

Approximating L^2 -invariants, and the Atiyah conjecture

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Abstract

Let G be a torsion free discrete group and let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in \mathbb{C} . We prove that $\overline{\mathbb{Q}}G$ fulfills the Atiyah conjecture if G lies in a certain class of groups \mathcal{D} , which contains in particular all groups which are residually torsion free elementary amenable or which are residually free. This result implies that there are no non-trivial zero-divisors in $\mathbb{C}G$. The statement relies on new approximation results for L^2 -Betti numbers over $\overline{\mathbb{Q}}G$, which are the core of the work done in this paper. Another set of results in the paper is concerned with certain number theoretic properties of eigenvalues for the combinatorial Laplacian on L^2 -cochains on any normal covering space of a finite CW complex. We establish the absence of eigenvalues that are transcendental numbers, whenever the covering transformation group is either amenable or in the Linnell class \mathcal{C} . We also establish the absence of eigenvalues that are Liouville transcendental numbers whenever the covering transformation group is either residually finite or more generally in a certain large bootstrap class \mathcal{G} .

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1 Introduction

Atiyah [1] defined in 1976 the L^2 -Betti numbers of a compact Riemannian manifold. They are defined in terms of the spectrum of the Laplace operator on the universal covering of M . By Atiyah's L^2 -index theorem [1], they can be used e.g. to compute the Euler characteristic of M .

Dodziuk [3] proved an L^2 -Hodge de Rham theorem which gives a combinatorial interpretation of the L^2 -Betti numbers, now in terms of the spectrum of the combinatorial Laplacian. This operator can be considered to be a matrix over the integral group ring of the fundamental group G of M . These L^2 -invariants have been successfully used to give new results in differential geometry, topology, and algebra. For example, one can prove certain cases of the Hopf conjecture about the sign of the Euler characteristic of negatively curved manifolds [8], or, in a completely different direction, certain cases of the zero divisor conjecture, which asserts that $\mathbb{Q}[G]$ has no non-trivial zero divisors if G is torsion-free [14, Lemma 2.4].

In this paper, we will be concerned with the computation of the spectrum of matrices not only over $\mathbb{Z}G$, but also over $\overline{\mathbb{Q}}G$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers in \mathbb{C} . This amends our understanding of the combinatorial version of L^2 -invariants. Moreover, it allows important generalizations of the algebraic applications. In particular, we can prove in new cases that $\mathbb{C}G$, and not only $\mathbb{Q}G$, has no non-trivial zero divisors.

Specifically, the goal of this note is to extend a number of (approximation) results of [13], [4], [7], [18] and [16] from $\mathbb{Q}G$ to $\overline{\mathbb{Q}}G$, and to give more precise information about the spectra of matrices over $\overline{\mathbb{Q}}G$ than was given there.

We deal with the following situation: G is a discrete group and K is a subfield of \mathbb{C} . The most important examples are $K = \mathbb{Q}$, $K = \overline{\mathbb{Q}}$, and $K = \mathbb{C}$. Assume $A \in M(d \times d, KG)$. We use the notation $l^2(G) = \{f: G \rightarrow \mathbb{C} \mid \sum_{g \in G} |f(g)|^2 < \infty\}$.

By left convolution, A induces a bounded linear operator $A: l^2(G)^d \rightarrow l^2(G)^d$ (using the canonical left G -action on $l^2(G)$), which commutes with the right G -action.

Let $\text{pr}_{\ker A}: l^2(G)^d \rightarrow l^2(G)^d$ be the orthogonal projection onto $\ker A$. Then

$$\dim_G(\ker A) := \text{tr}_G(\text{pr}_{\ker A}) := \sum_{i=1}^n \langle \text{pr}_{\ker A} e_i, e_i \rangle_{l^2(G)^n},$$

where $e_i \in l^2(G)^n$ is the vector with the trivial element of $G \subset l^2(G)$ at the i^{th} -position and zeros elsewhere.

To understand the type of results we want to generalize, we repeat the following definition:

1.1 Definition. Let K be a subring of \mathbb{C} . We say that a torsion free group G fulfills the *strong Atiyah conjecture over KG* if

$$\dim_G(\ker A) \in \mathbb{Z} \quad \forall A \in M(d \times d, KG);$$

where $\ker A$ is the kernel of the induced map $A: l^2(G)^d \rightarrow l^2(G)^d$.

It has long been observed that for a torsion free group G , the strong Atiyah conjecture over KG implies that KG does not contain non-trivial zero divisors. If $K = \overline{\mathbb{Q}}$, it does even imply that $\mathbb{C}G$ has no non-trivial zero divisors (we will recall the argument for this in Section 4).

Linnell proved the strong Atiyah conjecture over $\mathbb{C}G$ if G is torsion free and $G \in \mathcal{C}$, where \mathcal{C} contains all free groups and is closed under extensions with elementary amenable quotient and under directed unions.

In [16] (see also [17]), Linnell's results have been generalized to a larger class \mathcal{D} of groups, but only for $\mathbb{Q}G$ instead of $\mathbb{C}G$. One of the objectives of this paper is, to fix the flaw that only the coefficient ring \mathbb{Q} is allowed.

To get started, we have to repeat a number of definitions which are used in [16].

1.2 Definition. Assume that G is a finitely generated discrete group with a finite symmetric set of generators S (i.e. $s \in S$ implies $s^{-1} \in S$), and let H be an arbitrary discrete group. We say that G is a *generalized amenable extension of H* , if there is a set X with a free G -action (from the left) and a commuting free H -action (from the right), such that a sequence of H -subsets $X_1 \subset X_2 \subset X_3 \subset \dots \subset X$ exists with $\bigcup_{k \in \mathbb{N}} X_k = X$, and with $|X_k/H| < \infty$ for every $k \in \mathbb{N}$, and such that

$$\frac{|(S \cdot X_k - X_k)/H|}{|X_k/H|} \xrightarrow{k \rightarrow \infty} 0.$$

1.3 Definition. Let \mathcal{G} be the smallest class of groups which contains the trivial group and is closed under the following processes:

- If $H \in \mathcal{G}$ and G is a generalized amenable extension of H , then $G \in \mathcal{G}$.
- If $G = \lim_{i \in I} G_i$ is the direct or inverse limit of a directed system of groups $G_i \in \mathcal{G}$, then $G \in \mathcal{G}$.
- If $H \in \mathcal{G}$ and $U < H$, then $U \in \mathcal{G}$.

The class \mathcal{G} contains in particular all amenable groups, free groups, residually finite groups, and residually amenable groups.

1.4 Definition. Let \mathcal{D} be the smallest non-empty class of groups such that:

- (1) If G is torsion-free, A is elementary amenable, and we have an epimorphism $p: G \rightarrow A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A , then $G \in \mathcal{D}$.
- (2) \mathcal{D} is subgroup closed.
- (3) Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

Certainly \mathcal{D} contains all free groups, because these are residually torsion-free nilpotent. One of the main results of [16] (see also [17]) is

1.5 Theorem. *If $G \in \mathcal{D}$ then the strong Atiyah conjecture over $\mathbb{Q}G$ is true.*

We improve this to

1.6 Theorem. *If $G \in \mathcal{D}$ then the strong Atiyah conjecture over $\overline{\mathbb{Q}}G$ is true.*

As mentioned above, as a corollary we obtain that there are no non-trivial zero divisors in $\mathbb{C}G$, if $G \in \mathcal{D}$.

1.7 Corollary. *Suppose G is residually torsion free nilpotent, or residually torsion free solvable. Then the strong Atiyah conjecture is true for $\overline{\mathbb{Q}}G$.*

Proof. Under the assumptions we make, G belongs to \mathcal{D} . □

The proof of Theorem 1.5 relies on certain approximation results proved in [18] which give information about the spectrum of a self-adjoint matrix $A \in M(d \times d, \mathbb{Q}G)$. We have to improve these results to matrices over $\overline{\mathbb{Q}G}$. This will be done in Section 3, where also the precise statements are given. As a special case, we will prove the following result.

