Real versus complex K-theory using Kasparov's bivariant KK

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Abstract

In this paper, we use the KK-theory of Kasparov to prove exactness of sequences relating the K-theory of a real C^*-algebra and of its complexification.

We use this to relate the real version of the Baum-Connes conjecture for a discrete group to its complex counterpart. In particular, one implies the other, and, after inverting 2, the same is true for the injectivity or surjectivity part alone.

1 Motivation

In the majority of available sources about the subject, complex C^*-algebras and Banach algebras and their K-theory is studied. However, for geometrical reasons, the real versions also play a prominent role.

Before describing the results of this paper, we want to give the geometric motivation why both variants are necessary.

(1) Real K-Theory (meaning K-theory of real C^*-algebra) is more powerful since it contains additional information. Most notably this can be seen at Hitchins $\mathbb{Z}/2$-obstructions to positive scalar curvature in dimensions $8k+1$ and $8k+2$ [5]. They take values in $KO_j(\mathbb{R})$ for $j = 1, 2$. Related to this is the fact that there are 8 different groups, and not just 2, since real K-theory does not have the 2-periodicity of complex K-theory, but is 8-periodic.

In particular, we mention the following result of Stephan Stolz: if the real Baum-Connes map $\mu_{\mathbb{R},red}\colon RKO_*^\Gamma(\underline{E}\Gamma) \to KO_*(C^*_{\mathbb{R},red}\Gamma)$ is injective, then the stable Gromov-Lawson-Rosenberg conjecture is true for Γ. This means that a spin manifold with fundamental group Γ stably admits a metric with positive scalar curvature if and only if the Mishchenko-Fomenko index of its Dirac operator vanishes.

(2) Unfortunately, a real structure of some kind is needed to define indices in real K-theory. In particular, there is no good way to define a (higher) real index of the signature operator in dimension $4k+2$.

This explains why for the Dirac operator, and therefore for the study of metrics of positive scalar curvature on spin manifolds, one traditionally uses real K-theory, whereas complex K-theory is used for the signature operator and the study of higher signatures.

This issue came up in the paper [11] of Paolo Piazza and the author, where we studied both the signature operator and the spin Dirac operator. For a precursor of the latter paper compare [4] due to Nigel Higson and the author.

2 Real versus complex K-theory

In this paper, we give a theoretical comparison of real and complex K-theory. The results of this short note are essentially "folklore" knowledge. Early results date back to [1]. However, there only the special case of commutative C^*-algebras (in other words, spaces) is considered.

General results about the relation between real and complex K-theory are proved by Max Karoubi in [6], using some modern homotopy theory. The results of [6] are applied in [10] by Paul Baum and Max Karoubi to prove that, for discrete groups, the complex Baum-Connes conjecture implies the real Baum-Connes conjecture. Their proof is based, apart from [6], on the interpretation of the Baum-Connes map as a connecting homomorphism as explained by Roe in [12]. Our results are related and to in part equal to their results. We use, however, a different method entirely embedded in (real) bivariant K-theory (i.e. KK-theory), as developed by Kasparov (compare e.g. [8] and [7]).

In this paper, we prove in particular the following theorems.

2.1 Theorem. *Let A be a seperable* real *σ-unital C^*-algebra and $A_{\mathbb{C}} := A \otimes \mathbb{C}$. Then there is a long exact sequence in K-theory of C^*-algebras*

$$\cdots \to KO_{q-1}(A) \xrightarrow{\chi} KO_q(A) \xrightarrow{c} K_q(A_{\mathbb{C}}) \xrightarrow{\delta} KO_{q-2}(A) \to \cdots \tag{2.2}$$

Here, c is complexification, χ is multiplication with the generator $\eta \in KO_1(\mathbb{R}) \cong \mathbb{Z}/2$ (in particular $\chi^3 = 0$), and δ is the composition of the inverse of multiplication with the Bott element in $K_2(\mathbb{C})$ with "forgetting the complex structure".

2.3 Remark. Real and complex C^*-algebras and their K-theory are connected by "complexification" and "forgetting the complex structure". We use these terms throughout, precise definitions are given in Definitions 3.7 and 3.8.

2.4 Corollary. *In the situation of Theorem 2.1, if we invert 2, in particular if we tensor with $\mathbb{Q}$, the sequence splits into short* split *exact sequences*

$$0 \to KO_q(A) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{c} K_q(A_{\mathbb{C}}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\delta} KO_{q-2}(A) \otimes \mathbb{Z}[\frac{1}{2}] \to 0. \tag{2.5}$$

Proof. We obtain short exact sequences because $2\eta = 0$, i.e. the image of χ (and therefore the kernel of c) in (2.2) consists of 2-torsion.

