential operators and index
- classical Atiyah-Singer index theorem; cohomological formula and K-theoretic approach

(2) Second lecture: toward a more modern approach

- Family index theory
- Topological and geometric applications of index theory

- C^*-algebras and their K-theory
- Basics about Kasparov's KK-theory

(3) Third lecture: Kasparov;s KK-theory and Index theory and L^2-invariants

- Kasparov's KK-theory
- index in KK-theory
- K-homology; the Baum-Connes conjecture
- The Mishchenko-Fomenko index
- The Gromov-Lawson-Rosenberg conjecture about positive scalar curvature
- Atiyah's L^2-index theorem
- Vanishing results for L^2-rho invariants (for the concept of rho- and eta-invariants, compare Paolo Piazza's talks)
- L^2-Betti numbers and how they behave (a survey); integrality of L^2-Betti numbers and approximation results for L^2-Betti numbers (not covered).

(4) Fourth lecture: Miscellaneous and left over subjects (depending on time used up)

- The index obstruction to positive scalar curvature coming from enlargeability (following Gromov-Lawson), and its relation to the Mishchenko-Fomenko index (not covered)
- The Novikov conjecture for low dimensional cohomology classes; using a canonically associated twisting C^*-algebra (not covered)
- Codimension 2 index obstructions to positive scalar curvature (a reinterpretation of another result of Gromov-Lawson in the context of higher index theory) (not covered)
- The signature operator and homotopy invariance of (higher) signatures (not covered)

2 Talk 1: Classical index theory

The Atiyah-Singer index theorem is one of the great achievements of modern mathematics. It gives a formula for the index of a differential operator (the index is by definition the dimension of the space of its solutions minus the dimension of the solution space for its adjoint operator) in terms only of topological data associated to the operator and the underlying space. There are many good treatments of this subject available, apart from the original literature (most found in [1]). Much more detailed than the present notes can be, because of constraints of length and time, are e.g. [15, 3, 9].

2.1 Abstract Fredholm index theory

Given a bounded operator $T\colon H^+ \to H^-$ between two Hilbert space, it is called *Fredholm* if it is invertible modulo compact operators. Recall that one possible way to define the compact operators $K(H^+, H^-)$ is as norm closure of the linear span of operators of the form $x \mapsto b\langle a, x\rangle$ for $a \in H^+, b \in H^-$.

A Fredholm operator has a finite dimensional kernel and cokernel, and therefore a well defined index

$$\operatorname{ind}(T) := \dim(\ker(T)) - \dim(\operatorname{coker}(T)) \in \mathbb{Z},$$

where $\operatorname{coker}(T) = H^- / \operatorname{im}(T) \cong \ker(T^*)$.

This index is very stable: it is unchanged under norm-small perturbations of the operator T, and also under arbitrary compact perturbations. Therefore, it defines a homomorphism

$$\pi_0(Gl_1(\mathcal{B}(H^+, H^-)/K(H^+, H^-)))$$

2.1 Remark. Using polar decomposition, one can replace T by another operator T' of the same index, but such that T' is even unitary modulo compact operators.

For a more compact notation, we can use the graded Hilbert space $H := H^+ \oplus H^-$ and the odd operator $\tilde{T} := \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ which contains exactly the same information.

2.2 Elliptic operators and their index

We quickly review what type of operators we are looking at. This will also fix the notation.

2.2 Definition. Let M be a smooth manifold of dimension m; E, F smooth (complex) vector bundles on M. A *differential operator* (of order d) from E to F is a $\mathbb{C}$-linear map from the space of smooth sections $C^\infty(E)$ of E to the space of smooth sections of F:

$$D\colon C^\infty(E) \to C^\infty(F),$$

such that in local coordinates and with local trivializations of the bundles it can be written in the form

$$D = \sum_{|\alpha| \le d} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here $A_\alpha(x)$ is a matrix of smooth complex valued functions, $\alpha = (\alpha_1, \dots, \alpha_m)$ is an m-tuple of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. $\partial^{|\alpha|}/\partial x^\alpha$ is an abbreviation for $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}$. We require that $A_\alpha(x) \neq 0$ for some α with $|\alpha| = d$ (else, the operator is of order strictly smaller than d).

Let $\pi\colon T^*M \to M$ be the bundle projection of the cotangent bundle of M. We get pull-backs π^*E and π^*F of the bundles E and F, respectively, to T^*M.

The *symbol* $\sigma(D)$ of the differential operator D is the section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ on T^*M defined as follows:

In the above local coordinates, using $\xi = (\xi_1, \dots, \xi_m)$ as coordinate for the cotangent vectors in T^*M, in the fiber of (x, ξ), the symbol $\sigma(D)$ is given by multiplication with

$$\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}$.

The operator D is called *elliptic*, if $\sigma(D)_{(x,\xi)} \colon \pi^*E_{(x,\xi)} \to \pi^*F_{(x,\xi)}$ is invertible outside the zero section of T^*M, i.e. in each fiber over $(x, \xi) \in T^*M$ with $\xi \neq 0$. Observe that elliptic operators can only exist if the fiber dimensions of E and F coincide.

2.3 Example. Let $M = \mathbb{R}^m$ and $D = \sum_{i=1}^m (\partial/\partial_i)^2$ be the Laplace operator on functions. This is an elliptic differential operator of second order, with symbol $\sigma(D)_{(x,\xi)} = \sum_{i=1}^m \xi_i^2$.

Similarly, the Laplacian d^*d on functions of a Riemannian manifold M is elliptic with symbol $\sigma(d^*d)_{x,\xi} = |\xi|^2_{T_x^*M}$.

This operators are essentially self adjoint, therefore there index is zero.

More interesting are e.g.:

(1) On a Riemannian manifold the Euler characteristic operator $d+d^* \colon \Omega^{ev}(M) \to \Omega^{odd}(M)$ from differential forms of even degree to differential forms of odd degree. By the Hodge-de Rham theorem, its index is the Euler characteristic $\chi(M)$. Its square is the Laplacian (on forms) Δ, with $\sigma(\Delta)_{x,xi} = |\xi|^2_{T_x^*M} \,\mathrm{id}_{\Lambda_x^*M}$. Since the principal symbols is an algebra map, also the symbol of $d+d^*$ is invertible on S^*M, i.e. the Euler characteristic operator is elliptic.

(2) If M is an oriented Riemannian manifold, the operator $d + d^* \colon \Omega^+(M) \to \Omega^-(M)$ from even to odd forms with respect to the signature grading operator $\tau := i^{p(p-1)+2k}*$, where $*$ is the Hodge-$*$ operator given by the Riemannian metric, and $i^2 = -1$ on a $4k$-dimensional manifold is elliptic by the same argument as the Euler characteristic operator. Its index is the signature of the symmetric cohomology pairing $H^{2k}(M) \times H^{2k}(M) \xrightarrow{\cup} H^{4k}(M) \xrightarrow{\int} \mathbb{R}$ (where the interation uses the orientation). Since $\dim M$ is divisible by 4, an easy calculation shows that $\tau^2 = \mathrm{id}$. We then define $\Omega^\pm$ to be the ± 1 eigenspaces of τ.

(3) On a Riemannian spin manifold, the Dirac operator $D \colon \Gamma(S^+) \to \Gamma(S^-)$ is another elliptic operator.

Given any Hermitean bundle E on M with connection, we can define the twisted Dirac operator $D_E \colon C^i nfty(S^+ \otimes E) \to C^\infty(S^- \otimes E)$. It is also an elliptic differential operator. As a matter of fact, locally all geometrically relevant operators are of this type. Therefore it usually suffices to study them. More details about (generalized) Dirac operators are given in Paolo Piazzas talks.

We want to mention one key identity, the *Bochner-Weitzenböck* formula

$$D_E^2 = \nabla^*\nabla + \frac{\text{scal}}{4} + K_E.$$

This is important because $\nabla^*\nabla$ is a non-negative operator, which as a (total) covariant derivate has particularly nice properties. $\frac{\text{scal}}{4}$ stands for the multiplication operator with the function $\text{scal}/4$, and K_E is another fiberwise operator (smooth section of the bundle $\text{Hom}(S^+ \otimes E, S^- \otimes E)$), given in terms of the curvatrure of the connection on E.

The proof of the analytic properties of elliptic operators of Theorem 2.5 for Dirac type operators can be based on the Weitzenböck formula.

2.4 Remark. The class of differential operators is quite restricted. Many constructions one would like to carry out with differential operators automatically lead out of this class. Therefore, one often has to use *pseudodifferential operators.* Pseudodifferential operators are defined as a generalization of differential operators. There are many well written sources dealing with the theory of pseudodifferential operators. Since we will not discuss them in detail here, we omit even their precise definition and refer e.g. to [15] and [20].

We now want to state several important properties of elliptic operators.

2.5 Theorem. *Let M be a smooth manifold, E and F smooth finite dimensional vector bundles over M. Let $P\colon C^\infty(E) \to C^\infty(F)$ be an elliptic operator.*

Then the following holds.

(1) Elliptic regularity:
If $f \in L^2(E)$ is weakly in the null space of P, i.e. $\langle f, P^*g\rangle_{L^2(E)} = 0$ for all $g \in C_0^\infty(F)$, then $f \in C^\infty(E)$.

(2) The operator D^*D is essentially self adjoint (i.e. its closure is an unbounded self adjoint operator). Consequently, functional calculus for unbounded operators is available for it.