1.8 Theorem. *Suppose the group G has a sequence of normal subgroups $G \supset G_1 \supset G_2 \supset \dots$ with $\bigcap_{i \in \mathbb{N}} G_i = \{1\}$ and such that $G/G_i \in \mathcal{D}$ for every $i \in \mathbb{N}$. Assume that $B \in M(d \times d, \overline{\mathbb{Q}G})$. Let $p_i: G \rightarrow G/G_i$ denote the natural epimorphism, and let $B[i] \in M(d \times d, \overline{\mathbb{Q}G/G_i})$ be the image of B under the ring homomorphism induced by p_i . Then*

$$\dim_G(\ker(B)) = \lim_{i \rightarrow \infty} \dim_{G/G_i} \ker(B[i]).$$

We will also have to prove appropriate approximation results for generalized amenable extensions, whereas in [18] we only deal with the situation $H < G$ such that G/H is an amenable homogeneous space. Actually, in the amenable situation we are able to generalize Følner type approximation results to all matrices over the complex group ring.

These approximation results imply the following “stability” result about the Atiyah conjecture.

1.9 Proposition. *Assume G is a subgroup of the direct or inverse limit of a directed system of groups $(G_i)_{i \in I}$. Assume $G_i \in \mathcal{G}$, G_i is torsion-free, and G_i satisfies the strong Atiyah conjecture over $\mathbb{Q}G_i$ for every $i \in I$. Then G also satisfies the strong Atiyah conjecture over $\mathbb{Q}G$.*

In Section 5 we will address the question when Theorem 1.8 holds for all matrices over $\mathbb{C}G$. We will answer this question affirmatively in particular if G is torsion free and elementary amenable.

1.10 Proposition. *Let G be a torsion free elementary amenable group with a nested sequence of normal subgroups $G \supset G_1 \supset G_2 \supset \dots$ such that $\bigcap G_k = \{1\}$. Assume $G/G_i \in \mathcal{G}$ for every $i \in \mathbb{N}$. Let $A \in M(d \times d, \mathbb{C}G)$ and $A_k \in M(d \times d, \mathbb{C}[G/G_k])$ its image under the maps induced from the epimorphism $G \rightarrow G/G_k$. Then*

$$\dim_G(\ker(A)) = \lim_{k \rightarrow \infty} \dim_{G/G_k}(\ker(A_k)).$$

In section 6 we study the algebraic eigenvalue property for discrete groups G . This implies certain number theoretic properties of eigenvalues for the combinatorial Laplacian on L^2 -cochains on any normal covering space of a finite CW complex. We establish the absence of eigenvalues that are transcendental numbers, whenever the covering transformation group is either amenable or in the Linnell class \mathcal{C} . We also establish the absence of eigenvalues that are Liouville transcendental numbers whenever the covering transformation group is either residually finite or more generally in the bootstrap class \mathcal{G} . The statements rely on new approximation results for spectral density functions of the Laplacian on L^2 -cochains on any normal covering space having such covering transformation groups.

2 Notation and Preliminaries

Let G be a (finitely generated) group and $A \in M(d \times d, \mathbb{C}G)$. We consider A as a matrix indexed by $I \times I$ with $I = \{1, \dots, d\} \times G$. We can equally well regard A as a family of $d \times d$ complex matrices indexed by $G \times G$. With this interpretation, the following invariance condition holds

$$A_{x,y} = A_{xz,yz}. \tag{2.1}$$

for all $x, y, z \in G$. Similarly, elements of $l^2(G)^d$ will be regarded as sequences of elements of \mathbb{C}^d indexed by elements of G so that

$$(Af)_x = \sum_y A_{x,y} f_y.$$

Note that one can also identify $l^2(G)^d$ with $l^2(G) \otimes_{\mathbb{C}} \mathbb{C}^d$, and similarly for matrices, as we will do occasionally.

Observe also that there exists a positive number r depending on A only such that

$$A_{x,y} = 0 \quad \text{if} \quad \rho(x,y) > r \quad (2.2)$$

where $\rho(x,y)$ denotes the distance in the word metric between x and y for a given finite set of generators of G .

2.1 Free actions

If G acts freely on a space X (as in Definition 1.2), then every matrix A as above (and actually every operator on $l^2(G)^d$) induces an operator A_X on $l^2(X)^d$.

Choosing a free G -basis for X , $l^2(X)^d$ can be identified with a (possibly infinite) direct sum of copies of $l^2(G)^d$, and A_X is a diagonal operator with respect to this decomposition, where each operator on $l^2(G)^d$ is equal to A . Consequently, if we apply the functional calculus, $f(A)_X = f(A_X)$ for every measurable function f (provided A is self adjoint).

Pick $x \in X$ and define

$$\text{tr}_G(A_X) := \sum_{i=1}^d \langle A_X(x \otimes e_i), x \otimes e_i \rangle = \sum_{i=1}^d \langle A(1 \otimes e_i), 1 \otimes e_i \rangle,$$

where (e_1, \dots, e_d) is the standard basis of \mathbb{C}^d and where we identify $l^2(X)^d$ with a direct sum of a number of copies of $l^2(G)^d$ as above and use the identification $l^2(X)^d = l^2(X) \otimes_{\mathbb{C}} \mathbb{C}^d$. Then $\text{tr}_G(A_X) = \text{tr}_G(A)$ (in particular the expression does not depend on $x \in X$).

2.2 Spectral density functions and determinants

Assume $A \in M(d \times d, \mathbb{C}G)$ as above, and assume that A , considered as an operator on $l^2(G)^d$, is positive self adjoint. This is the case e.g. if $A = B^*B$ for some $B \in M(d \times d, \mathbb{C}G)$. Here B^* is the adjoint in the sense of operators on $l^2(G)^d$. If $B = (b_{ij})$ with $b_{ij} \in \mathbb{C}G$, then $B^* \in M(d \times d, \mathbb{C}G)$ is the matrix with entry b_{ij}^* at the position (j, i) , where $(b_{ij}^*)_g = \overline{b_{ij} g^{-1}}$.

For $\lambda \geq 0$ let pr_λ be the spectral projection of A corresponding to the interval $[0, \lambda]$. This operator also commutes with the right G -action on $l^2(G)^d$. Using exactly the same definition of the regularized trace as before for finite matrices, we define the spectral density function

$$F_A(\lambda) := \text{tr}_G(\text{pr}_\lambda).$$

This function is right continuous monotonic increasing, with

$$F_A(0) = \dim_G(\ker(A)).$$

We will be interested in the behavior of this function near 0.

Using the spectral density function, we can define a normalized determinant as follows.

2.3 Definition. If $A \in M(d \times d, \mathbb{C}G)$ is positive self adjoint, we define (the logarithm) of its *normalized determinant* by the Riemann-Stieltjes integral

$$\ln \det_G(A) = \int_{0^+}^{\infty} \ln(\lambda) dF_A(\lambda).$$

Using integration by parts, if $K \geq \|A\|$, we can rewrite

$$\ln \det_G(A) = \ln(K)(F_A(K) - F_A(0)) - \int_{0^+}^K \frac{F_A(\lambda) - F_A(0)}{\lambda} d\lambda.$$

Observe that, if $G = \{1\}$, $\det_G(A)$ is the product of the eigenvalues of A which are different from zero. In general, $\det_G(A)$ is computed from the spectrum of A on the orthogonal complement of its kernel.

2.3 Generalized amenable extensions

Let G be a generalized amenable extension of H with a finite set of symmetric generators S . Let $X_1 \subset X_2 \subset \cdots \subset X$ be a generalized amenable exhaustion as in Definition 1.2.

2.4 Definition. For $r \in \mathbb{N}$, we define the r -neighborhood of the boundary of X_i (with respect to S) as

$$\mathfrak{N}_r(X_i) := S^r X_i \cap S^r(X - X_i).$$

Note that all these sets are right H -invariant.