The sequence is split exact, with split being given by "forgetting the complex structure" $K_*(A_{\mathbb{C}}) \to KO_*(A)$, since the composition of "complexification" with "forgetting the complex structure" induces multiplication with 2 in $KO_*(A)$, i.e. an automorphism after inverting 2. For more details, compare Definition 3.7, Definition 3.8 and Lemma 3.9. $\square$

2.6 Theorem. *Assume that Γ is a discrete group and X is a proper Γ-space. Let B be a separable real σ-unital Γ-C^*-algebra. Then we have a long exact sequence in equivariant representable K-homology with coefficients in B (defined e.g. via Kasparov's KK-theory)*

$$\cdots \to RKO^\Gamma_{q-1}(X;B) \xrightarrow{\chi} RKO^\Gamma_q(X;B) \xrightarrow{c} RK^\Gamma_q(X;B_\mathbb{C}) \xrightarrow{\delta} RKO^\Gamma_{q-2}(X;B) \to \cdots \tag{2.7}$$

Here, c is again complexification, and χ is given by multiplication with the generator in $KO_1(pt) = \mathbb{Z}/2$, i.e. $\chi^3 = 0$. δ is the composition of (the inverse of) the complex Bott periodicity isomorphism with 'forgetting the complex structure".

2.8 Corollary. *In the situation of Theorem 2.6, after inverting 2, in particular after tensor product with $\mathbb{Q}$, we obtain split short exact sequences*

$$0 \to KO^\Gamma_q(X;B)\otimes\mathbb{Z}[\frac{1}{2}] \xrightarrow{c} K^\Gamma_q(X;B_\mathbb{C})\otimes\mathbb{Z}[\frac{1}{2}] \xrightarrow{\delta} KO^\Gamma_{q-2}(X;B)\otimes\mathbb{Z}[\frac{1}{2}] \to 0. \tag{2.9}$$

Proof. Compare the proof of Corollary 2.4. □

2.10 Theorem. *Let Γ be a discrete group. Consider the special case of Theorem 2.1 where $A = C^*_{\mathbb{R},red}(\Gamma;B)$ is the crossed product of B by Γ, and the special case of Theorem 2.6 where $X = \underline{E}\Gamma$, the universal space for proper Γ-actions. We have (Baum-Connes) index maps*

$$\mu_{red}\colon RK^\Gamma_p(\underline{E}\Gamma;B_\mathbb{C}) \to K_p(C^*_{red}(\Gamma;B_\mathbb{C})); \tag{2.11}$$
$$\mu_{\mathbb{R},red}\colon RKO^\Gamma_p(\underline{E}\Gamma;B) \to K_p(C^*_{\mathbb{R},red}(\Gamma;B)). \tag{2.12}$$

Using the canonical identification $C^*_{\mathbb{R},red}(\Gamma;B)_\mathbb{C} = C^*_{red}(\Gamma;B_\mathbb{C})$, the index maps (2.11) commute with the maps in the long exact sequences (2.2) and (2.9).

2.13 Corollary. *The real Baum-Connes conjecture is true if and only if the complex Baum-Connes conjecture is true, i.e. μ_{red} of (2.11) is an isomorphism if and only if $\mu_{\mathbb{R},red}$ is an isomorphism.*

After inverting 2, in particular after tensoring with 2, injectivity and surjectivity are separately equivalent in the real and complex case, i.e. in

$$\mu_{red}\otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{2}]}\colon RK^\Gamma_p(\underline{E}\Gamma;B_\mathbb{C})\otimes\mathbb{Z}[\frac{1}{2}] \to K_p(C^*_{red}(\Gamma;B_\mathbb{C}))\otimes\mathbb{Z}[\frac{1}{2}];$$
$$\mu_{\mathbb{R},red}\otimes\mathbb{Z}[\frac{1}{2}]\colon RKO^\Gamma_p(\underline{E}\Gamma;B)\otimes\mathbb{Z}[\frac{1}{2}] \to K_p(C^*_{\mathbb{R},red}(\Gamma;B))\otimes\mathbb{Z}[\frac{1}{2}],$$

one of the maps is injective for all p if and only if the other maps is injective for all p, and is surjective for all p if and only if the other map is surjective for all p.