(3) A generalization of elliptic regularity holds: the operator $D(1+D^*D)^{-1/2}$ is bounded and, on a closed manifold, Fredholm (even unitary modulo compact operators).

We define $\text{ind}(D) := \text{ind}(D(1+D^*D)^{-1/2}$. *Actually, by elliptic regularity, the index can also be calculated from the kernel and cokernel of D, acting on smooth sectons.*

2.3 Index and K-theory

Recall the following definition:

2.6 Definition. Let X be a compact topological space. We define the K-theory of X, $K^0(X)$, to be the Grothendieck group of (isomorphism classes of) complex vector bundles over X (with finite fiber dimension). More precisely, $K^0(X)$ consists of equivalence classes of pairs (E,F) of (isomorphism classes of) vector bundles over X, where $(E,F) \sim (E',F')$ if and only if there exists another vector bundle G on X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. One often writes $[E]-[F]$ for the element of $K^0(X)$ represented by (E,F).

Let Y now be a closed subspace of X. The *relative K-theory* $K^0(X,Y)$ is given by equivalence classes of triples (E,F,ϕ), where E and F are complex vector bundles over X, and $\phi\colon E|_Y \to F|_Y$ is a given isomorphism between the restrictions of E and F to Y. Then (E,F,ϕ) is isomorphic to (E',F',ϕ') if we find isomorphisms $\alpha\colon E \to E'$ and $\beta\colon F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} E|_Y & \xrightarrow{\phi} & F|_Y \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ E'|_Y & \xrightarrow{\phi'} & F'|_Y \end{array}$$

Two pairs (E,F,ϕ) and (E',F',ϕ') are equivalent, if there is a bundle G on X such that $(E\oplus G, F\oplus G, \phi\oplus \mathrm{id})$ is isomorphic to $(E'\oplus G, F'\oplus G, \phi'\oplus\mathrm{id})$.

2.7 Example. The symbol of an elliptic operator gives us two vector bundles over T^*M, namely π^*E and π^*F, together with a choice of an isomorphism $\sigma(D)_{(x,\xi)}\colon E_{(x,\xi)} \to F_{(x,\xi)}$ for $(x,\xi)\in ST^*M$ of the fibers of these two bundles outside the zero section. If M is compact, this gives an element of the relative K-theory group $K^0(DT^*M, ST^*M)$, where DT^*M and ST^*M are the disc bundle and sphere bundle of T^*M, respectively (with respect to some arbitrary Riemannian metric).

It turns out that the index of the elliptic operator D does only depend on the corresponding K-theory class $[\sigma(D)] \in K^0(D^*M, S^M)$. Since (at least when one also allows to use pseudodifferential operators) every such K-theory class occurs as a symbol class, this defines a homomorphism

$$\mathrm{ind}_a\colon K^0(D^*M, S^*M) \to \mathbb{Z},$$

the *analytic index*.

Using Bott preriodicity and the Thom homomorphism in vector bundle K-theory, Atiyah and Singer define, in a purely topological way, another homomorphism

$$\mathrm{ind}_t\colon K^0(D^*M, S^*M) \to K^0(*) = \mathbb{Z}.$$

This is done as follows: Take the element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic operator. Embed M into high dimensional Euclidean space $\mathbb{R}^N$. This gives an embedding of T^*M into $\mathbb{R}^{2N}$, and further into its one point compatification S^{2N}, with normal bundle ν. In this situation, ν has a canonical complex structure. The embedding now defines a transfer map

$$K^0(DT^*M, ST^*M) \to K^0(S^{2N}, \infty),$$

by first using the Thom isomorphism to map to the (compactly supported) K-theory of the normal bundle, and then push forward to the K-theory of the sphere. The latter map is given by extending a vector bundle on the open subset ν of S^{2N} which is trivialized outside a compact set (i.e. represents an element in compactly supported K-theory) trivially to all of S^{2N}.

Compose with the Bott periodicity isomorphism to map to $K^0(pt) = \mathbb{Z}$. The image of the symbol element under this homomorphism is denoted the *topological index* $\mathrm{ind}_t(D) \in K^0(*) = \mathbb{Z}$. The reason for the terminology is that it is obtained from the symbol only, using purely topological constructions. The Atiyah-Singer index theorem states that analytical and topological index coincide:

2.8 Theorem. $\mathrm{ind}_t(D) = \mathrm{ind}_a(D)$.

To prove this, one can show that the topological index is uniquely characterized by a couple of properties it has more or less by definition. Then, one shows with a lot of effort; relying on general properties of the analytic index and a very small number of index calculations, that ind_a also satisfies these properties.

2.4 Characteristic classes

For explicit formulas for the index of a differential operator —which can be derived from the topological index— we will have to use characteristic classes of certain bundles involved. Therefore, we quickly review the basics about the theory of characteristic classes.

2.9 Theorem. *Given a compact manifold M (or actually any finite CW-complex), there is a bijection between the isomorphism classes of n-dimensional complex vector bundles on M, and the set of homotopy classes of maps from M to $BU(n)$, the classifying space for n-dimensional vector bundles. $BU(n)$ is by definition the space of n-dimensional subspaces of $\mathbb{C}^\infty$ (with an appropriate limit topology).*

The isomorphism is given as follows: On $BU(n)$ there is the tautological n-plane bundle $E(n)$, the fiber at each point of $BU(n)$ just being the subspace of $\mathbb{C}^\infty$ which represents this point. Any map $f\colon M \to BU(n)$ gives rise to the pull back bundle $f^*E(n)$ on M. The theorem states that each bundle on M is isomorphic to such a pull back, and that two pull backs are isomorphic if and only the maps are homotopic.

2.10 Definition. A characteristic class c of vector bundles assigns to each vector bundle E over M an element $c(E) \in H^*(M)$ which is natural, i.e. which satisfies

$$c(f^*E) = f^*c(E) \qquad \forall f\colon M \to N, \quad E \text{ vector bundle over } N.$$

It follows that characteristic classes are given by cohomology classes of $BU(n)$.

2.11 Theorem. *The integral cohomology ring $H^*(BU(n))$ is a polynomial ring in generators $c_0 \in H^0(BU(n))$, $c_1 \in H^2(BU(n))$, ..., $c_n \in H^{2n}(BU(n))$. We call these generators the* Chern classes *of the tautological bundle $E(n)$ of Theorem 2.9, $c_i(E(n)) := c_i$.*

2.12 Definition. Write a complex vector bundle E over M as $f^*E(n)$ for $f\colon M \to BU(n)$ appropriate. Define $c_i(fE) := f^*(c_i) \in H^{2i}(M;\mathbb{Z})$, this is called the *i-th Chern class* of the bundle $E = f^*E(n)$.

If F is a real vector bundle over M, define the Pontryagin classes

$$p_i(F) := c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(M;\mathbb{Z}).$$

(The odd Chern classes of the complexification of a real vector bundle are two torsion and therefore are usually ignored).

2.4.1 Splitting principle

2.13 Theorem. *Given a manifold M and a vector bundle E over M, there is another manifold N together with a map $\phi\colon N \to M$, which induces a monomorphism $\phi^U\colon H^*(M;\mathbb{Z}) \to H^*(N;\mathbb{Z})$, and such that $\phi^*E = L^1 \oplus \dots L^n$ is a direct sum of line bundles.*

Using Theorem 2.13, every question about characteristic classes of vector bundles can be reduced to the corresponding question for line bundles, and questions about the behavior under direct sums.

In particular, the following definitions makes sense:

2.14 Definition. The Chern character is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space M a cohomology class $\mathrm{ch}(E) \in H^*(M;\mathbb{Q})$. It is characterized by the following properties:

(1) Normalization: If L is a complex line bundle with first Chern class x, then

$$\mathrm{ch}(L) = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in H^*(M;\mathbb{Q}).$$

Observe that in particular $\mathrm{ch}(\mathbb{C}) = 1$.

(2) Additivity: $L(E \oplus F) = L(E) + L(F)$.

2.15 Proposition. *The Chern character is not only additive, but also multiplicative in the following sense: for two vector bundles E, F over M we have*

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cup \mathrm{ch}(F).$$

2.16 Definition. The *Hirzebruch L-class* as normalized by Atiyah and Singer is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space M a cohomology class $L(E) \in H^*(M;\mathbb{Q})$. It is characterized by the following properties:

(1) Normalization: If L is a complex line bundle with first Chern class x, then

$$L(L) = \frac{x/2}{\tanh(x/2)} = 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \in H^*(M;\mathbb{Q}).$$

Observe that in particular $L(\mathbb{C}) = 1$.

(2) Multiplicativity: $L(E \oplus F) = L(E)L(F)$.

2.17 Definition. The *Todd-class* is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space M a cohomology class $\mathrm{Td}(E) \in H^*(M;\mathbb{Q})$. It is characterized by the following properties:

(1) Normalization: If L is a complex line bundle with first Chern class x, then

$$\mathrm{Td}(L) = \frac{x}{1 - \exp(-x)} \in H^*(X;\mathbb{Q}).$$

Observe that in particular $\mathrm{Td}(\mathbb{C}) = 1$.

(2) Multiplicativity: $L(E \oplus F) = L(E)L(F)$.