2.5 Lemma. *We have*

$$\frac{|(S^n \cdot X_k - X_k)/H|}{|X_k/H|} \xrightarrow{k \rightarrow \infty} 0 \quad \forall n \in \mathbb{N}, \quad (2.6)$$

$$\frac{|(\mathfrak{N}_n(X_k)/H)|}{|X_k/H|} \xrightarrow{k \rightarrow \infty} 0 \quad \forall n \in \mathbb{N}. \quad (2.7)$$

In particular, for any finite subset T of G ,

$$\frac{|(T \cdot X_k \cap T(X - X_k))/H|}{|X_k/H|} \xrightarrow{k \rightarrow \infty} 0. \quad (2.8)$$

Proof. Observe that $S^n X_k = S^{n-1} X_k \cup S(S^{n-1} X_k - X_k)$. Consequently,

$$\frac{|(S^n X_k - X_k)/H|}{|X_k/H|} \leq \frac{|(S^{n-1} X_k - X_k)/H|}{|X_k/H|} + |S| \frac{|(S^{n-1} X_k - X_k)/H|}{|X_k/H|}.$$

Since $|S|$ is fixed, (2.6) follows by induction.

Next, $\mathfrak{N}_n(X_k) = S^n X_k \cap S^n(X - X_k)$ is contained in the union of $S^n X_k - X_k$ and $S^n(X - X_k) \cap X_k$. For the latter observe that $vy = x$ with $v \in S^n$, $y \notin X_k$ and $x \in X_k$ means that $y = v^{-1}x$, and since S is symmetric $v^{-1} \in S^n$, i.e. $y \in S^n X_k - X_k$. Consequently, $|(S^n(X - X_k) \cap X_k)/H| \leq |S^n| \cdot |(S^n X_k - X_k)/H|$, such that (2.7) follows from (2.6).

(2.8) follows, since $T \subset S^n$ for sufficiently large $n \in \mathbb{N}$. \square

2.9 Example. If $H \subset G$ and G/H is an amenable Schreier graph (e.g. if H is a normal subgroup of G such that G/H is amenable), then G is a generalized amenable extension of H . In these cases, we can take X to be G , and $X_k := p^{-1}Y_k$ if $p: G \rightarrow G/H$ is the natural epimorphism, and Y_k are a Følner exhaustion of the amenable Schreier graph (or amenable group, respectively) G/H .

The following result follows from an idea of Warren Dicks.

2.10 Example. Assume G is a finitely generated group with symmetric set of generators S which acts on a set X (not necessarily freely). Assume there is an exhaustion $X_1 \subset X_2 \subset \dots \subset X$ of X by finite subsets such that

$$\frac{|SX_k - X_k|}{|X_k|} \xrightarrow{k \rightarrow \infty} 0.$$

For $x \in X$ let $G_x := \{g \in G \mid gx = x\}$ be the stabilizer of x . If a group H contains an abstractly isomorphic copy of G_x for every $x \in X$, e.g. if H is the free product or the direct sum of all G_x ($x \in X$), then G is a generalized amenable extension of H .

This follows immediately from the following Lemma, which is (with its proof) due to Warren Dicks.

2.11 Lemma. *Let X be a left G -set.*

If, for every $x \in X$, the G -stabilizer G_x embeds in H , then

(1) *there exists a family of maps*

$$(\gamma_x: G \rightarrow H \mid x \in X)$$

such that, for all $x \in X$, for all $g_1, g_2 \in G$,

$$\gamma_{g_1x}(g_2)\gamma_x(g_1) = \gamma_x(g_2g_1),$$

and the restriction of γ_x to G_x is injective (and a group homomorphism).

If (1) holds, then there is a G -free H -free (G, H) -bi-set structure on $X \times H$ such that

$$g(x, y)h = (gx, \gamma_x(g)yh),$$

for all $g \in G$, $x \in X$, $y, h \in H$. Moreover, for each $x \in X$, if Y is a left $\gamma_x(G_x)$ -transversal in H , then $\{x\} \times Y$ is a left G -transversal in $Gx \times H$.

Proof. Let $x_0 \in X$, and let $\alpha: G_{x_0} \rightarrow H$ be an injective group homomorphism.

We claim that there exists a right α -compatible map $\beta: G \rightarrow H$, that is, $\beta(gk) = \beta(g)\alpha(k)$ for all $k \in G_{x_0}$ and all $g \in G$. Since G is free as right G_{x_0} -set, we can construct β as follows. Choose a right G_{x_0} -transversal in G , and map each element of this transversal arbitrarily to an element of H , and extend by the G_{x_0} -action to all of G .

Now let $x \in Gx_0$, and choose $g_x \in G$ such that $x = g_x x_0$.

Define $\gamma_x: G \rightarrow H$ by $\gamma_x(g) = \beta(gg_x)\beta(g_x)^{-1}$ for all $g \in G$. Since β is right α -compatible, we see that γ_x is independent of the choice of g_x .

For all $g_1, g_2 \in G$,

$$\begin{aligned} \gamma_{g_1x}(g_2)\gamma_x(g_1) &= \beta(g_2g_1g_x)\beta(g_1g_x)^{-1}\beta(g_1g_x)\beta(g_x)^{-1} \\ &= \beta(g_2g_1g_x)\beta(g_x)^{-1} = \gamma_x(g_2g_1) \end{aligned}$$

We now show that γ_x restricted to G_x is an injective group homomorphism. Suppose that $g \in G_x$, so $g_x^{-1}gg_x \in G_{x_0}$, and

$$\begin{aligned}\gamma_x(g) &= \beta(gg_x)\beta(g_x)^{-1} = \beta(g_xg_x^{-1}gg_x)\beta(g_x)^{-1} \\ &= \beta(g_x)\alpha(g_x^{-1}gg_x)\beta(g_x)^{-1}.\end{aligned}$$

Since α is injective, we see that γ_x is injective on G_x .

Since Gx_0 is an arbitrary G -orbit in X , we have proved that (1) holds.

Now suppose that (1) holds. It is straightforward to check that we have the desired (G, H) -bi-set structure on $X \times H$, and that it is G -free and H -free. Finally, for any $x \in X$, there is a well-defined, injective map from $\gamma_x(G_x) \setminus H$ to $G \setminus (X \times H)$, with $\gamma_x(G_x)h$ mapping to $G(x, h)$, for all $h \in H$. It is surjective, since $G(gx, h) = G(x, \gamma_x(g)^{-1}h)$ for all $g \in G$. \square

One should observe that such a G -space X of course is the disjoint union of Schreier graphs G/G_x . However, X being an amenable G -set in the sense of Example 2.9 does not necessarily imply that any of these G/G_x is an amenable Schreier graph. If, for a fixed set of generators S of G one can find quotient groups with arbitrarily small exponential growth, then the disjoint union of these quotients becomes an amenable G -set.

2.4 Direct and inverse limits

Suppose that a group G is the direct or inverse limit of a directed system of groups G_i , $i \in I$. The latter means that we have a partial ordering $<$ on I such that for all $i, j \in I$ there is $k \in I$ with $i < k$ and $j < k$, and maps $p_{ij}: G_i \rightarrow G_j$ in the case of the direct limit and $p_{ji}: G_j \rightarrow G_i$ in the case of the inverse limit whenever $i < j$, satisfying the obvious compatibility conditions. In the case of a direct limit we let $p_i: G_i \rightarrow G$ be the natural maps and similarly, for an inverse limit, $p_i: G \rightarrow G_i$. The maps between groups induce mappings between matrices with coefficients in group rings in the obvious way. Let $B \in M(d \times d, \mathbb{C}G)$. If G is an inverse limit we set $B[i] = p_i(B)$.

In the case of a direct limit it is necessary to make some choices. Namely, let $B = (a_{kl})$ with $a_{kl} = \sum_{g \in G} \lambda_{kl}^g g$. Then, only finitely many of the λ_{kl}^g are nonzero. Let V be the corresponding finite collection of $g \in G$. Since G is the direct limit of G_i , we can find $j_0 \in I$ such that $V \subset p_{j_0}(G_{j_0})$. Choose an inverse image for each g in G_{j_0} . This gives a matrix $B[j_0] \in M(d \times d, G_{j_0})$ which is mapped to $B[i] := p_{j_0 i} B[j_0] \in M(d \times d, G_i)$ for $i > j_0$.

Observe that in both cases $B^*[i] = B[i]^*$.