Proof. Using the long exact sequences (2.2) and (2.7) and the 5-lemma, if $\mu_{\mathbb{R},red}$ is an isomorphism then also μ_{red} is an isomorphism. For the converse, we use the algebraic Lemma 3.1 and the fact that $\chi^3 = 0$.

After inverting 2, the long exact sequences split into short exact sequences, and consequently we can deal with injectivity and surjectivity separately, using e.g. the general form of the 5-lemma [3, Proposition 1.1]. □

2.14 Theorem. *Corresponding results to the ones stated above hold if we replace the reduced C^*-algebras with the maximal ones (and the reduced index map with the maximal assembly map).*

Corresponding results also hold if we replace the classifying space for proper actions $\underline{E}\Gamma$ with the classifying space for free actions $E\Gamma$. If $B\Gamma := E\Gamma/\Gamma$ is a finite CW-complex, and $B = \mathbb{R}$, then $RKO_p(B\Gamma, B) = KO_p(B\Gamma)$ is the real K-homology of the space $B\Gamma$. We get the new index map as composition of the index map of Theorem 2.10 with a canonical map $RKO_p^\Gamma(E\Gamma; B) \to RKO_p^\Gamma(\underline{E}\Gamma; B)$.

2.15 Remark. Of course, in Theorem 2.14, the assembly map will in many cases *not* be an isomorphism —whereas no example is known such that the assembly map of Theorem 2.10 is not an isomorphism. In Theorem 2.14 we only claim that it is an isomorphism for the real version if and only if is an isomorphism for the complex version.

The long exact sequences of Theorem 2.1 and Theorem 2.6 are special cases of the following bivariant theorem.

2.16 Theorem. *Let Γ be a discrete group and A, B separable real σ-unital Γ-C^*-algebras. Then there is a long exact sequence*

$$\cdots \to KKO_{q-1}^\Gamma(A;B) \xrightarrow{\chi} KKO_q^\Gamma(A;B) \xrightarrow{c} KK_q^\Gamma(A_\mathbb{C};B_\mathbb{C}) \xrightarrow{\delta} KKO_{q-2}^\Gamma(A;B) \to \cdots \tag{2.17}$$

Here, χ is given by Kasparov product with the generator of $KKO_1^\Gamma(\mathbb{R},\mathbb{R}) = \mathbb{Z}/2$, c is given by complexification *as defined in Definition 3.7, and δ is the composition of the inverse of the complex Bott periodicity isomorphism with "forgetting the complex structure" as defined in Definition 3.8.*

In particular, $2\eta = 0$, $2\chi = 0$, and $\chi^3 = 0$.

3 Proofs of the theorems

Note first that Theorem 2.1 and Theorem 2.6 indeed are special cases of Theorem 2.16. For Theorem 2.1 we simply have to take $\Gamma = \{1\}$, $A = \mathbb{R}$ (and then B of Theorem 2.16 is A of Theorem 2.1). For Theorem 2.6 let first Y be a Γ-compact Γ-invariant subspace of X, and set $A = C_0(Y)$. By definition,

$$RKO_p^\Gamma(X;B) = \lim KKO_p^\Gamma(C_0(Y), B),$$

where the (direct) limit is taken over all Γ-compact subspaces of X. The corresponding sequence for each Y is exact. Since the direct limit functor is exact, the same is true for the sequence (2.7).

We therefore only have to prove Theorem 2.16, Theorem 2.10 and Lemma 3.1 (which was used in the proof of Corollary 2.13).

An algebraic lemma

3.1 Lemma. *Assume that one has a commutative diagram of abelian groups with exact rows which are 3-periodic*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\chi} & A & \xrightarrow{c} & B & \xrightarrow{\delta} & A & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu_A} & & \\
\cdots & \longrightarrow & U & \xrightarrow{\chi_U} & U & \xrightarrow{c_U} & V & \xrightarrow{\delta_V} & U & \longrightarrow & \cdots .
\end{array} \tag{3.2}$$

Let χ and χ_U be endomorphisms of finite order. Then μ_B is an isomorphism if and only if the same is true for μ_A.

Proof. If μ_A is an isomorphism so is μ_B by the 5-lemma.