Note that ch as well as L and Td take values in the even dimensional part

$$H^{ev}(M;\mathbb{Q}) := \oplus_{k=0}^{\infty} H^{2k}(M;\mathbb{Q}).$$

2.4.2 Chern-Weyl theory

Chern-Weyl theory can be used to explicitly compute characteristic classes of finite dimensional vector spaces. For a short description compare [16]. To carry out the Chern-Weyl procedure, one has to choose a connection on the given vector bundle E. This connection has a curvature Ω, which is a two form with values in the endomorphism bundle of the given vector bundle. Using the cup product of differential forms and the composition of Endomorphisms, we can also form the powers Ω^k, forms of degree $2k$ with values in the endomorphism bundle of E. Applying the ordinary trace of Endomorphisms, we get from this an ordinary differential form $\mathrm{tr}(\Omega^k)$ of degree $2k$ (with complex coefficients).

2.18 Theorem. *For any finite dimension vector bundle E (over a smooth manifold M) with connection with curvature Ω, its Chern character is given by*

$$\mathrm{ch}(E) = \sum_k \frac{1}{(2\pi i)^k} \frac{\mathrm{tr}(\Omega^k)}{k!} \in H^{2k}(M;\mathbb{C}).$$

Since all other (complex valued) characteristic classes of complex vector bundles are given in terms of the Chern character, this gives an explicit way to calculate arbitrary characteristic classes.

2.4.3 Stable characteristic classes and K-theory

The elements of $K^0(X)$ are represented by vector bundles. Therefore, it makes sense to ask whether a characteristic class of vector bundles can be used to define maps from $K^0(X)$ to $H^*(X)$.

It turns out, that this is not always the case. The obstacle is, that two vector bundles E, F represent the *same* element in $K^0(X)$ if (and only if) there is $N \in \mathbb{N}$ such that $E \oplus \mathbb{C}^N \cong F \oplus \mathbb{C}^N$. Therefore, we have to make sure that $c(E) = c(F)$ in this case. A characteristic class which satisfies this property is called *stable*, and evidently induces a map

$$c\colon K^0(X) \to H^*(X).$$

We deliberately did not specify the coefficients to be taken for cohomology, because most stable characteristic classes will take values in $H^*(X;\mathbb{Q})$ instead of $H^*(X;\mathbb{Z})$.

The following proposition is an immediate consequence of the definition:

2.19 Proposition. *Assume a characteristic class c is multiplicative, i.e.* $c(E \oplus F) = c(E) \cup c(F) \in H^*(X)$, *and* $c(\mathbb{C}) = 1$. *Then c is a stable characteristic class.*

Assume a characteristic class c is additive, i.e. $c(E \oplus F) = c(E) + c(F)$. *Then c is a stable characteristic class.*

It follows in particular that the Chern character, as well as Hirzebruch's L-class are stable characteristic classes, i.e. they define maps from the K-theory $K^0(X) \to H^*(X;\mathbb{Q})$.

The relevance of the Chern character becomes apparent by the following theorem.

2.20 Theorem. *For a finite CW complex X,*

$$\mathrm{ch} \otimes \mathrm{id}_{\mathbb{Q}}\colon K^0(X) \otimes \mathbb{Q} \to H^{ev}(X;\mathbb{Q}) \otimes \mathbb{Q} = H^{ev}(X;\mathbb{Q})$$

is an isomorphism.

We have constructed *relative* K-theory $K^0(X,A)$ in terms of pairs of vector bundles on X with a given isomorphism of the restrictions to A. We can always find representatives such that one of the bundles is trivialized, and the other one E has in particular a trivialization $E|_A = \mathbb{C}^n$ of its restriction to A. Such vector bundles correspond to homotopy classes $[(X,A);(BU(n),pt)]$ of maps from X to $BU(n)$ which map A to a fixed point *pt* in $BU(n)$.

For $k > 0$, we define relative Chern classes $c_k(E, E|_A = \mathbb{C}^n) \in H^{2k}(X,A;\mathbb{Z})$ as pull back of $c_k \in H^{2k}(BU(n),pt) \cong H^{2k}(BU(n))$. The splitting principle also holds for such relative vector bundles, and therefore all the definitions we have made above go through in this relative situation. In particular, we can define a Chern character

$$\mathrm{ch}\colon K^0(X,A) \to H^{ev}(X,A;\mathbb{Q}).$$

Given an elliptic differential operator D, we can apply this to our symbol element

$$\sigma(D) \in K^0(DT^*M, ST^*M),$$

to obtain $\mathrm{ch}(\sigma(D))$.

2.21 Proposition. *Given a smooth manifold M of dimension m, there is a homomorphism*

$$\pi_!\colon H^{k+m}(DT^*M, ST^*M; \mathbb{R}) \to H^k(M),$$

called integration along the fiber. *It is defined as follows: let $\omega \in \Omega^{k+m}(DT^*M)$ be a closed differential form representing an element in $H^{k+m}(DT^*M, ST^*M)$ (i.e. with vanishing restriction to the boundary). Locally, one can write $\omega = \sum \alpha_i \cup \beta_i$, where β_i are differential forms on M pulled back to DT^*M via the projection map $\pi\colon DT^*M \to M$, and α_i are pulled back from the fiber in a local trivialization. Then $\pi_!\omega$ is represented by*

$$\sum_i (\int_{DT_x^*M} \alpha)\beta_i.$$

For more details about integration along the fiber, consult [5, Section 6]

2.5 Cohomological version of the Atiyah-Singer index theorem

There are different variants of the Atiyah-Singer index theorem. We start with a cohomological formula for the index.

2.22 Theorem. *Let M be a compact oriented manifold of dimension m, and $D\colon C^\infty(E) \to C^\infty(F)$ an elliptic operator with symbol $\sigma(D)$. Define the* Todd character $\mathrm{Td}(M) := \mathrm{Td}(TM \otimes \mathbb{C}) \in H^*(M; \mathbb{Q})$. *Then*

$$\mathrm{ind}(D) = (-1)^{m(m+1)/2} \int_M \pi_! \,\mathrm{ch}(\sigma(D)) \cup \mathrm{Td}(M).$$

For the characteristic classes, compare Subsection 2.4.

This formula is obtained from the topological index by observing that

$$\int_M \pi_!\colon H^*(D^*M, S^*M; \mathbb{Q}) \to H^*(*; \mathbb{Q}) = \mathbb{Q}$$

is a close relative of the topological index map in K-theory, but that the evident map involving the Chern character which connects this map and ind_t is not commutative; the defect is given precisely by multiplication with the Todd class of the complexified tangent bundle.

If we start with specific operators given by the geometry, explicit calculation usually give more familiar terms on the right hand side.

For example, for the signature operator we obtain Hirzebruch's signature formula expressing the signature in terms of the L-class, for the Euler characteristic operator we obtain the Gauss-Bonnet formula expressing the Euler characteristic in terms of the Pfaffian, and for the spin or spin^c Dirac operator we obtain an $\hat{A}$-formula. For applications, these formulas prove to be particularly useful.

The Atiyah-Singer index theorem now specializes to

$$\operatorname{sign}(M) = \operatorname{ind}(D_{sig}) = \langle 2^{2k} L(TM), [M] \rangle,$$

with $\dim M = 4k$ as above.

2.23 Remark. One direction to generalize the Atiyah-Singer index theorem is to give an index formula for manifolds with boundary. Indeed, this is achieved in the Atiyah-Patodi-Singer index theorem. However, these results are much less topological than the results for manifolds without boundary. They are not discussed in these notes, but by the talks of Paolo Piazza.

2.6 Families of operators and their index

Another important generalization is given if we don't look at one operator on one manifold, but a family of operators on a family of manifolds.

This is an interesting topic in its own right, and has useful applications (we will mention one application to positive scalar curvature later). Moreover, it turns out that the true nature of a mathematical question often becomes much more transparent when one is studying deformations/families of the objects in question.

Therefore, let X be any compact topological space, $\pi\colon Y \to X$ a locally trivial fiber bundle with fibers $Y_x := \pi^{-1}(x) \cong M$ smooth compact manifolds ($x \in X$), and structure group the diffeomorphisms of the typical fiber M. Let E, F be families of smooth vector bundles on Y (i.e. vector bundles which are smooth for each fiber of the fibration $Y \to X$), and $C^\infty(E)$, $C^\infty(F)$ the continuous sections which are smooth along the fibers. More precisely, E and F are smooth fiber bundles over X, the typical fiber is a vector bundle over M, and the structure group consists of diffeomorphisms of this vector bundle which are fiberwise smooth.

Assume that $D\colon C^\infty(E) \to C^\infty(F)$ is a family $\{D_x\}$ of elliptic differential operator along the fiber $Y_x \cong M$ ($x \in X$), i.e., in local coordinates D becomes

$$\sum_{|\alpha| \le m} A_\alpha(y, x) \frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

with $y \in M$ and $x \in X$ such that $A_\alpha(y, x)$ depends continuously on x, and each D_x is an elliptic differential operator on Y_x.

If $\dim_{\mathbb{C}} \ker(D_x)$ is independent of $x \in X$, then all of these vector spaces patch together to give a vector bundle called $\ker(D)$ on X, and similarly for the (fiberwise) adjoint D^*. This then gives a K-theory element $[\ker(D)] - [\ker(D^*)] \in K^0(X)$.

Unfortunately, it does sometimes happen that these dimensions jump. However, using appropriate perturbations or stabilizations, one can always define the K-theory element

$$\mathrm{ind}(D) := [\ker(D)] - [\ker(D^*)] \in K^0(X),$$

the analytic index of the family of elliptic operators D. For details on this and the following material, consult e.g. [15, Paragraph 15].