3 Approximation with algebraic coefficients

3.1 Lower bounds for determinants

We will define in Definition 3.18 an invariant $\kappa(A)$ of matrices $A \in M(d \times d, \mathbb{C}G)$ which occurs in the lower bounds of generalized determinants we want to establish.

3.1 Definition. We say that a group G has the *determinant bound property* if, for every $B \in M(d \times d, o(\overline{\mathbb{Q}})G)$ and $\Delta = B^*B$, where $o(\overline{\mathbb{Q}})$ is the ring of algebraic integers over \mathbb{Z} in \mathbb{C} , the

following is true: Choose a finite Galois extension $i: L \subset \mathbb{C}$ of \mathbb{Q} such that $B \in M(d \times d, LG)$. Let $\sigma_1, \dots, \sigma_r: L \rightarrow \mathbb{C}$ be the different embeddings of L in \mathbb{C} with $\sigma_1 = i$. Then

$$\ln \det_G(\Delta) \geq -d \sum_{k=2}^r \ln(\kappa(\sigma_k(\Delta))). \quad (3.2)$$

We say that G has the *algebraic continuity property*, if always

$$\dim_G(\ker(B)) = \dim_G(\ker(\sigma_k B)) \quad \forall k = 1, \dots, r. \quad (3.3)$$

3.4 Theorem. *If $G \in \mathcal{G}$, then G has the determinant bound property and the algebraic continuity property.*

3.5 Corollary. *If $B = B^* \in M(d \times d, \mathbb{Q}G)$ and $\lambda \in \mathbb{R}$ is algebraic and an eigenvalue of B , then λ is totally real, i.e. $\sigma(\lambda) \in \mathbb{R}$ for every automorphism $\sigma: \mathbb{C} \rightarrow \mathbb{C}$. The same is true if \mathbb{Q} is replaced by any totally real algebraic extension $\mathbb{Q} \subset R \subset \mathbb{C}$, i.e. $\sigma(R) \subset \mathbb{R}$ for every automorphism $\sigma: \mathbb{C} \rightarrow \mathbb{C}$.*

Estimate (3.2) implies a sometimes more convenient estimate for the spectral density functions, which we want to note now.

3.6 Corollary. *Let $C \in \mathbb{R}$. Suppose $A \in M(d \times d, \mathbb{C}G)$ with $\|A\| \geq 1$ is positive self adjoint and satisfies*

$$\ln \det_G(A) \geq -C.$$

Then, for $0 < \lambda < \|A\|$,

$$F_A(\lambda) - F_A(0) \leq \frac{C + d \ln(\|A\|)}{-\ln(\lambda/\|A\|)}. \quad (3.7)$$

In particular, in the situation of Theorem 3.4, for all λ such that $0 < \lambda < \|A\|$, we have

$$\begin{aligned} F_A(\lambda) - F_A(0) &\leq \frac{d \cdot (\ln(\|A\|) + \sum_{k=2}^r \ln(\kappa(\sigma_k(A))))}{-\ln(\lambda/\|A\|)} \\ &\leq \frac{d \sum_{k=1}^r \ln(\kappa(\sigma_k(A)))}{-\ln(\lambda/\|A\|)}. \end{aligned} \quad (3.8)$$

If $r = 1$, i.e. $A \in M(d \times d, \mathbb{Z}G)$, then this simplifies to

$$F_A(\lambda) - F_A(0) \leq \frac{d \ln(\|A\|)}{-\ln(\lambda/\|A\|)}. \quad (3.9)$$

Proof. The proof of (3.7) is done by an elementary estimate of integrals. Fix $0 < \lambda < \|A\|$. Then

$$\begin{aligned} -C &\leq \ln \det_G(A) = \int_{0^+}^{\|A\|} \ln(\tau) dF_A(\tau) \\ &= \int_{0^+}^{\lambda} \ln(\tau) dF_A(\tau) + \int_{\lambda^+}^{\|A\|} \ln(\tau) dF_A(\tau) \\ &\leq \ln(\lambda)(F_A(\lambda) - F_A(0)) + \ln(\|A\|)(F_A(\|A\|) - F_A(\lambda)) \\ &\leq \ln(\lambda/\|A\|)(F_A(\lambda) - F_A(0)) + d \ln(\|A\|). \end{aligned}$$

This implies that

$$F_A(\lambda) - F_A(0) \leq \frac{C + d \ln(\|A\|)}{-\ln(\lambda/\|A\|)}.$$

For (3.8) we use (3.2) and the fact that $\|A\| \leq \kappa(\sigma_1(A))$. \square

3.2 Approximation results

We will prove Theorem 3.4 together with certain approximation results for L^2 -Betti numbers in an inductive way. We now formulate precisely the approximation results, which were already mentioned in the introduction. To do this, we first describe the situation that we will consider.

3.10 Situation. Given is a group G and a matrix $B \in M(d \times d, \mathbb{C}G)$. We assume that one of the following applies:

- (1) G is the inverse limit of a directed system of groups G_i . Let $B[i] \in M(d \times d, \mathbb{C}G)$ be the image of B under the natural map $p_i : G \rightarrow G_i$.
- (2) G is the direct limit of a directed system of groups G_i . For $i \geq i_0$ we choose $B[i] \in M(d \times d, \mathbb{C}G)$ as described in subsection 2.4.
- (3) G is a generalized amenable extension of U with free G - U space X and Følner exhaustion $X_1 \subset X_2 \subset \cdots \subset X$ of X as in Definition 1.2 (with $|X_i/U| := N_i < \infty$). Let $P_i : l^2(X)^d \rightarrow l^2(X_i)^d$ be the corresponding projection operators. Recall that B_X is the operator on $l^2(X)^d$ induced by B , using the free action of G on X . We set $B[i] := P_i B_X P_i^*$.

Moreover, we set $\Delta := B^*B$, and $\Delta[i] = B[i]^*B[i]$ (for $i \geq i_0$).

In the first two cases we write \dim_i , tr_i and \det_i for \dim_{G_i} , tr_{G_i} and \det_{G_i} , respectively. In the third case we use a slight modification. We set $\dim_i := \frac{1}{N_i} \dim_U$ and $\text{tr}_i := \frac{1}{N_i} \text{tr}_U$, $\ln \det_i := \frac{1}{N_i} \ln \det_U$, and write $G_i := U$ to unify the notation.

We also define the spectral density functions $F_{\Delta[i]}(\lambda)$ as in Subsection 2.2, but using tr_i instead of the trace used there.

We are now studying approximation results for the dimensions of eigenspaces. We begin with a general approximation result in the amenable case. In particular, we generalize the main result of [4] from the rational group ring to the complex group ring of an amenable group, with a simpler proof. Eckmann [6] uses related ideas in his proof of the main theorem of [4].

3.11 Theorem. *Assume G is a generalized amenable extension of U , and $B \in M(d \times d, \mathbb{C}G)$. Adopt the notation and situation of 3.10(3). Then*

$$\dim_G(\ker(B)) = \lim_{i \rightarrow \infty} \dim_i \ker(B[i]) = \lim_{i \rightarrow \infty} \frac{\dim_U \ker(B[i])}{N_i}.$$

We can generalize this to 3.10(1) and 3.10(2), but, for the time being, only at the expense of restricting the coefficient field to $\overline{\mathbb{Q}}$.

3.12 Theorem. *Suppose $B \in M(d \times d, \overline{\mathbb{Q}}G)$ is the limit of matrices $B[i]$ over $\overline{\mathbb{Q}}G_i$ as described in 3.10. Assume $G_i \in \mathcal{G}$ for all i . Then*

$$\dim_G(\ker(B)) = \lim_{i \in I} \dim_i(\ker(B[i])).$$

Note that it is sufficient to prove Theorem 3.12 for $\Delta = B^*B$, since the kernels of B and of Δ coincide, and since by Lemma 3.17 $B[i]^* = (B^*)[i]$. Moreover, every algebraic number is the product of a rational number and an algebraic integer (compare [5, 15.24]). Hence we can multiply B by a suitable integer N (without changing the kernels) and therefore assume that B , and Δ , is a matrix over $o(\overline{\mathbb{Q}})G$.