Perhaps the most elegant way to prove the converse is to observe that the rows from exact couples in the sense of [9, Section 2.2.3]. Consequently, we get derived commutative diagrams of abelian groups with exact rows which are 3-periodic

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \chi^n(A) & \xrightarrow{\chi|} & \chi^n(A) & \xrightarrow{c_n} & B_n & \xrightarrow{\delta_n} & \chi^n(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mu_A|} & & \downarrow{\scriptstyle \mu_A|} & & \downarrow{\scriptstyle (\mu_B)_n} & & \downarrow{\scriptstyle \mu_A|} & & \\ \cdots & \longrightarrow & \chi^n_U(U) & \xrightarrow{\chi_U|} & \chi^n_U(U) & \xrightarrow{(c_U)_n} & V_n & \xrightarrow{(\delta_V)_n} & \chi^n_U(U) & \longrightarrow & \cdots . \end{array} \tag{3.3}$$

Here, $\chi^n(A)$ is the image of A under the n-fold iterated map χ. One defines inductively $B_n := \ker(c_{n-1} \circ \delta_{n-1}) / \operatorname{im}(c_{n-1} \circ \delta_{n-1})$; this is a certain homology of B_{n-1}, c_n is the composition of c_{n-1} with the restriction map, and δ_n and $(\mu_B)_n$ are the maps δ_{n-1} and $(\mu_B)_{n-1}$, respectively, which one proves are well defined on homology classes.

In particular, $(\mu_B)_n$ is an isomorphism if μ_B is (an isomorphism induces an isomorphism on homology).

We prove now by reverse induction that $\mu_A|\colon \chi^n(A) \to \chi^n_U(V)$ is an isomorphism for each n. Since χ and χ_U have finite order, there images are eventually zero, so the assertion is true for n large enough.

Under the assumption that $\mu_A|_{\chi^n(A)}$ is an isomorphism, we have to prove the same for $\mu_A|\colon \chi^{n-1}(A) \to \chi^{n-1}_U(U)$. For this, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} \chi^{n-1}(A)/\chi^n(A) & \xrightarrow{c_{n-1}} & B_{n-1} & \xrightarrow{\delta_{n-1}} & \chi^{n-1}(A) & \xrightarrow{\chi} & \chi^n(A) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mu_A|} & & \downarrow{\scriptstyle (\mu_B)_n} & & \downarrow{\scriptstyle \mu_A|_{\chi^{n-1}(A)}} & & \downarrow{\scriptstyle \mu_A|_{\chi^n(A)}} & & \downarrow \\ \chi^{n-1}_U(U)/\chi^n_U(U) & \xrightarrow{c_{U\,n-1}} & V_{n-1} & \xrightarrow{(\delta_V)_{n-1}} & \chi^{n-1}_U(U) & \xrightarrow{\chi_U} & \chi^n_U(U) & \longrightarrow & 0 \end{array} \tag{3.4}$$

obtained by cutting the long exact sequence (3.3). We have just argued that $(\mu_B)_{n-1}$ is an isomorphism since μ_B is one by assumption, and by the induction assumption $\mu_A|_{\chi^n(A)}$ is an isomorphism. By the 5-lemma [3, Proposition 1.1] $\mu_A|_{\chi^{n-1}(A)}$ therefore is onto. This implies that the leftmost vertical map in (3.4) also is onto. Now we can use the 5-lemma [3, Proposition 1.1] again to conclude that $\mu_A|_{\chi^{n-1}(A)}$ also is injective.

Induction concludes the proof. □

3.5 Remark. The proof of "injectivity implies injectivity" in Corollary 2.10 follows from the fact that, after inverting 2,

$$K_p(B_{\mathbb{C}}) \otimes \mathbb{Z}[\frac{1}{2}] \cong (KO_p(B) \oplus KO_{p-2}(B)) \otimes \mathbb{Z}[\frac{1}{2}]$$

for any real C^*-algebra B in a natural way, with a similar assertion for the left hand side of the Baum-Connes assembly map.

In particular, injectivity or surjectivity, respectively, in degree p for the complex Baum-Connes map is (after inverting 2) equivalent to injectivity or surjectivity, respectively, in the two degrees p and $p-2$ for the real Baum-Connes map.

We should note that not only the proof of this assertion in Theorem 2.10 does not work if 2 is not inverted. Worse: the underlying algebraic statement is actually false. The easiest example is given by the short exact sequence for K-theory of a point. If we tensor this with $\mathbb{Z}[1/2]$, it remains exact. The natural map $M \to M \otimes \mathbb{Z}[1/2]$ connects the original exact sequence with the new exact sequence. For complex K-theory, the relevant maps are the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[1/2]$ and $0 \to 0$. In particular, this is injective in all degrees. For real K-theory, we also get (in degrees 1 and 2 mod 8) the map $\mathbb{Z}/2 \to 0$, i.e. the map here is not exact in all degrees.