We define the symbol of D (or rather a family of symbols) exactly as in the non-parametrized case. This gives now rise to an element in $K^0(DT_v^*Y, ST_v^*Y)$, where T_v^*Y is the cotangent bundle along the fibers. Note that all relevant spaces here are fiber bundles over X, with typical fiber T^*M, DT^*M or ST^*M, respectively.

Now we proceed with a family version of the construction of the topological index, copying the construction in the non-family situation, and using

- a (fiberwise) embedding of Y into $\mathbb{R}^N \times X$ (which is compatible with the projection maps to X)
- the Thom isomorphism for families of vector bundles
- the family version of Bott periodicity, namely

  $$K^0(S^{2N} \times X, \{\infty\} \times X) \xrightarrow{\cong} K^0(X).$$

  (Instead, one could also use the Künneth theorem together with ordinary Bott periodicity.)

This gives rise to $\mathrm{ind}_t(D) \in K^0(X)$. The Atiyah-Singer index theorem for families states:

2.24 Theorem. $\mathrm{ind}(D) = \mathrm{ind}_t(D) \in K^0(X)$.

The upshot of the discussion of this and the last section (for the details the reader is referred to the literature) is that the natural receptacle for the index of differential operators in various situations are appropriate K-theory groups, and much of todays index theory deals with investigating these K-theory groups.

Moreover, we have seen a particular problem occur: the index bundle is not always defined (because of the dimension jumps). I suggest the following solution of this problem: find a different, more flexible definition of K-theory where this (and other) indices are automatically contained. Actually, the idea is to define the group as group of equivalence classes of index problems —this will be done via KK-theory. Of course, the big taks then is to calculate these KK-groups, in particular to show that (in special cases) they give the K-groups we know. It turns out that KK-theory is particularly powerful because it allows efficient calculations using the Kasparov product.

2.7 Geometric and Topological consequence of the index theorem

The index theorem has many powerful applications. We only lift a few of those.

(1) If we apply the theorem to the signature operator or the Euler characteristic operator, we get interesting formulas for the signature and Euler characteristic, respectively, in terms of characteristic classes of the tangent bundle. From this, we can e.g. derive immediately, that both invariants vanish for manifolds with a flat unitary connection on the tangent bundle, because then the corresponding characteristic classes vanish by Chern-Weyl theory.

(2) Since the index of an operator is by definition an integer, the same is true for the cohomological expression of the index, a priori a rational number. E.g., if M is a closed spin manifold, $\int_M \hat{A}(M) \in \mathbb{Z}$. It turns out that this integrality does not hold for non-spin manifolds in general. It was one of the motivations for Atiyah-Singer to explain such integrality results, which let them develop the index theorem.

(3) The Weitzenböck formula $D^2 = \nabla^*\nabla + \operatorname{scal}/4$ for the Dirac operator on a spin manifold M implies the following: if $\operatorname{scal} > 0$ then the operator D^2 is positive, therefore invertible, therefore the index of D^+ vanishes (because both the kernel of D^+ and of $(D^+)^* = D^-$ are trivial). Consequenly, under this assumption, $\int_M \hat{A}(M) = 0$.

    In other words, if $\int_M \hat{A}(M) \neq 0$ for a spin manifold M, like the K3-surface, then this manifolds does not admit a Riemannian metric with positive scalar curvature.

    The spin condition is essential, as shows $\mathbb{C}P^2$, which has a metric of positive sectional curvature, but non-trivial $\hat{A}$-genus (and which does not have a spin structure).

    This simple method does not apply to flat manifolds like T^n because by Chern-Weyl theory $\int T^n \hat{A}(T^n) = 0$. However, using the family index theorem, one can show that T^n does not admit a metric of positive scalar curvature, either.

3 Survey on C^*-algebras and their K-theory

More detailed references for this section are, among others, [23], [9], and [4].

3.1 C^*-algebras

3.1 Definition. A *Banach algebra* A is a complex algebra which is a complete normed space, and such that $|ab| \leq |a|\,|b|$ for each $a, b \in A$.

A $*$*-algebra* A is a complex algebra with an anti-linear involution $*\colon A \to A$ (i.e. $(\lambda a)^* = \overline{\lambda} a^*$, $(ab)^* = b^*a^*$, and $(a^*)^* = a$ for all $a, b \in A$).

A *Banach $*$-algebra* A is a Banach algebra which is a $*$-algebra such that $|a^*| = |a|$ for all $a \in A$.

A *C^*-algebra* A is a Banach $*$-algebra which satisfies $|a^*a| = |a|^2$ for all $a \in A$.

Alternatively, a C^*-algebra is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed subalgebra of the algebra of bounded operators on some Hilbert space H (this is the Gelfand-Naimark representation theorem, compare e.g. [9, 1.6.2]).

A C^*-algebra A is called separable if there exists a countable dense subset of A.

3.2 Example. If X is a compact topological space, then $C(X)$, the algebra of complex valued continuous functions on X, is a commutative C^*-algebra (with unit). The adjoint is given by complex conjugation: $f^*(x) = \overline{f(x)}$, the norm is the supremum-norm.

Conversely, it is a theorem that every abelian unital C^*-algebra is isomorphic to $C(X)$ for a suitable compact topological space X [9, Theorem 1.3.12].

Assume X is locally compact, and set

$$C_0(X) := \{f\colon X \to \mathbb{C} \mid f \text{ continuous}, f(x) \xrightarrow{x\to\infty} 0\}.$$

Here, we say $f(x) \to 0$ for $x \to \infty$, or f *vanishes at infinity*, if for all $\epsilon > 0$ there is a compact subset K of X with $|f(x)| < \epsilon$ whenever $x \in X - K$. This is again a commutative C^*-algebra (we use the supremum norm on $C_0(X)$), and it is unital if and only if X is compact (in this case, $C_0(X) = C(X)$).

3.2 K_0 of a ring

Suppose R is an arbitrary ring with 1 (not necessarily commutative). A module M over R is called finitely generated projective, if there is another R-module N and a number $n \geq 0$ such that

$$M \oplus N \cong R^n.$$

This is equivalent to the assertion that the matrix ring $M_n(R) = End_R(R^n)$ contains an idempotent e, i.e. with $e^2 = e$, such that M is isomorphic to the image of e, i.e. $M \cong eR^n$.

3.3 Example. Description of projective modules.

(1) If R is a field, the finitely generated projective R-modules are exactly the finite dimensional vector spaces. (In this case, every module is projective).

(2) If $R = \mathbb{Z}$, the finitely generated projective modules are the free abelian groups of finite rank

(3) Assume X is a compact topological space and $A = C(X)$. Then, by the Swan-Serre theorem [22], M is a finitely generated projective A-module if and only if M is isomorphic to the space $\Gamma(E)$ of continuous sections of some complex vector bundle E over X.

3.4 Definition. Let R be any ring with unit. $K_0(R)$ is defined to be the Grothendieck group of finitely generated projective modules over R, i.e. the group of equivalence classes $[(M,N)]$ of pairs of (isomorphism classes of) finitely generated projective R-modules M, N, where $(M,N) \equiv (M',N')$ if and only if there is an $n \geq 0$ with

$$M \oplus N' \oplus R^n \cong M' \oplus N \oplus R^n.$$

The group composition is given by

$$[(M,N)] + [(M',N')] := [(M \oplus M', N \oplus N')].$$

We can think of (M,N) as the formal difference of modules $M - N$.

Any unital ring homomorphism $f\colon R \to S$ induces a map

$$f_*\colon K_0(R) \to K_0(S)\colon [M] \mapsto [S \otimes_R M],$$

where S becomes a right R-module via f. We obtain that K_0 is a covariant functor from the category of unital rings to the category of abelian groups.

3.5 Example. Calculation of K_0.

- If R is a field, then $K_0(R) \cong \mathbb{Z}$, the isomorphism given by the dimension: $\dim_R(M,N) := \dim_R(M) - \dim_R(N)$.
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$, given by the rank.
- If X is a compact topological space, then $K_0(C(X)) \cong K^0(X)$, the topological K-theory given in terms of complex vector bundles. To each vector bundle E one associates the $C(X)$-module $\Gamma(E)$ of continuous sections of E.
- Let G be a discrete group. The group algebra $\mathbb{C}G$ is a vector space with basis G, and with multiplication coming from the group structure, i.e. given by $g \cdot h = (gh)$.

  If G is a finite group, then $K_0(\mathbb{C}G)$ is the complex representation ring of G.

3.3 K-Theory of C^*-algebras

3.6 Definition. Let A be a unital C^*-algebra. Then $K_0(A)$ is defined as in Definition 3.4, i.e. by forgetting the topology of A.

3.3.1 K-theory for non-unital C^*-algebras

When studying (the K-theory of) C^*-algebras, one has to understand morphisms $f\colon A \to B$. This necessarily involves studying the kernel of f, which is a closed ideal of A, and hence a *non-unital* C^*-algebra. Therefore, we proceed by defining the K-theory of C^*-algebras without unit.