We start with one half of the proof of Theorem 3.11.

3.13 Lemma. *In the situation of Theorem 3.11,*

$$\dim_G(\ker(B)) \leq \liminf_{i \rightarrow \infty} \frac{\dim_U \ker(B[i])}{|X_i/U|}.$$

Proof. First observe that, in the situation of 3.10(3), we have to discuss the relation between B and B_X . Since the action of G on X is free, B_X is defined not only for $B \in M(d \times d, \mathbb{C}G)$, but even for $B \in M(d \times d, \mathcal{N}G)$, where $\mathcal{N}G$ is the group von Neumann algebra of G , i.e. the weak closure of $\mathbb{C}G$ in $\mathcal{B}(l^2(G))$, or equivalently (using the bicommutant theorem) the set of all bounded operators on $l^2(G)$ which commute with the right G -action.

Let P be the orthogonal projection onto $\ker(B)$. Then $\dim_G(\ker(B)) = \text{tr}_G(P) = \text{tr}_G(P_X)$ (where P_X is the orthogonal projection onto $\ker(B_X)$, as explained above).

Let P_i be the orthogonal projection of $l^2(X)^d$ onto the set of functions with values in \mathbb{C}^d and supported on X_i . We need to compare $P_i B_X$ and $B_X P_i$. Let T be the support of B , i.e. the set of all $g \in G$ such that the coefficient of g in at least one entry in the matrix B is non-zero, and fix $r \in \mathbb{N}$ such that $T \subseteq S^r$. Also let A be the matrix of B_X with respect to the basis X . A calculation shows that

$$(P_i B_X f - B_X P_i f)_x = \begin{cases} \sum_{y \in X \setminus X_i} A_{x,y} f_y, & \text{if } x \in X_i \\ \sum_{y \in X_i} A_{x,y} f_y, & \text{if } x \notin X_i. \end{cases} \quad (3.14)$$

It follows that the value of the difference is determined by the values f_y on the r -neighborhood $\mathfrak{N}_r(X_i)$ of the boundary of X_i .

We have

$$\text{tr}_G P_X = \frac{1}{|X_i/U|} \text{tr}_U P_i P_X$$

by the invariance property (2.1). Since $\|P_i P\| \leq 1$, $\text{tr}_U P_i P \leq \dim_U \overline{\text{im } P_i P}$ and $\text{im } P_i P = P_i(\ker(B_X))$, we have

$$\dim_G \ker(B_X) \leq \frac{1}{|X_i/U|} \dim_U \overline{P_i(\ker(B_X))}.$$

To simplify the notation we identify the functions on X supported on $Y \subset X$ with functions on X . With this convention, $P_i = P_i^*$ and $B[i] = P_i B_X P_i$.

Consider $f \in \ker(B_X)$, i.e. $B_X f = 0$. It follows that

$$B[i]P_i f = P_i B_X P_i f - P_i B_X f = P_i(B_X P_i f - P_i B_X f).$$

By (3.14), the right-hand side is determined uniquely by the restriction of f to $\mathfrak{N}_r(X_i)$. Thus, if the restriction of $f \in \ker(B_X)$ to $\mathfrak{N}_r(X_i)$ is zero, then

$$B[i]P_i f = 0. \quad (3.15)$$

The bounded U -equivariant operator $P_i: l^2(X)^d \rightarrow l^2(X_i)^d$ restricts to a bounded operator $P_i: \ker(B_X) \rightarrow l^2(X_i)^d$. Let $V \subset \ker(B_X)$ be the orthogonal complement of the kernel of this restriction inside $\ker(B_X)$. Then P_i restricts to an injective map $V \rightarrow l^2(X_i)$ with image $P_i(\ker(B_X))$. Consequently, $\dim_U(V) = \dim_U \overline{P_i(\ker(B_X))}$ [14, see §2].

Let $\text{pr}_i: l^2(X_i)^d \rightarrow \ker(B[i])$ be the orthogonal projection, and let $Q_i: l^2(X)^d \rightarrow l^2(\mathfrak{N}_r(X_i))^d$ be the restriction map to the subset $\mathfrak{N}_r(X_i)$ (this again is the orthogonal projection if we use the above convention).

The bounded U -equivariant linear map

$$\phi: l^2(X)^d \rightarrow l^2(\mathfrak{N}_r(X_i))^d \oplus \ker(B[i]): f \mapsto (Q_i(f), \text{pr}_i P_i(f))$$

restricts to a map $\alpha: V \rightarrow l^2(\mathfrak{N}_r(X_i))^d \oplus \ker(B[i])$. We claim that α is injective. In fact, if $f \in V \subset \ker(B_X)$ and $Q_i(f) = 0$ then by (3.15) $P_i(f) \in \ker(B[i])$. Therefore, $\pi_i P_i(f) = P_i(f)$. Since $f \in \ker(\alpha)$ this implies $P_i(f) = 0$. But the restriction of P_i to V is injective, hence $f = 0$ as claimed.

It follows that

$$\dim_G(\ker(B)) \leq \frac{1}{|X_i/U|} \dim_U(l^2(\mathfrak{N}_r(X_i))^d \oplus \ker(B[i])).$$

Since $\mathfrak{N}_r(X_i)$ is a free U -space, $\dim_U l^2(\mathfrak{N}_r(X_i))^d = d |\mathfrak{N}_r(X_i)/U|$, therefore

$$\dim_G(\ker(B)) \leq d \frac{|\mathfrak{N}_r(X_i)/U|}{|X_i/U|} + \frac{\dim_U \ker(B[i])}{|X_i/U|}.$$

By Lemma 2.5, the first summand on the right hand side tends to zero for $i \rightarrow \infty$. Consequently

$$\dim_G(\ker(B)) \leq \liminf_{i \rightarrow \infty} \frac{\dim_U \ker(B[i])}{|X_i/U|}.$$

□

Using similar ideas, it would be possible to finish the proof of Theorem 3.11. However, we will need more refined estimates on the size of the spectrum *near* zero to establish Theorem 3.4. Moreover, what follows is in the same way also needed for the proof of Theorem 3.12, so we will not finish the proof now. Instead, we will continue with the preparation for the proofs of all the results of Subsections 3.1 and 3.2, which will comprise the rest of this and the following three subsections.

We start with two easy observations.

3.16 Lemma. *With the definitions given in 3.10, tr_i in each case is a positive and normal trace, which is normalized in the following sense: if $\Delta = \text{id} \in M(d \times d, \mathbb{Z}G)$ then $\text{tr}_i(\Delta[i]) = d$.*

Proof. This follows since tr_{G_i} has the corresponding properties. □

3.17 Lemma. *Suppose we are in the situation described in 3.10. Let K and L be subrings of \mathbb{C} . If B is defined over KG , then $B[i]$ is defined over KG_i for all i in the first and third case, and for $i \geq i_0$ in the second case.*

Let $\sigma: K \rightarrow L$ be a ring homomorphism. Then it induces homomorphisms from matrix rings over KG or $KG[i]$ to matrix rings over LG or $LG[i]$ respectively; we shall also indicate by σ any one of these homomorphisms. Furthermore we get $\sigma(B[i]) = (\sigma B)[i]$ and $B[i]^ = (B^*)[i]$ whenever $B[i]$ is defined. In particular, the two definitions of $\Delta[i]$ agree.*

Since we are only interested in $B[i]$ for large i , without loss of generality, we assume that the statements of Lemma 3.17 are always fulfilled.

Proof. The construction of $A[i]$ involve only algebraic (ring)-operations of the coefficients of the group elements, and every σ is a ring-homomorphism, which implies the first and second assertion. Taking the adjoint is a purely algebraic operation, as well: if $B = (b_{ij})$ then $B^* = (b_{ji}^*)$, where for $b = \sum_{g \in G} \lambda_g g \in \mathbb{C}G$ we have $b^* = \sum_{g \in G} \overline{\lambda_{g^{-1}}} g$. Since these algebraic operations in the construction of $B[i]$ commute with complex conjugation, the statement about the adjoint operators follows. \square

3.3 An upper bound for the norm of a matrix

The approximation results we want to prove here have a sequence of predecessors in [13], [4], and [18]. The first two rely on certain estimates of operator norms (which are of quite different type in the two cases). We want to use a similar approach here, but have to find bounds which work in both the settings considered in [13] and [4]. (These problems were circumvented in [18] using a different idea, which however we didn't manage to apply here). The following definition will supply us with such a bound.