Exterior product with "small" KK-elements

3.6 Definition. In the sequel, we will frequently encounter homomorphisms $f\colon KKO_p^\Gamma\Gamma(A, B\otimes M_1) \to KKO_{p+l}^\Gamma(A, B\otimes M_2)$, where M_1, M_2 are "elementary" C^*-algebras (with trivial Γ-action), e.g. $M_i \in \{\mathbb{R}, \mathbb{C}, M_2(\mathbb{R}), \cdots\}$. In most cases, f will be induced by exterior Kasparov product with an element $[f] \in KK_l(M_1, M_2)$.

Such a homomorphism will be called *small*, or given by *Kasparov product with a small element*. It is clear that the composition of small homomorphisms is again a small homomorphism.

Complexification and forgetting the real structure

3.7 Definition. Let A be a real Γ-C^*-algebra with complexification $A_\mathbb{C} := A\otimes\mathbb{C}$. Note that $A_\mathbb{C}$ can also be viewed as a real Γ-C^*-algebra, with a canonical natural inclusion $A \hookrightarrow A_\mathbb{C}$. This map, and the maps it induces on K-theory are called "complexification" and denoted by c.

For A and B separable real Γ-C^*-algebras, A unital, the map induced by $c\colon A \hookrightarrow A_\mathbb{C}$ can be composed with the isomorphism of Proposition 3.10, to get

$$c_\mathbb{C}\colon KKO_n^\Gamma(B, A) \xrightarrow{[c]} KKO_n^\Gamma(B, A_\mathbb{C}) \cong KK_n^\Gamma(B_\mathbb{C}, A_\mathbb{C}).$$

This is what we call the complexification homomorphism in KK-theory, it induces corresponding maps in K-theory and K-homology. Note that c is a small homomorphism, induced from $[c\colon \mathbb{R} \to \mathbb{C}] \in KKO_0(\mathbb{R}, \mathbb{C})$.

3.8 Definition. Let A be a real separable Γ-C^*-algebra, $A_\mathbb{C}$ its complexification. We have a canonical natural inclusion $A_\mathbb{C} \hookrightarrow M_2(A)$, using the usual inclusion $i\colon \mathbb{C} \to M_2(\mathbb{R})$. If A is σ-unital and B is another separable Γ-C^*-algebra, using Morita equivalence, we get the induced homomorphisms in K-theory

$$f_\mathbb{C}\colon KK_n^\Gamma(B_\mathbb{C}, A_\mathbb{C}) \cong KKO_n^\Gamma(B, A_\mathbb{C}) \xrightarrow{f} KKO_n^\Gamma(B, M_2(A)) \xrightarrow[M]{\cong} KKO_n^\Gamma(B, A),$$

called "forgetting the complex structure". Note that f is a small homomorphism, induced by $[i\colon \mathbb{C} \hookrightarrow M_2(\mathbb{R})] \in KKO_0(\mathbb{C}, M_2(\mathbb{R}))$. Also M is a small homomorphism, as explained in the proof of Lemma 3.9. We define $f_\mathbb{R} := [i] \bullet [M_\mathbb{R}] \in KKO_0(\mathbb{C}, \mathbb{R})$ to be the corresponding composition of small KKO-elements.

3.9 Lemma. *In the situation of Definitions 3.7 and 3.8, the composition of first complexification and then forgetting the complex structure is multiplication by* 2*.*

Proof. By definition, this composition is the small homomorphism given by exterior Kasparov product with $[i\colon \mathbb{R} \hookrightarrow M_2(\mathbb{R})]$ (i the diagonal inclusion) composed with the small Morita equivalence homomorphism $[M_{\mathbb{R}}] \in KKO_0(M_2(\mathbb{R}), \mathbb{R})$. It is known that $[i] \bullet [M_{\mathbb{R}}] = 2 \in KKO_0(\mathbb{R}, \mathbb{R})$, which by associativity implies the claim. For a short KK-theoretic proof, observe that $[M_{\mathbb{R}}] = [\mathbb{R}^2 \oplus 0, 0] \in KKO_0(M_2(\mathbb{R}, \mathbb{R})$, with the obvious left $M_2(\mathbb{R})$ and right $\mathbb{R}$-module structure on $\mathbb{R}^2$, and with operator 0. On the other hand, $[i] = (M_2(\mathbb{R}) \oplus 0, 0)$. Since both operators in our representatives are 0 we get (compare e.g. [2])

$$[i] \bullet [M_{\mathbb{R}}] = [\mathbb{R}^2 \oplus 0, 0] = 2[\mathbb{R} \oplus 0, 0] \in KK_0^{\Gamma}(\mathbb{R}, \mathbb{R}).$$

Since $[\mathbb{R} \oplus 0, 0] = 1 \in KK_0^{\Gamma}(\mathbb{R}, \mathbb{R})$, the claim follows. □

Lemma 3.9 implies that, after inverting 2, the long exact sequences of Section 2 give rise to the split short exact sequences we claim to get.