3.7 Definition. To any C^*-algebra A, with or without unit, we assign in a functorial way a new, unital C^*-algebra A_+ as follows. As $\mathbb{C}$-vector space, $A_+ := A \oplus \mathbb{C}$, with product

$$(a, \lambda)(b, \mu) := (ab + \lambda a + \mu b, \lambda\mu) \qquad \text{for } (a, \lambda), (b, \mu) \in A \oplus \mathbb{C}.$$

The unit is given by $(0, 1)$. The star-operation is defined as $(a, \lambda)^* := (a^*, \overline{\lambda})$, and the new norm is given by

$$|(a, \lambda)| = \sup\{|ax + \lambda x| \mid x \in A \text{ with } |x| = 1\}$$

3.8 Remark. A is a closed ideal of A_+, the kernel of the canonical projection $A_+ \twoheadrightarrow \mathbb{C}$ onto the second factor. If A itself is unital, the unit of A is of course different from the unit of A_+.

3.9 Example. Assume X is a locally compact space, and let $X_+ := X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X)_+ \cong C(X_+).$$

The ideal $C_0(X)$ of $C_0(X)_+$ is identified with the ideal of those functions $f \in C(X_+)$ such that $f(\infty) = 0$.

3.10 Definition. For an arbitrary C^*-algebra A (not necessarily unital) define

$$K_0(A) := \ker(K_0(A_+) \to K_0(\mathbb{C})).$$

Any C^*-algebra homomorphisms $f\colon A \to B$ (not necessarily unital) induces a unital homomorphism $f_+\colon A_+ \to B_+$. The induced map

$$(f_+)_*\colon K_0(A_+) \to K_0(B_+)$$

maps the kernel of the map $K_0(A_+) \to K_0(\mathbb{C})$ to the kernel of $K_0(B_+) \to K_0(\mathbb{C})$. This means it restricts to a map $f_*\colon K_0(A) \to K_0(B)$. We obtain a covariant functor from the category of (not necessarily unital) C^*-algebras to abelian groups.

Of course, we need the following result.

3.11 Proposition. *If A is a unital C^*-algebra, the new and the old definition of $K_0(A)$ are canonically isomorphic.*

3.3.2 Higher topological K-groups

3.12 Definition. Let A be a unital C^*-algebra. Then $Gl_n(A)$ becomes a topological group, and we have continuous embeddings

$$Gl_n(A) \hookrightarrow Gl_{n+1}(A)\colon X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We set $Gl_\infty(A) := \lim_{n\to\infty} Gl_n(A)$, and we equip $Gl_\infty(A)$ with the direct limit topology.

3.13 Proposition. *Let A be a unital C^*-algebra. If $k \geq 1$, then*

$$K_k(A) = \pi_{k-1}(Gl_\infty(A))(\cong \pi_k(BGl_\infty(A))).$$

Observe that any unital morphism $f\colon A \to B$ of unital C^*-algebras induces a map $Gl_n(A) \to Gl_n(B)$ and therefore also between $\pi_k(Gl_\infty(A))$ and $\pi_k(Gl_\infty(B))$. This map coincides with the previously defined induced map in topological K-theory.

3.14 Remark. Note that the topology of the C^*-algebra enters the definition of the higher topological K-theory of A, and in general the topological K-theory of A will be vastly different from the algebraic K-theory of the algebra underlying A. For connections in special cases, compare [21].

3.15 Example. It is well known that $Gl_n(\mathbb{C})$ is connected for each $n \in \mathbb{N}$. Therefore

$$K_1(\mathbb{C}) = \pi_0(Gl_\infty(\mathbb{C})) = 0.$$

A very important result about K-theory of C^*-algebras is the following long exact sequence. A proof can be found e.g. in [9, Proposition 4.5.9].

3.16 Theorem. *Assume I is a closed ideal of a C^*-algebra A. Then, we get a short exact sequence of C^*-algebras $0 \to I \to A \to A/I \to 0$, which induces a long exact sequence in K-theory*

$$\to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(I) \to \cdots \to K_0(A/I).$$

3.4 Bott periodicity and the cyclic exact sequence

One of the most important and remarkable results about the K-theory of C^*-algebras is Bott periodicity, which can be stated as follows.

3.17 Theorem. *Assume A is a C^*-algebra. There is a natural isomorphism, called the Bott map*

$$K_0(A) \to K_0(S^2A),$$

which implies immediately that there are natural isomorphism

$$K_n(A) \cong K_{n+2}(A) \qquad \forall n \geq 0.$$

3.18 Remark. Bott periodicity allows us to define $K_n(A)$ for each $n \in \mathbb{Z}$, or to regard the K-theory of C^*-algebras as a $\mathbb{Z}/2$-graded theory, i.e. to talk of $K_n(A)$ with $n \in \mathbb{Z}/2$. This way, the long exact sequence of Theorem 3.16 becomes a (six-term) cyclic exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d, "\mu_*"] \\
K_1(A/I) \arrow[u] & K_1(A) \arrow[l] & K_1(I). \arrow[l]
\end{tikzcd}$$

The connecting homomorphism μ_* is the composition of the Bott periodicity isomorphism and the connecting homomorphism of Theorem 3.16.

3.5 Survey on KK-theory

We now want to give the "natural" construction of K-theory in terms of index problems. We first have to consider the "spaces" on which these index problems are given. In the classical case, these are just operators on Hilbert space. However, if we have a family index problem, we actually have for each point in the base a Hilbert space (sections of some bundle along the fiber), and these Hilbert spaces depend contiuously along the fiber; they form indeed a bundle of Hilbert spaces on the base, and we have to study continuous sections of this bundle. This is axiomatized and generalized in the notion of a Hilbert A-module, for a C^*-algebra A.

From now on, for technical reasons we assume that our C^*-algebras are separable and sigma-unital.

3.19 Definition. Given a C^*-algebra A, a Hilbert A-module H is a right A-module together with an A-valued innerproduct $\langle \cdot, \cdot \rangle \colon H \times H \to A$ with the following properties:

(1) the inner product is A-linear in the second variable

(2) sesqui-linearity: $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in H$, where we use the $*$ of the C^*-algebra A

(3) Positivity: $\langle f, f \rangle > 0$ for $f \in 0$, where we use the concept of positivity in the C^*-algebra A.

(4) This implies that $f \mapsto |\langle f, f \rangle|_A^{1/2}$ is a norm on H. We require that H is complete, i.e. a Banach space, with respect to this norm.

3.20 Example. If $A = \mathbb{C}$, Hilbert A-modules are just Hilbert spaces.

For an arbitrary A, define $l^2(A) := \{(a_k)_{k \in \mathbb{N}} \mid \sum_{k \in \mathbb{N}} a_k^* a_k \text{ converges in } A\}$. This is a Hilbert A-module with the obvious right A-module structure, and with inner product $\langle (a_k), (b_k) \rangle = \sum_k a_k^* b_k$.

If X is a space and E a bundle of Hilbert spaces on X (e.g. a finite dimensional unitary vector bundle), the the space H of continuous sections (vanishing at infinity) of this bundle is a Hilbert $C_0(X)$-module. The module structure is given by pointwise multiplication, and the $C_0(X)$-valued inner product by pointwise taking the inner product.

3.21 Definition. Given a Hilbert A-module H, $B(H)$ is the space of bounded A-linear maps $T \colon H \to H$ which admit an adjoint T^*. $B(H)$ is C^*-algebra, with $*$ given by the adjoint.

$K(H)$ is defined as the norm closure of the linear span of operators of the form $f \mapsto a\langle b, f \rangle$ for $a, b \in H$. This is an closed $*$-ideal of $B(H)$.

3.22 Remark. In constrast to the case of Hilbert spaces, adjoints of bounded operators do not always exist. Neither do orthogonal complements of closed A-submodules (both phenomena are closely related).

We now define $KK(\mathbb{C}, A)$ as equivalence classes of tuples (H, T) where $H = H^+ \oplus H^-$ is a $\mathbb{Z}/2$-graded Hilbert A-module, and $T \in \mathcal{B}(H)$ is an odd Hilbert A-module morphism, such that the following properties are satisfied:

(1) $T^* = T$

(2) $T^2 - 1 \in K(H)$.

Such a cycle is called *degenerate*, if $T^2 - 1 = 0$. We consider unitary isomorphism classes of such cycles. Direct sum defines an obvious semigroup structure on this set. We now define the equivalence relation of stable homotopy, where two cycles (H_1, T_1) and (H_2, T_2) are stably homotopy equivalent, if there are degenerate cycles (D_1, S_1), (D_2, S_2) such that $H_1 \oplus D_1$ is isomorphic to $H_2 \oplus D_2$ and, via this isomorphism, there is a norm continuous homotopy of $T_1 \oplus S_1$ to $T_2 \oplus S_2$ through odd self adjoint operators which are unitary module compacts.

3.23 Definition. The set of such equivalence classes is actually a group, the group $KK(\mathbb{C}, A)$.

3.24 Example. Let $T \colon H^+ \to H^-$ be a Fredholm operator, made unitary module the compact operators. Then $(H = H^+ \oplus H^-, \begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix})$ defines an element in $KK(\mathbb{C}, \mathbb{C})$.

More generally, if we have a family D_b of graded self adjoint elliptic operators parametrized by a space B $(H, D_b(1 + D_b^2)^{-1/2})$ defines an element in $KK(\mathbb{C}, C_0(B))$. Here H is the space of continuous sections (vanishing at infinity) of the bundle of Hilbert spaces $L^2(S_b)$, S_b the fiberwise define bundle on which D_b acts.

3.25 Theorem. *There is an map $K_0(A) \to KK(\mathbb{C}, A)$. If A is unital, given a finitely generated projective module P, realizing P as direct summand of A^n provides P by restriction with an A-valued inner product, and this way P becomes a Hilbert A-module. The class $[P] - [Q]$ is then sent to $[(H = H^+ \oplus H^- = P \oplus Q, 0)]$.*

This map is an isomorphism (and the result extends to non-unital A).