3.18 Definition. Let I be an index set. For $A := (a_{ij})_{i,j \in I}$ with $a_{ij} \in \mathbb{C}$ set

$$S(A) := \sup_{i \in I} |\text{supp}(z_i)|,$$

where z_i is the vector $z_i := (a_{ij})_{j \in I}$ and $\text{supp}(z_i) := \{j \in I \mid a_{ij} \neq 0\}$. Set

$$|A|_\infty := \sup_{i,j} |a_{ij}|.$$

Set $A^* := (\overline{a_{ji}})_{i,j \in I}$. If $S(A) + S(A^*) + |A|_\infty < \infty$ set

$$\kappa(A) := \sqrt{S(A)S(A^*)} \cdot |A|_\infty.$$

Else set $\kappa(A) := \infty$.

If G is a discrete group and $A \in M(d \times d, \mathbb{C}G)$ consider A as a matrix with complex entries indexed by $I \times I$ with $I := \{1, \dots, d\} \times G$, and define $\kappa(A)$ using this interpretation.

3.19 Definition. Let I and J be two index sets. Two matrices $A = (a_{is})_{i,s \in I}$ and $B = (b_{jt})_{j,t \in J}$ are called *of the same shape*, if for every row or column of A there exists a row or column, respectively, of B with the same number of non-zero entries, and if the sets of non-zero entries of A and B coincide, i.e.

$$\{0\} \cup \{a_{is} \mid i, s \in I\} = \{0\} \cup \{b_{jt} \mid j, t \in J\}.$$

3.20 Lemma. *If two matrices A and B have the same shape, then*

$$\kappa(A) = \kappa(B).$$

Proof. This follows immediately from the definitions. \square

3.21 Lemma. *If $A \in M(d \times d, \mathbb{C}G)$ as above then $\kappa(A) < \infty$.*

Proof. If $\sum \alpha_g g \in \mathbb{C}G$ then $(\sum \alpha_g g)^* = \sum \overline{\alpha_{g^{-1}}} g$, and correspondingly for matrices.

The assertion follows from the finite support condition and from G -equivariance $a_{(k,g),(l,h)} = a_{(k,gu),(l,hu)}$ for $g, h, u \in G$. \square

3.22 Lemma. *Assume $A = (a_{ij})_{i,j \in I}$ fulfills $\kappa(A) < \infty$. Then A induces a bounded operator on $l^2(I)$ and for the norm we get*

$$\|A\| \leq \kappa(A).$$

This applies in particular to $A \in M(d \times d, \mathbb{C}G)$ acting on $l^2(G)^d$.

Proof. Assume $v = (v_j) \in l^2(I)$. With $z_i := (a_{ij})_{j \in I}$ we get

$$|Av|^2 = \sum_{i \in I} |\langle z_i, v \rangle_{l^2(I)}|^2.$$

We estimate

$$\begin{aligned} |\langle z_i, v \rangle|^2 &= \left| \sum_{j \in \text{supp}(z_i)} a_{ij} \overline{v_j} \right|^2 \leq \sum_{j \in \text{supp}(z_i)} |a_{ij}|^2 \cdot \sum_{j \in \text{supp}(z_i)} |v_j|^2 \\ &\leq S(A) \sup_{i,j \in I} |a_{ij}|^2 \cdot \sum_{j \in \text{supp}(z_i)} |v_j|^2. \end{aligned}$$

Here we used the Cauchy-Schwarz inequality, but took the size of the support into account. It follows that

$$|Av|^2 \leq S(A) \sup_{i,j \in I} |a_{ij}|^2 \cdot \sum_{i \in I} \sum_{j \in \text{supp}(z_i)} |v_j|^2. \quad (3.23)$$

Observe that

$$j \in \text{supp}(z_i) \iff a_{ij} \neq 0 \iff \overline{a_{ij}} \neq 0 \iff i \in \text{supp}(y_j),$$

where y_j is the j -th row of A^* . Consequently, for each fixed $j \in I$, there are not more than $S(A^*)$ elements $i \in I$ with $j \in \text{supp}(z_i)$, and hence

$$\sum_{i \in I} \sum_{j \in \text{supp}(z_i)} |v_j|^2 = \sum_{j \in I} \sum_{i \in \text{supp}(y_j)} |v_j|^2 \leq S(A^*) |v|^2. \quad (3.24)$$

Now (3.24) and (3.23) give the desired inequality

$$|Av| \leq \kappa(A) \cdot |v|. \quad \square$$

3.25 Lemma. *Assume $J \subset I$ and $P^*: l^2(J) \rightarrow l^2(I)$ is the induced isometric embedding, with adjoint orthogonal projection $P: l^2(I) \rightarrow l^2(J)$. If A is a matrix indexed by I , then PAP^* is a matrix indexed by J , and*

$$\kappa(PAP^*) \leq \kappa(A).$$

Proof. This is obvious from the definition, since we only remove entries in the original matrix A to get PAP^* . \square

3.26 Lemma. *Suppose $U < G$. For $A \in M(d \times d, \mathbb{C}U)$ let $i(A) \in M(d \times d, \mathbb{C}G)$ be the image of A under the map induced by the inclusion $U \rightarrow G$. Then*

$$\kappa(A) = \kappa(i(A)).$$

Proof. Using the fact that G is a free left U -set, and a set of representatives of the translates of U , we see that the matrix for $i(A)$ has copies of A on the “diagonal” and zeros elsewhere (compare the proof of [18, 3.1]). Therefore the two matrices have the same shape and the statement follows from Lemma 3.20. \square

3.27 Lemma. *Let G be a generalized amenable extension of H , and X a corresponding free G - H -set as in Definition 1.2. To $A \in M(d \times d, \mathbb{C}G)$ we associate the operator $A_X : l^2(X)^d \rightarrow l^2(X)^d$. It is now natural to consider A_X as a matrix indexed by $I \times I$ with $I := \{1, \dots, d\} \times X$ (similar to A , where X is replaced by G). With this convention $\kappa(A_X)$ is defined. Then*

$$\kappa(A_X) = \kappa(A).$$

Proof. The argument is exactly the same as for Lemma 3.26, replacing the free U -set G there by the free G -set X here. \square

We can combine these results to get a uniform estimate for $\kappa(B[i])$, if we are in the situation described in 3.10.

3.28 Lemma. *Suppose we are in the situation described in 3.10. Then*

$$\kappa(B[i]) \leq \kappa(B)$$

for i sufficiently large.

Proof. In the cases 3.10(1) and 3.10(2), we actually prove that $\kappa(B[i]) = \kappa(B)$ for i sufficiently large. Observe that only finitely many elements of G occur as coefficients in A . Let Y be the corresponding subset of G . If G is the inverse limit of G_i and $G \rightarrow G_i$ restricted to Y is injective (which is the case for i sufficiently big) the claim follows because the matrices describing B and $B[i]$ then have the same shape, and we can apply Lemma 3.20.

If G is the direct limit of the groups G_i then, since $A[j_0]$ has finite support, there is $j_1 > j_0$ such that for $i \geq j_1$ the map $G_i \rightarrow G$ is injective if restricted to the support of $A[i]$. We arrive at the conclusion as before.