To relate the K-theory of a complex C^*-algebra with the K-theory of the same C^*-algebra, considered as a real C^*-algebra, we already used the following results.

3.10 Proposition. *Let Γ be a discrete group. If A is a σ-unital complex Γ-C^*-algebra (which can also be considered as a real C^*-algebra) and B is a separable real Γ-C^*-algebra, then the inclusion $B \hookrightarrow B_{\mathbb{C}}$ induces a natural isomorphisms*

$$b\colon KK_n^{\Gamma}(B_{\mathbb{C}}, A) \xrightarrow{\cong} KKO_n^{\Gamma}(B, A) \tag{3.11}$$

Proof. The isomorphism of (3.11) is given by the fact that there is a one to one correspondence already on the level of Kasparov triples: since A is σ-unital, every Hilbert $A_{\mathbb{C}}$-module E is a complex vector space, and therefore the same is true for the set of bounded operators on E. Therefore, the real linear maps $B \to \mathcal{B}(E)$ are in one-to-one correspondence with the complex linear maps $B_{\mathbb{C}} \to \mathcal{B}(E)$. All the other conditions on equivariant Kasparov triples, and the equivalence relations are preserved by this correspondence. All this follows directly by inspecting Definitions 2.1 to 2.3 in [8]. □

Proof of Theorem 2.16

Special cases of Theorem 2.16 are well known, compare e.g. [1, (3.4)], which we are going to use below.

Following the notation of [1, Section 2] and [6, Section 7], let $\mathbb{R}^{1,0}$ be the real line with the involution $x \mapsto -x$, and $D^{1,0}$, $S^{1,0}$ the unit disc and sphere, respectively, with the induced involution.

Given any real C^*-algebra A and a locally compact space X with involution $x \mapsto \overline{x}$, following [6, Section 6] we define

$$A(X) := \{f\colon X \to A \otimes \mathbb{C} \mid f(\overline{x}) = \overline{f(x)}; f(x) \xrightarrow{x\to\infty} 0\}.$$

This is again a real C^*-algebra.

We have the short exact sequence

$$0 \to \mathbb{R}(\mathbb{R}^{1,0}) \to \mathbb{R}(D^{1,0}) \to \mathbb{R}(S^{1,0}) \to 0, \tag{3.12}$$

where we identify $\mathbb{R}^{1,0}$ with the open unit interval in $D^{1,0}$. This short exact sequence admits a completely positive cross section, using a cutoff function $\rho\colon D^1 \to [0,1]$ with value 1 at the boundary and 0 at the origin to extend functions on $S^{1,0}$ to the disc.

For an arbitrary real Γ-C^*-algebra A, tensoring (3.12) with A, we get a short exact sequence (using the canonical isomoprhism $A \otimes_{\mathbb{R}} \mathbb{R}(X) \cong A(X)$)

$$0 \to A(\mathbb{R}^{1,0}) \to A(D^{1,0}) \to A(S^{1,0}) \to 0, \tag{3.13}$$

which again admits a completely positive cross section induced from the completely positive cross section of (3.12).

Moreover, evaluation at 1 gives a natural and canonical C^*-algebra *isomorphism*

$$\phi\colon A(S^{1,0}) \xrightarrow{\cong} A_{\mathbb{C}}.$$

We also note that the evaluation map

$$A(D^{1,0}) \to A; f \mapsto f(0)$$

is a homotopy equivalence in the sense of KK-theory. The homotopy inverse maps $x \in A$ to the constant map with value x. In particular, we have natural isomorphisms

$$\psi\colon KKO_*^\Gamma(B,A) \cong KKO_*^\Gamma(B, A(D^{1,0})),$$

where the maps in both directions are small homomorphisms in the sense of Definition 3.6.

Concerning $A(\mathbb{R}^{1,0})$, by [7, Paragraph 5, Theorem 7]

$$\alpha\colon KKO_n^\Gamma(B, A(\mathbb{R}^{1,0})) \xrightarrow{\cong} KKO_{n-1}^\Gamma(B,A),$$

where the map α and its inverse are given by exterior Kasparov product with small elements.