3.26 Remark. The definition of KK-theory and Theorem 3.25 does not make index theory trivial. Calculating indices now means that one to find an inverse of the isomorphism of Theorem 3.25, i.e. find the canonical simple form (the index) of the KK-element represented by the operator.

3.27 Remark. There are versions odd KK-groups which are not more complicated than the even ones, such that Theorem 3.25 extends, and it is very important to have the complete picture (e.g. for the long exact sequence in K-theory and for Bott periodicity). However, for reasons of brevity we concentrate here on the even case.

KK-theory is very powerful because the groups are actually calculable. Most important is the Kasparov product. To introduce this, and to make more efficient use of the KK-groups, we have to extend the definition to the bivariant theory.

Given two C^*-algebras B, A, cycles for $KK(B, A)$ are tuples (ϕ, H, T) where H is a graded Hilbert A-module as before, but now additionally equipped with an even C^*-algebra homomorphism $\phi\colon B \to B(H)$ (i.e. $\phi(b)$ is even for each $b \in B$). In particular, H is a B-A bimodule. $T \in B(H)$ is an odd self-adjoint Hilbert A-module operator which has the following additional property:

(1) $\phi(b)(T^2 - 1) \in K(H)$ for each $b \in B$.

(2) $\phi(b)[T, \phi(a)] \in K(H)$ for each $a, b \in B$.

The first condition is, if B is non-unital, a weakening of the corresponding condition for $KK(\mathbb{C}, A)$. It means that T is only locally required to be unitary modulo compacts. If B and phi are unital, it is equivalent to $T^2 - 1 \in K(H)$, since $K(H)$ is an ideal.

A cycle (phi, H, T) is called *degenerate*, if the above conditions are satisfied not modulo $K(H)$, but on the nose.

As before, we define the equivalence relation "stable homotopy".

3.28 Definition. $KK(B, A)$ is the set of such equivalence classes.

3.29 Example. Assume that M is a, not necessarily compact Riemannian spin manifold.

Then $(L^2(S), T = D(1 + D^2)^{-1/2})$ defines a cycle for $KK(C_0(M), \mathbb{C})$, where ϕ is given by pointwise multiplication. The trick is that $T^2 - 1$ might not be compact, but it is so after multiplication with a compactly supported function (or one which vanihes at infinity). Similarly, the fact that D is a differential operator of order 1 implies that T is a pseudodifferential operator of order zero. Its commutator with a multiplication operator has degree -1 and consequently is locally compact, again.

3.30 Remark. It is somewhat inconvenient to always have to transform the differential operators to bounded operators, as seen in the examples. Baaj-Julg have developed a variant of KK-theory which directly allows to work with unbounded operators. A technical difficulty consists in explaining what exactly are the appropriate unbounded operators in general, and to develop the correct tools to manipulate them.

The most powerful part of KK-theory is the Kasparov product. This is a homomorphism

$$KK(A, B) \otimes KK(B, C) \to KK(A, C),$$

more generally

$$KK(A, B_1 \otimes B_2) \otimes KK(B_2 \otimes B_3, C) \to KK(A \otimes B_3, C \otimes B_2)$$

with many very nice properties (and technically very hard to construct in general).

3.31 Example. We have just seen that, given a (compact) spin manifold M of even dimension, the Dirac operator defines a class $[D] \in KK(C(M), \mathbb{C})$. On the other hand, we have also seen that a vector $E \to M$ bundle defines a class $[E] = (\Gamma(E), 0) \in KK(\mathbb{C}, C(M))$. The Kasparov product $[E] \otimes_{C(M)} [D] \in KK(\mathbb{C}, \mathbb{C})$ is represented by $(L^2(S \otimes E), D_E(1 + D_E^2)^{-1/2})$ and is exactly the index of the Dirac operator D^+ twisted by E.

Mpre generally, if A is any other C^*-algebra, a bundle E of finitely generated projective A-modules over M defines a class $[E] = (\Gamma(E), 0) \in KK(\mathbb{C}, C(M, A) = C(M) \otimes A)$ (and every class is represented by a difference of two such cycles).

We "define"

$$\mathrm{ind}(D_E^+) := [E] \otimes_{C(M)} [D] \in KK(\mathbb{C}, A) = K_0(A).$$

It is represented by a cycle similar as in the case of finite dimensional fibers.

Mishchenko-Fomenko give a direct definition of this index, and prove an index theorem in this case (using a reduction to classical Atiyah-Singer index theorem).

We now consider a special example of such a bundle for a suitable C^*-algebra, which canonically exists on every manifold M.

3.5.1 The C^*-algebra of a group

Let Γ be a discrete group, which you can think of as being the fundamental group of the manifold M in question. Define $l^2(\Gamma)$ to be the Hilbert space of square summable complex valued functions on Γ. We can write an element $f \in l^2(\Gamma)$ as a sum $\sum_{g\in\Gamma} \lambda_g g$ with $\lambda_g \in \mathbb{C}$ and $\sum_{g\in\Gamma} |\lambda_g|^2 < \infty$.

We defined the *complex group algebra* (often also called the *complex group ring*) $\mathbb{C}\Gamma$ to be the complex vector space with basis the elements of Γ (this can also be considered as the space of complex valued functions on Γ with finite support, and as such is a subspace of $l^2(\Gamma)$). The product in $\mathbb{C}\Gamma$ is induced by the multiplication in Γ, namely, if $f = \sum_{g\in\Gamma} \lambda_g g, u = \sum_{g\in\Gamma} \mu_g g \in \mathbb{C}\Gamma$, then

$$(\sum_{g\in\Gamma} \lambda_g g)(\sum_{g\in\Gamma} \mu_g g) := \sum_{g,h\in\Gamma} \lambda_g \mu_h (gh) = \sum_{g\in\Gamma} \left(\sum_{h\in\Gamma} \lambda_h \mu_{h^{-1}g} \right) g.$$

This is a convolution product.

We have the *left regular representation* λ_Γ of Γ on $l^2(\Gamma)$, given by

$$\lambda_\Gamma(g) \cdot (\sum_{h\in\Gamma} \lambda_h h) := \sum_{h\in\Gamma} \lambda_h gh$$

for $g \in \Gamma$ and $\sum_{h\in\Gamma} \lambda_h h \in l^2(\Gamma)$.

This unitary representation extends linearly to $\mathbb{C}\Gamma$.

The *reduced C^*-algebra* $C_r^*\Gamma$ of Γ is defined to be the norm closure of the image $\lambda_\Gamma(\mathbb{C}\Gamma)$ in the C^*-algebra of bounded operators on $l^2(\Gamma)$.

3.32 Remark. It's no surprise that there is also a *maximal C^*-algebra* $C_{max}^*\Gamma$ of a group Γ. It is defined using not only the left regular representation of Γ, but simultaneously all of its representations. We will not make use of $C_{max}^*\Gamma$ in these notes, and therefore will not define it here.

3.33 Example. If Γ is finite, then $C_r^*\Gamma = \mathbb{C}\Gamma$ is the complex group ring of Γ.

In particular, in this case $K_0(C_r^*\Gamma) \cong R\Gamma$ coincides with the (additive group of) the complex representation ring of Γ.

3.34 Definition. There is a canonical trace on the group ring $\mathbb{C}\Gamma$, sending $\sum \lambda_g g$ to $\tau(\sum \lambda_g g) = \lambda_e$, the coefficient of the identity. An easy calculation shows that this is indeed a trace. In terms of the left regular representation, $\tau(x) = \langle xe, e\rangle_{l^2\Gamma}$, which shows that this trace extends to a continuous trace on $C_r^*\Gamma$.

3.35 Definition. Given a manifold M with fundamental group Γ, we can form the bundle $\tilde{M} \times_\Gamma C_r^*\Gamma$ (using the left multiplicaton of Γ on $C_r^*\Gamma$). This is a canonically defined flat bundle of free $C_r^*\Gamma$ of rank 1 for the right action of $C_r^*\Gamma$ on itself.

This bundle is called the *Mishchenko* line bundle L. Given any elliptic differential operator D on an even dimensional manifold M, we can then form as above the index $\operatorname{ind}(D_L) \in K_0(C_r^*\Gamma)$.

Our goal is to get more information about this index, and about its applications. One first observation:

3.36 Remark. The Weitzenböck formula, being a local calculation, still holds, with no extra term because the Mishchenko line bundle is flat:

$$D_L^2 = \nabla^*\nabla + \operatorname{scal}/4.$$

Moreover, because we are working in a C^*-algebra, positivity of scal implies that D_L^2, and therefore also D_L, is invertible. Using continuous functional calculus, we can therefore homotop $D_L(1+D_L^2)^{-1/2}$ to a unitary operator (preserving all the KK-conditions), simply replacing D_L by $h_t(D_L)$, with $h_0(\lambda) = \lambda$, and $h_1(\lambda) = sgn(\lambda)$ for λ in the sectrum of D_L (which does not contain zero). By definition of KK, this means that $\operatorname{ind}(D_L) = 0 \in KK(C_r^*\Gamma)$. In other words, this index is an obstructio to the existence of a metric with positive scalar curvature.

3.37 Remark. In fact, the index of the Dirac operator on a torus twisted with the Mishchenko line bundle can be shown to be non-zero (it is equivalent to a certain family index which can be computed), so this is one way to see that there is no metric with positive scalar curvature on T^n.