In the case 3.10(3), first observe that by Lemma 3.27 $\kappa(B) = \kappa(B_X)$. Because of Lemma 3.25, on the other hand, $\kappa(B[i]) \leq \kappa(B_X)$. \square

3.4 Convergence results for the trace

3.29 Lemma. *Suppose we are in the situation described in 3.10. Assume that $p(x) \in \mathbb{C}[x]$ is a polynomial. Then*

$$\mathrm{tr}_G p(B) = \lim_{i \in I} \mathrm{tr}_i p(B[i])$$

Proof. For 3.10(1) and 3.10(2), this is proved in [18, Lemma 5.5]. For 3.10(3), this is essentially the content of [18, Lemma 4.6]. Actually, there it is proved that $\mathrm{tr}_G p(B_X) = \lim_{i \rightarrow \infty} \mathrm{tr}_i p(B[i])$, but observe that by Subsection 2.1, $\mathrm{tr}_G p(B_X) = \mathrm{tr}_G p(B)$. Moreover, the assumption that G is a generalized amenable extension of U is slightly more general than the assumptions made in [18, Lemma 4.6], but our assumptions are exactly what is needed in the proof given there. \square

3.5 Kazhdan's inequality

We continue to adopt the situation described in 3.10.

3.30 Definition. Define

$$\begin{aligned} \overline{F}_\Delta(\lambda) &:= \limsup_i F_{\Delta[i]}(\lambda), \\ \underline{F}_\Delta(\lambda) &:= \liminf_i F_{\Delta[i]}(\lambda). \end{aligned}$$

Recall that $\limsup_{i \in I} \{x_i\} = \inf_{i \in I} \{\sup_{j > i} \{x_j\}\}$.

3.31 Definition. Suppose $F : [0, \infty) \rightarrow \mathbb{R}$ is monotone increasing (e.g. a spectral density function). Then set

$$F^+(\lambda) := \lim_{\epsilon \rightarrow 0^+} F(\lambda + \epsilon)$$

i.e. F^+ is the right continuous approximation of F . In particular, we have defined \overline{F}_Δ^+ and \underline{F}_Δ^+ .

3.32 Remark. Note that by our definition a spectral density function is right continuous, i.e. unchanged if we perform this construction.

We need the following functional analytical lemma (compare [13] or [2]):

3.33 Lemma. *Let \mathcal{N} be a finite von Neumann algebra with positive normal and normalized trace $\mathrm{tr}_\mathcal{N}$. Choose $\Delta \in M(d \times d, \mathcal{N})$ positive and self-adjoint.*

If for a function $p_n : \mathbb{R} \rightarrow \mathbb{R}$

$$\chi_{[0, \lambda]}(x) \leq p_n(x) \leq \frac{1}{n} \chi_{[0, K]}(x) + \chi_{[0, \lambda + 1/n]}(x) \quad \forall 0 \leq x \leq K \quad (3.34)$$

and if $\|\Delta\| \leq K$ then

$$F_\Delta(\lambda) \leq \mathrm{tr}_\mathcal{N} p_n(\Delta) \leq \frac{1}{n} d + F_\Delta(\lambda + 1/n).$$

Here $\chi_S(x)$ is the characteristic function of the subset $S \subset \mathbb{R}$.

Proof. This is a direct consequence of positivity of the trace, of the definition of spectral density functions and of the fact that $\mathrm{tr}_\mathcal{N}(1 \in M(d \times d, \mathcal{N})) = d$ by the definition of a normalized trace. \square

3.35 Proposition. *For every $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned}\overline{F_\Delta}(\lambda) &\leq F_\Delta(\lambda) = \underline{F_\Delta}^+(\lambda), \\ F_\Delta(\lambda) &= \underline{F_\Delta}^+(\lambda) = \overline{F_\Delta}^+(\lambda).\end{aligned}$$

Proof. The proof only depends on the key lemmata 3.28 and 3.29. These say (because of Lemma 3.22)

- $\|\Delta[i]\| \leq \kappa(\Delta[i]) \leq \kappa(\Delta) \forall i \in I$
- For every polynomial $p \in \mathbb{C}[x]$ we have $\text{tr}_G(p(\Delta)) = \lim_i \text{tr}_i(p(\Delta[i]))$.

For each $\lambda \in \mathbb{R}$ choose polynomials $p_n \in \mathbb{R}[x]$ such that inequality (3.34) is fulfilled. Note that by the first key lemma we find the uniform upper bound $\kappa(\Delta)$ for the spectrum of all of the Δ_i . Then by Lemma 3.34

$$F_{\Delta_i}(\lambda) \leq \text{tr}_i(p_n(\Delta_i)) \leq F_{\Delta_i}(\lambda + \frac{1}{n}) + \frac{d}{n}$$

We can take \liminf and \limsup and use the second key lemma to get

$$\overline{F_\Delta}(\lambda) \leq \text{tr}_G(p_n(\Delta)) \leq \underline{F_\Delta}(\lambda + \frac{1}{n}) + \frac{d}{n}.$$

Now we take the limit as $n \rightarrow \infty$. We use the fact that tr_G is normal and $p_n(\Delta)$ converges strongly inside a norm bounded set to $\chi_{[0,\lambda]}(\Delta)$. Therefore the convergence even is in the ultra-strong topology.

Thus

$$\overline{F_\Delta}(\lambda) \leq F_\Delta(\lambda) \leq \underline{F_\Delta}^+(\lambda).$$

For $\epsilon > 0$ we can now conclude, since $\underline{F_\Delta}$ and $\overline{F_\Delta}$ are monotone, that

$$F_\Delta(\lambda) \leq \underline{F_\Delta}(\lambda + \epsilon) \leq \overline{F_\Delta}(\lambda + \epsilon) \leq F_\Delta(\lambda + \epsilon).$$

Taking the limit as $\epsilon \rightarrow 0^+$ gives (since F_Δ is right continuous)

$$F_\Delta(\lambda) = \overline{F_\Delta}^+(\lambda) = \underline{F_\Delta}^+(\lambda).$$

Therefore both inequalities are established. □

We are now able to finish the proof of Theorem 3.11.

Proof of Theorem 3.11. We specialize Proposition 3.35 to $\lambda = 0$ and see that

$$\dim_G(\ker(\Delta)) \geq \limsup_i \dim_i(\ker(\Delta[i])).$$

On the other hand, by Lemma 3.13,

$$\dim_G(\ker(\Delta)) \leq \liminf_i \dim_i \ker(\Delta[i]),$$

and consequently

$$\dim_G(\ker(\Delta)) = \lim_i \dim_i \ker(\Delta[i]).$$

As remarked above, the conclusion for an arbitrary B follows by considering $\Delta = B^*B$. □

To prove an estimate similar to Lemma 3.13 in the general situation of 3.10, we need additional control on the spectral measure near zero. This can be given using uniform bounds on determinants.

3.36 Theorem. *In the situation described in 3.10, assume that there is $C \in \mathbb{R}$ such that*

$$\ln \det_i(\Delta[i]) \geq C \quad \forall i \in I.$$

Then $\ln \det_G(\Delta) \geq C$, and

$$\dim_G(\ker(\Delta)) = \lim_{i \in I} \dim_i(\ker(\Delta[i])).$$

Proof. Set $K := \kappa(\Delta)$. Then, by Lemma 3.22 and Lemma 3.28, $K > \|\Delta\|$ and $K > \|\Delta[i]\| \forall i$. Hence,

$$\ln \det_i(\Delta[i]) = \ln(K)(F_{\Delta[i]}(K) - F_{\Delta[i]}(0)) - \int_{0^+}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda.$$

If this is (by assumption) $\geq C$, then since $F_{\Delta[i]}(K) = \text{tr}_i(\text{id}_d) = d$ by our normalization

$$\int_{0^+}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda + C \leq \ln(K)(d - F_{\Delta[i]}(0)) \leq \ln(K)d.$$

We want to establish the same estimate for Δ . If $\epsilon > 0$ then

$$\int_{\epsilon}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} d\lambda = \int_{\epsilon}^K \frac{F_{\underline{\Delta}}^+(\lambda) - F_{\Delta}(0)}{\lambda} d\lambda = \int_{\epsilon}^K \frac{F_{\underline{\Delta}}(\lambda) - F_{\Delta}(0)}{\lambda}$$