(We remark that the corresponding, but slightly different statements in [13, 2.5.1] are partly *wrong*, since Schröder is disregarding the gradings.)

Therefore, the short exact sequence (3.12) induces via the short exact sequence (3.13) for separable real σ-unital Γ-C^*-algebras A and B a long exact sequence in equivariant bivariant real K-theory

$$\begin{tikzcd}
KKO_{n+1}^\Gamma(B, A(S^{1,0})) \arrow[r, "\delta"] \arrow[d, "\phi_*", "\cong"'] & KKO_n^\Gamma(B, A(\mathbb{R}^{1,0})) \arrow[r, "i"] \arrow[d, "="] & KKO_n^\Gamma(B, A(D^{1,0})) \arrow[r, "j"] & KKO_n^\Gamma(B, A(S^{1,0})) \arrow[d, "\phi_*", "\cong"'] \\
KKO_{n+1}^\Gamma(B, A_{\mathbb{C}}) \arrow[r, "\delta'"] & KKO_n^\Gamma(B, A(\mathbb{R}^{1,0})) \arrow[r, "\rho"] \arrow[d, "\alpha", "\cong"'] & KKO_n^\Gamma(B,A) \arrow[r, "i_2"] \arrow[u, "\psi", "\cong"'] \arrow[d, "="] & KKO_n^\Gamma(B, A_{\mathbb{C}}) \\
KK_{n+1}^\Gamma(B_{\mathbb{C}}, A_{\mathbb{C}}) \arrow[r, "\delta''"] \arrow[u, "b", "\cong"'] & KKO_{n-1}^\Gamma(B,A) \arrow[r, "\chi"] & KKO_n^\Gamma(B,A) \arrow[r, "c"] & KK_n^\Gamma(B_{\mathbb{C}}, A_{\mathbb{C}}) \arrow[u, "b", "\cong"']
\end{tikzcd}$$

In this diagram, all the vertical maps are isomorphisms which have been explained above. They are in particular small homomorphisms where the inverse is also a small homomorphism. In particular ϕ_* is induced from the isomorphism of C^*-algebras $\phi\colon \mathbb{R}(S^{1,0}) \to \mathbb{C}$ (and therefore is small). Here, b is a somewhat exceptional homomorphism: it is the identification of KK-groups of Proposition 3.10 which is true on the level of cycles.

The construction of the long exact sequence in KK-theory implies that δ, i and j are small homomorphisms (compare [13, Theorem 2.5.6]), and i and j are induced from the maps in the short exact sequence (3.12).

Since the compositions of small homomorphisms are small, the same follows for δ', ρ, i_2 and ρ'. Composing the maps, we see that i_2 is induced from the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$, and therefore c is the complexification homomorphism of Definition 3.7. Since δ'' equals $\alpha \circ \delta'$ (upto the canonical identification b) we can also consider δ'' as a small homomorphism.

To identify the small homomorphisms $\tilde{\chi} \in KK_1(\mathbb{R},\mathbb{R}) = \mathbb{Z}/2$ and δ'', it suffices to study the case $A = B = \mathbb{R}$ and $\Gamma = \{1\}$. Then we get

$$\cdots \to K^{n+1}(\mathbb{C}) \xrightarrow{\delta''} KO^{n-1}(\mathbb{R}) \xrightarrow{\chi} KO^n(\mathbb{R}) \xrightarrow{c} K^n(\mathbb{C}) \to \cdots \tag{3.14}$$

This exact sequence, including the identification of χ as multiplication with the generator of $KKO^1(\mathbb{R}) = KKO_1(\mathbb{R},\mathbb{R}) = \mathbb{Z}/2$ and of δ'' as composition of complex Bott periodicity with “forgetting the complex structure” is already established in [1]. Alternatively, a careful analysis of the constructions in this special case also identifies χ and δ'' without much difficulty. Note that without these computations we nevertheless identify $\tilde{\chi} = \chi$, since it has to be non-zero by the known K-theory of $\mathbb{R}$ and $\mathbb{C}$, and there is only one non-zero element in $KK_1(\mathbb{R},\mathbb{R})$. Since, in this paper, we don't use the explicit description of δ'', the main results of this paper are established without using [1] (or Remark 3.15).

Finally, observe that χ has additive order 2. Moreover, if we iterate χ three times, we take the Kasparov product with third power of the generator in $KK_1(\mathbb{R},\mathbb{R})$ which is zero, and therefore $\chi^3 = \chi \circ \chi \circ \chi = 0$.