3.5.2 A degree zero index theorem

We next want to get some numerical invariants for operators of the form D_L, which are computable.

We will study very simply invariants, among them the L^2-index, and derive a very simple index formula for them, by reducing to the Atiyah-Singer index theorem.

3.38 Definition. Given a trace $\tau\colon A \to R$ on an algebra A, it induces a homomorphism
$$\tau\colon K_0(A) \to R$$
as follows: let $p = (p_{ij}) \in M_n(A)$ be a projection (whose image is a finitely generated projective module and therefore represents an element of $K_0(A)$). Set $\tau([\mathrm{im}(p)]) := \sum_i \tau(p_{ii})$.

3.39 Definition. Given a (smooth) bundle L of finitely generated projective Hilbert A-modules, choose a structure preserving connection ∇ on L. This connection then has a curvature Ω, a 2-form on M with values in the bundle of endomorphisms of L. Given a trace $\tau\colon A \to \mathbb{C}$ on A, we can now, as in the case of finite dimensional vector bundles, take the trace of the powers of Ω and that way define a Chern character form
$$\mathrm{ch}_\tau(L, \nabla) := \sum \tau(-\frac{\Omega^k}{(2\pi i)^k k!}) \in \Omega^{ev}(M).$$
The de Rham cohomology class of this form does not depend on the chosen connection, but only on L.

3.40 Theorem. *If D is the Dirac operator on M, L is a bundle of finitely generated projective A-modules, and $\tau\colon A \to \mathbb{C}$ a trace then*
$$\tau(\mathrm{ind}(D_L)) = \int_M \hat{A}(M) \cup \mathrm{ch}_\tau(L).$$
If D is an other elliptic operator on M, one has to replace $\hat{A}(M)$ by the corresponding Atiyah-Singer integrand and the formula remains correct.

Proof. We have two maps $K_0(C(M) \otimes A) \to \mathbb{C}$. The first one is given by sending $[L]$ to $\tau(\mathrm{ind}(D_L))$, the second one by sending $[L]$ to the right hand side $\int_M \hat{A}(M) \cup \mathrm{ch}_\tau(L)$. Some not very hard deformation arguments show that the second map really is well defined. We want to show that they coincide.

But now observe that we have a Künneth homomorphism
$$K_0(C(M)) \otimes K_0(A) \oplus K_1(C(M)) \otimes K_1(A) \to K_0(C(M) \otimes A) \qquad (3.41)$$
which is, because $C(M)$ is commutative, surjective upto torsion. It therefore suffices to check that the two homomorphisms coincide on the image of the Künneth homomorphism. For $K_1(C(M)) \otimes K_1(A)$, one checks that both maps vanish (not hard, but requires to look at Bott periodicity a little bit). For $K_0(C(M)) \otimes K_0(A)$, the image of the Künneth homomorphism of $[E] \otimes [P]$, where E is a finite dimensional vector bundle and P a projective module, is $E \otimes_{\mathbb{C}} P$. Because P is a constant "dummy factor" $\mathrm{ind}(D_{E\otimes P}) = \mathrm{ind}(D_E) \cdot P$, and therefore $\mathrm{ch}_\tau \mathrm{ind}(E_{E\otimes P}) = \mathrm{ind}(D_E) \cdot \tau(P)$. Moreover, we can put a "product" connection on $E \otimes P$ which is the trivial connection on the factor P, so that $\mathrm{ch}_\tau(E \otimes P) = \mathrm{ch}(E) \cdot \tau(P)$.

Using the classical Atiyah-Singer index theorem, the required equality follows immediately. □

3.42 Remark. The proof of Mishchenko-Fomenko of their index theorem uses the Künneth homomorphism in a similar way.

The main point of the Chern-Weyl theory employed here is that it gives some kind of a "partial converse" of the rational Künneth isomorphism, at least as long as only the τ-information is concerned. Observe that it is not easy in general to obtain the splitting information, which expresses a general A-bundle L as image under the Künneth isomorphism.

3.43 Corollary. *If L is flat, $\tau(\mathrm{ind}(D_L))$ contains exactly the same information as $\mathrm{ind}(D)$ (plus the "dimension" of the fiber measured by τ).*

For example, if we twist with the Mishchenko line bundle L and apply the canonical trace $\tau\colon C_r^*\Gamma \to \mathbb{C}$, because $\tau([C_r^*\Gamma]) = 1$, $\tau(\mathrm{ind}(D_L)) = \mathrm{ind}(D) \in \mathbb{Z}$.

This is Atiyah's L^2-index theorem, the L^2-index being precisely $\tau(\mathrm{ind}(D_L))$ (in fact, Atiyah's definition is different and one has to check equality here, as well).

3.44 Definition. Given a manifold M with fundamental group Γ, the map

$$KK(C(M), \mathbb{C}) \to KK(\mathbb{C}, C_r^*\Gamma),$$

sending the class of an elliptic operator $[D]$ on M to $\mathrm{ind}(D_L)$, where L is the Mishchenko line bundle, is called the assembly map.

Strictly speaking, one has to allow manifolds with a map f to M with an operator D, and then looks at $\mathrm{ind}(D_{f^*L})$. It is a theorem of Baum-Douglas that every element of $KK(C(M), \mathbb{C})$ is obtained that way, and then is even allows to substitute M be spaces which are not manifolds.

3.45 Definition. Let Γ be a discrete group. A classifying space $B\Gamma$ for Γ is a CW-complex with the property that $\pi_1(B\Gamma) \cong \Gamma$, and $\pi_k(B\Gamma) = 0$ if $k \neq 1$. A classifying space always exists, and is unique up to homotopy equivalence. Its universal covering $E\Gamma$ is a contractible CW-complex with a free cellular Γ-action, the so called *universal space for Γ-actions*.

Recall that a group Γ is called torsion-free, if $g^n = 1$ for $g \in \Gamma$ and $n > 0$ implies that $g = 1$.

We can now formulate the Baum-Connes conjecture for torsion-free discrete groups.

3.46 Conjecture. *Assume Γ is a dicsrete group such that $B\Gamma$ is a finite CW-complexs (this implies that it is torsion-free. Then the assembly map*

$$\overline{\mu}_*\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma) \tag{3.47}$$

is predicted to be an isomorphism.

3.48 Example. The map $\overline{\mu}_*$ of Equation (3.47) is also defined if Γ is not torsion-free. However, in this situation it will in general not be an isomorphism. This can already be seen if $\Gamma = \mathbb{Z}/2$. Then $C_r^*\Gamma = \mathbb{C}\Gamma \cong \mathbb{C} \oplus \mathbb{C}$ as a $\mathbb{C}$-algebra. Consequently,

$$K_0(C_r^*\Gamma) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{3.49}$$

On the other hand, using the homological Chern character,

$$K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{n=0}^{\infty} H_{2n}(B\Gamma; \mathbb{Q}) \cong \mathbb{Q}. \tag{3.50}$$

(Here we use the fact that the rational homology of every finite group is zero in positive degrees, which follows from the fact that the transfer homomorphism $H_k(B\Gamma; \mathbb{Q}) \to H_k(\{1\}; \mathbb{Q})$ is (with rational coefficients) up to a factor $|\Gamma|$ a left inverse to the map induced from the inclusion, and therefore is injective.)

The calculations (3.49) and (3.50) prevent μ_0 of (3.47) from being an isomorphism.

There is a variant of the Baum-Connes conjecture (where the left hand side is modified appropriately), which can be formuated for all groups and for which no counterexample is known.

3.5.3 Obstructions to positive scalar curvature

The Baum-Connes conjecture implies the so called "stable Gromov-Lawson-Rosenberg" conjecture. This implication is a theorem due to Stephan Stolz.

3.51 Theorem. *Fix a group Γ. Assume that μ in the real version of the Baum-Connes conjecture is injective (which follows e.g. if the Baum-Connes map above is an isomorphism), and assume that M is a closed spin manifold with $\pi_1(M) = \Gamma$. Assume that a certain (index theoretic) invariant $\alpha(M) \in K_{\dim M}(C^*_{\mathbb{R},r}\Gamma)$ vanishes. Then there is an $n \geq 0$ such that $M \times B^n$ admits a metric with positive scalar curvature.*

Here, B is any simply connected 8-dimensional spin manifold with $\hat{A}(M) = 1$. Such a manifold is called a *Bott manifold*.

The converse of Theorem 3.51, i.e. positive scalar curvature implies vanishing of $\alpha(M)$, is true for arbitrary groups and without knowing anything about the Baum-Connes conjecture.

3.6 The status of the Baum-Connes conjecture

The Baum-Connes conjecture is known to be true for the following classes of groups.

(1) discrete subgroups of $SO(n,1)$ and $SU(n,1)$ [10]

(2) Groups with the *Haagerup property*, sometimes called *a-T-menable groups*, i.e. which admit an isometric action on some affine Hilbert H space which is proper, i.e. such that $g_n v \xrightarrow{n\to\infty} \infty$ for every $v \in H$ whenever $g_n \xrightarrow{n\to\infty} \infty$ in G [6]. Examples of groups with the Haagerup property are amenable groups, Coxeter groups, groups acting properly on trees, and groups acting properly on simply connected CAT(0) cubical complexes

(3) One-relator groups, i.e. groups with a presentation $G = \langle g_1, \dots, g_n \mid r \rangle$ with only one defining relation r [2].

(4) Cocompact lattices in $Sl_3(\mathbb{R})$, $Sl_3(\mathbb{C})$ and $Sl_3(\mathbb{Q}_p)$ ($\mathbb{Q}_p$ denotes the p-adic numbers) [14]

(5) Word hyperbolic groups in the sense of Gromov [17].

(6) Artin's full braid groups B_n [19].

3.52 Definition. A finitely generated discrete group Γ is called amenable, if for any given finite set of generators S (where we require $1 \in S$ and require that $s \in S$ implies $s^{-1} \in S$) there exists a sequence of finite subsets X_k of Γ such that

$$\frac{|SX_k := \{sx \mid s \in S, x \in X_k\}|}{|X_k|} \xrightarrow{k\to\infty} 1.$$

$|Y|$ denotes the number of elements of the set Y.

An arbitrary discrete group is called amenable, if each finitely generated subgroup is amenable.

Examples of amenable groups are all finite groups, all abelian, nilpotent and solvable groups. Moreover, the class of amenable groups is closed under taking subgroups, quotients, extensions, and directed unions.

The free group on two generators is not amenable. "Most" examples of non-amenable groups do contain a non-abelian free group.

There is a certain stronger variant of the Baum-Connes conjecture, the *Baum-Connes conjecture with coefficients*. It has the following stability properties:

(1) If a group Γ acts on a tree such that the stabilizer of every edge and every vertex satisfies the Baum-Connes conjecture with coefficients, the same is true for Γ [18].

(2) If a group Γ satisfies the Baum-Connes conjecture with coefficients, then so does every subgroup of Γ [18]

(3) If we have an extension $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$, Γ_3 is torsion-free and Γ_1 as well as Γ_3 satisfy the Baum-Connes conjecture with coefficients, then so does Γ_2.

It should be remarked that in the above list, all groups except for word hyperbolic groups, and cocompact subgroups of Sl_3 actually satisfy the Baum-Connes conjecture with coefficients.

The Baum-Connes assembly map μ is known to be rationally injective for considerably larger classes of groups, in particular the following.

(1) Discrete subgroups of connected Lie groups [11]

(2) Discrete subgroups of p-adic groups [12]

(3) Bolic groups (a certain generalization of word hyperbolic groups) [13].

(4) Groups which admit an amenable action on some compact space [8].

Last, it should be mentioned that recent constructions of Gromov show that certain variants of the Baum-Connes conjecture, among them the Baum-Connes conjecture with coefficients, and an extension called the *Baum-Connes conjecture for groupoids*, are false [7]. At the moment, no counterexample to the Baum-Connes conjecture seems to be known. However, there are many experts in the field who think that such a counterexample eventually will be constructed [7].

4 Some newer results

To finish, we want to indicate a couple of somewhat newer results, partially obtained by the author (with coauthors).

We want to have a closer look at the L^2-index. So far, we have only seen that by Atiyah's L^2-index theorem it does not contain any new information.

However, we want to remark that it is also possible to define an L^2-index for Atiyah-Patodi-Singer type index problems for manifolds with boundary, and then the story again becomes interesting. In this situation, an index theorem for this L^2-index exists (due to Ramachandran). It has the usual form, with a correction eta-invariant term.

4.1 Definition. For an elliptic differential operator D twisted with a $C_r^*\Gamma$-module bundle L, we define the L^2-eta invariant

$$\eta_{(2)}(D_L) := \frac{1}{\sqrt{\pi}} \int_0^\infty \tau(D_L e^{-(tD)^2})\, dt.$$

We extend τ to the operator in question using an integral over the diagonal of the integral kernel. It is a non-trivial fact that the eta-integral always converges, which relies in part also on the fact that τ is not an arbitrary trace, but has the rather strong continuity property of being "normal".

4.2 Theorem. *Given a APS-boundary value problem for the twisted Dirac operator D_L as above, assume that the boundary operator is invertible. Then*

$$\operatorname{ind}_{(2)}(D_L) = \int_M AS(D) \cup \operatorname{ch}_\tau(L) - \frac{\eta_{(2)}(D_L^\partial)}{2}.$$

The theorem even extends to the case of non-invertible boundary operator (with an additional contribution from the kernel of this operator). This indeed only works because the trace τ is normal, so that it extends to the von Neumann closure of $C_r^*\Gamma$ and therefore can (via measurable functional calculus) be used to measure the size of any kernel as trace of the projection onto it.

Using this result and topological considerations about K-homology, Paolo Piazza and the author prove the following

4.3 Theorem. *Assume that M is a closed spin manifold of positive scalar curvature such that the fundamental group Γ is torsion-free and satisfies the*

Baum-Connes conjecture for the maximal C^*-algebra. *This is the case e.g. if* Γ *is amenable. Then*

$$\eta(D) = \eta_{(2)}(D_L).$$

Here L is the Mishchenko line bundle.

Proof. By injectivity of the Baum-Connes map, and since D_L is invertible because of positive scalar curvature, the class of $[D] \in KK(C(B\Gamma), \mathbb{C})$ is zero. This means (by topology) essentially that M is spin bordant (with fixed fundamental group Γ) to a manifold N of a very simple structure (and with positive scalar curvature), for which by a direct calculation $\eta(D(N)) = \eta_{(2)}(D_L(N))$.

Now this bordism W is a manifold with boundary and its boundary Dirac operator is invertible. Consequently, we can apply the two APS-index theorem; they have the same local term (because L is flat and $\tau([L]) = 1$).

Consequently: $\operatorname{ind}(D(W)) - \operatorname{ind}_{(2)}(D(W)_L) = -\frac{1}{2}(\eta(D(M)) - \eta_{(2)}(D_L(M))$.

Now, because the boundary operators are invertible, the operator $D(W)$ actually represents a class in $KK(C(W), \mathbb{C})$ and we get an index in $K(C^*_{max}\Gamma)$. The two numbers we want to compute are the images of this index class under two homomorphisms defined on $K(C^*_{max}\Gamma)$. By the Atiyah-Singer index theorem, these homomorphisms coincide on the subgroup generated by indexes of closed manifold. But because we assumed surjectivity of the Baum-Connes map, this is all of $KK(C^*_{maqx}\Gamma)$. This finishes the proof.

If we only know the Baum-Connes isomorphism for the reduced C^*-algebra, this doesn't suffice, because from the index element in $KK(C^*_r\Gamma)$ one can not a priori read off the ordinary integer valued index (this is tied to the trivial representation of the group). $\square$

The result is enlighting in view of the fact that for groups with torsion, the same invariant $\eta - \eta_{(2)}$ is quite efficient to distinguish different metrics with positive scalar curvature, in particular is not identically zero.

This is the content of a second theorem of Piazza and the author:

4.4 Theorem. *Assume that Γ contains an element of finite order. Assume that M is a spin manifold of positive scalar curvature of dimension $4k+3$, $k > 0$, with fundamental group Γ.*

Then there are infinitely many different metric with positive scalar curvature on M, such that the L^2-rho invariant on all of these are pairwise different. Between no pair of such metrics, there is a spin bordism of metrics of positive scalar curvature (and with fundamental group Γ). In particular, this metrics lie in different components of the space of metrics of positive scalar curvature (even upto the action of the diffeomorphism group by pullback).

Here, a bordism between metrics with positive scalar curvature (possibly on different manifolds M_1, M_2) is a manifold W with boundary $M_1 \cup -M_2$, together with a metric of positive scalar curvature on W which is of product type near the boundary, and restricts to the given metrics on the two boundary components. A special bordism is the cylinder $M \times [0,1]$, and a path of metrics of positive scalar curvature can be used to put a corresponding bordism metric on $M \times [0,1]$.

Proof. Given a bordism W as above, the Dirac operator with APS-boundary conditions (also twisted with the Mishchenko line bundle L) is invertible, so that the index vanishes. Since the local integrands in the two APS-index theorems coincide, the differences of eta-invariants of the two sides also have to coincide (taking the inverse orientiation on one side into account).

It follows that we can use the L^2-rho invariants to distinguish bordism classes of metrics of positive scalar curvature.

Now, there are examples of Botvinnik-Gilkey for finite cyclic fundamental groups, and on very special manifolds, with different metrics of positive scalar curvature where these rho-invariants differ.

We can then use "induction" and Gromov-Lawson-Schoen-Yau surgery arguments, to transport these examples to any manifold with the right dimension and the right kind of fundamental group. $\square$

4.5 Remark. For quite restricted classes of fundamental groups (e.g. such which contain a central element of odd order), the method in a modified way can be used in dimensions $4k+1$ as well. To our knowledge, this gives ther first examples in such dimensions, where we know that the moduli space of metrics of positive scalar curvature is not connected, but even has infinitely many components (in dimensions $4k+3$ this is know to be always the case, if the space is not empty, of course).

Similar methods and results can be applied to the signature operator. The relative of the vanishing result for torsion free fundamental groups there is a homotopy invariance result; a result originially due to Keswani for the same kind of assumptions. We give a new (and from our point of view more conceptual) proof, relying heavily on the work of Hilsum-Skandalis about the homotopy invariance of the index of the signature operator twisted with an flat A-module bundle.

Relatives of the non-triviality of the space of metrics of positive scalar curvature have also been obtained in this context. The original results are due to Chang-Weinberger, they say that under similar assumptions as above, for a the given manifold there always exist infinitely many non-diffeomorphic (or even homeomorphic) manifold which are homotopy equivalent.

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