3.15 Remark. A possible way to calculate δ'' is the following: the same arguments which lead to the K-theoretic exact sequence (3.14) give rise to a corresponding sequence in K-homology. This implies that $KKO_{-2}(\mathbb{C},\mathbb{R}) \cong \mathbb{Z}$. Moreover, using the fact that “forgetting the complex structure” gives a unital ring homomorphism $KK_*(\mathbb{C},\mathbb{C}) \to KKO_*(\mathbb{C},\mathbb{C})$ (using the arguments of Proposition 3.10) we get Bott periodicity $KKO_{-2}(\mathbb{C},\mathbb{R}) \cong KKO_0(\mathbb{C},\mathbb{R})$, where the map is given by multiplication with the complex Bott periodicity element $\beta \in KKO_2(\mathbb{C},\mathbb{C})$. In particular, the generators of $KKO_{-2}(\mathbb{C},\mathbb{R})$ are products of the inverse $\beta^{-1} \in KKO_{-2}(\mathbb{C},\mathbb{C})$ of the complex Bott periodicity element with the generators of $KKO_0(\mathbb{C},\mathbb{R})$. Using the K-homology version of (3.14) again (where $\mathbb{Z} \cong KKO_0(\mathbb{C},\mathbb{R}) \to KKO_0(\mathbb{R},\mathbb{R}) \cong \mathbb{Z}$ is induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$), the element “forgetting the complex structure” $f_\mathbb{R}$ (defined in 3.8) defines a generator of $KKO_0(\mathbb{C},\mathbb{R})$ (since this element is mapped to 2, as we conclude from the long exact sequence also happens to a generator). It follows that the Kasparov product of β^{-1} with $f_\mathbb{R}$ is an additive generator of $KKO_{-2}(\mathbb{C},\mathbb{R})$.

The exact sequence (3.14) implies that δ'' can not be divisible. Since it is given by some element of $\mathbb{Z} \cong KKO_{-2}(\mathbb{C},\mathbb{R})$, it has (up to a sign which we don't have to determine) to coincide with the generator, i.e. the product of β^{-1} and $f_\mathbb{R}$

Proof of Theorem 2.10

It remains to prove that the Baum-Connes assembly maps are compatible with the maps in the long exact sequences. To do this, we have to recall this assembly (or index) map. It is in the real and complex case given by the same procedure, which we describe for complex K-theory.

μ is given as composition of two maps, namely

$$\mathit{descent}\colon KK_n^\Gamma(C_0(\underline{E}\Gamma),\mathbb{C}) \to KK_n(C^*_{red}(\Gamma;C_0(\underline{E}\Gamma)),C^*_{red}(\Gamma;\mathbb{C})). \tag{3.16}$$

To be precise, we have to apply this map to Γ-compact subsets of $\underline{E}\Gamma$ and then pass to the limit. We avoid this to simplify notation. By [2, Theorem 20.6.2] (and [13, Theorem 2.4.13] for the real case) this descent is compatible with Kasparov products. Since Γ acts trivially on the right hand factor $\mathbb{C}$, $C^*_{red}(\Gamma;\mathbb{C}) \cong \mathbb{C}\otimes_{\mathbb{C}} C^*_{red}\Gamma \cong C^*_{red}\Gamma$. Note that the construction of descend for trivial actions just amounts to the exterior Kasparov product with the identity on the level of KK. In other words, if A and B have a trivial Γ-action, then

$$\mathit{descent}\colon KK(A,B) \to KK(A\otimes C^*_{red}\Gamma, B\otimes C^*_{red}\Gamma)$$

is given by exterior tensor product with the identity.

Since descent is compatible with the intersection product, it follows that descent commutes with exterior Kasparov product with small elements in the sense of Definition 3.6 in (3.16).

The second map

$$KK(C^*_{red}(\Gamma;C_0(\underline{E}\Gamma)),C^*_{red}\Gamma) \to KK(\mathbb{C},C^*_{red}\Gamma) \tag{3.17}$$

is given by left Kasparov product with a certain element, the so called Mishchenko line bundle, in $KK(\mathbb{C},C_0(\underline{E}\Gamma))$. This also commutes with exterior Kasparov product with small KK-elements.

Now the Baum-Connes assembly map μ is the composition of the two homomorphisms just described, and therefore also commutes with small homomorphisms.

Since the homomorphisms in the long exact sequences are all small, the Baum-Connes maps are compatible with them, which is the assertion of Theorem 2.10.

Variations

It is clear that all the arguments given in this section apply in exactly the same way in the situations described in Theorem 2.14, which is therefore also true.

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