(since the integrand is bounded, the integral over the left continuous approximation is equal to the integral over the original function)

$$\begin{aligned} &\leq \int_{\epsilon}^K \frac{F_{\underline{\Delta}}(\lambda) - \overline{F_{\Delta}}(0)}{\lambda} \\ &= \int_{\epsilon}^K \frac{\liminf_i F_{\Delta[i]}(\lambda) - \limsup_i F_{\Delta[i]}(0)}{\lambda} \\ &\leq \int_{\epsilon}^K \frac{\liminf_i (F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0))}{\lambda} \\ &\leq \liminf_i \int_{\epsilon}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} \leq d \ln(K) - C. \end{aligned}$$

Since this holds for every $\epsilon > 0$, we even have

$$\begin{aligned} \int_{0^+}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} &\leq \int_{0^+}^K \frac{F_{\underline{\Delta}}(\lambda) - \overline{F_{\Delta}}(0)}{\lambda} d\lambda \\ &\leq \sup_{\epsilon > 0} \liminf_i \int_{\epsilon}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda \\ &\leq d \ln(K) - C. \end{aligned}$$

The second integral would be infinite if $\lim_{\delta \rightarrow 0} F_{\Delta}(\delta) \neq \overline{F_{\Delta}(0)}$. It follows that $\limsup_i F_{\Delta[i]}(0) = F_{\Delta}(0)$. Since we can play the same game for every subnet of I , also $\liminf_i F_{\Delta[i]}(0) = F_{\Delta}(0)$ i.e. the approximation result is true.

For the estimate of the determinant note that in the above inequality

$$\begin{aligned} & \sup_{\epsilon > 0} \liminf_i \int_{\epsilon}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda \\ & \leq \liminf_i \sup_{\epsilon > 0} \int_{\epsilon}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda = \liminf_i \int_{0+}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda \\ & \leq \ln(K)(d - F_{\Delta[i]}(0)) - C. \end{aligned}$$

Therefore

$$\begin{aligned} \ln \det_G(\Delta) &= \ln(K)(d - F_{\Delta}(0)) - \int_{0+}^K \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda} d\lambda \\ &\geq \ln(K)(d - \lim_i F_{\Delta[i]}(0)) - \liminf_i \int_{0+}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda \\ &= \limsup_i \left(\ln(K)(d - F_{\Delta[i]}(0)) - \int_{0+}^K \frac{F_{\Delta[i]}(\lambda) - F_{\Delta[i]}(0)}{\lambda} d\lambda \right) \\ &\geq C. \quad \square \end{aligned}$$

3.6 Proof of “determinant bound” and “algebraic continuity” by induction

Theorems 3.4 and 3.12 are generalizations of [18, Theorem 6.9], where the corresponding statements are proved for matrices over $\mathbb{Z}G$. The proof will be done by transfinite induction (using the induction principle [18, 2.2]) and is very similar to the proof of [18, 6.9]. Therefore, we will not give all the details here but concentrate on the necessary modifications.

We will use the following induction principle.

3.37 Proposition. *Suppose a property C of groups is shared by the trivial group, and the following is true:*

- *whenever U has property C and G is a generalized amenable extension of U , then G has property C as well;*
- *whenever G is a direct or inverse limit of a directed system of groups with property C , then G has property C*
- *If $U < G$, and U has property C , then also G has property C .*

Then property C is shared by all groups in the class \mathcal{G} .

Proof. The proof of the induction principle is done by transfinite induction in a standard way, using the definition of \mathcal{G} . The result corresponds to [18, Proposition 2.2]. \square

We are going to use the induction principle to prove the determinant bound property and the algebraic continuity property of Definition 3.1.

3.6.1 Trivial group

First we explain how the induction gets started:

3.38 Lemma. *The trivial group $G = \{1\}$ has the determinant bound property and the algebraic continuity property.*

Proof. Remember that $\Delta = B^*B$ and $\det_{\{1\}}(B^*B)$ in this case is the product of all non-zero eigenvalues of B^*B , which is positive since B^*B is a non-negative operator. If $q(t)$ is the characteristic polynomial of Δ and $q(t) = t^r \bar{q}(t)$ with $\bar{q}(0) \neq 0$ then $\det_{\{1\}}(\Delta) = \bar{q}(0)$. In particular it is contained in $o(\overline{\mathbb{Q}})$. Moreover $\det_{\{1\}}(\sigma_k(A)) = \sigma_k(\det_{\{1\}}(A))$ for $k = 1, \dots, r$. The product of all these numbers is fixed under $\sigma_1, \dots, \sigma_r$, therefore a rational number and is an algebraic integer, hence a (non-zero) integer. Taking absolute values, we get a positive integer, and consequently

$$\sum_{k=1}^r \ln |\det_{\{1\}}(\sigma_k(A))| \geq 0.$$

Using Lemma 3.39 we get the desired estimate of the determinant.

Since \mathbb{C} is flat over L ,

$$\dim_{\{1\}}(\ker(B)) = \dim_{\mathbb{C}}(\ker(B)) = \dim_L(\ker(B|_{L^d})).$$

Choose $\sigma \in \{\sigma_1, \dots, \sigma_r\}$. Since $\sigma: L \rightarrow \sigma(L)$ is a field automorphism,

$$\dim_L(\ker(B|_{L^d})) = \dim_{\sigma(L)}(\ker(\sigma(B)|_{\sigma(L)^d})),$$

which by the same reasoning as above coincides with $\dim_{\{1\}}(\ker(\sigma B))$. \square

3.39 Lemma. *Let $A \in M(d \times d, \mathbb{C})$. Then*

$$|\det_{\{1\}}(A)| \leq \|A\|^d,$$

where $\|A\|$ denotes the Euclidean operator norm.

Proof. The result is well known if A has no kernel and $\det_{\{1\}}(A) = \det(A)$. By scaling, we may assume that $\|A\| \geq 1$. If A has a non-trivial kernel, choose a basis whose first elements span the kernel of A . Then $A = \begin{pmatrix} 0 & * \\ 0 & A_0 \end{pmatrix}$ and A_0 has a smaller dimension than A . It is obvious from the definitions that $q_A(t) = t^{\dim(\ker(A))} q_{A_0}(t)$, where q_A is the characteristic polynomial of A . Therefore $\det_{\{1\}}(A) = \det_{\{1\}}(A_0)$ (since this is the first non-trivial coefficient in the characteristic polynomial). By induction $\det_{\{1\}}(A) \leq \|A_0\|^{d - \dim \ker(A)}$. Since $\|A_0\| \leq \|A\|$, the result follows. Observe that the statement is trivial if $d = 1$. \square

3.6.2 Subgroups

We have to check that, if G has the determinant bound property and the algebraic continuity property, then the same is true for any subgroup U of G .

This is done as follows: a matrix over $\mathbb{C}U$ can be induced up to (in other words, viewed as) a matrix over $\mathbb{C}G$. By Lemma 3.26, the right hand side of (3.2) is unchanged under this process, and it is well known that the spectrum of the operator and therefore (3.3) and the left hand side of (3.2) is unchanged, as well, compare e.g. [18, 3.1]. Since the spectrum is unchanged, the algebraic continuity also is immediately inherited.

3.6.3 Limits and amenable extensions

Now assume G is a generalized amenable extension of U , or the direct or inverse limit of a directed system of groups $(G_i)_{i \in I}$, and assume $B \in M(d \times d, o(\overline{\mathbb{Q}})G)$, and $\Delta = B^*B$. That means, we are exactly in the situation described in 3.10 (as we have seen before, without loss of generality we can assume throughout that the coefficients are algebraic integers instead of algebraic numbers).

We want to use the induction hypothesis to establish a uniform lower bound on $\ln \det_i(\Delta[i])$.

3.40 Lemma. *In the given situation,*

$$\ln \det_i(\Delta[i]) \geq -d \sum_{k=2}^r \ln(\kappa(\sigma_k(\Delta))) \quad \text{for } i \text{ sufficiently large.} \quad (3.41)$$

Proof. In case of 3.10(1) or 3.10(2), since $\det_i = \det_{G_i}$ in this situation, by the induction hypothesis we have

$$\ln \det_i(\Delta[i]) \geq -d \sum_{k=2}^r \ln(\kappa(\sigma_k(\Delta[i])))$$

However, by Lemma 3.17, $\sigma_k(\Delta[i]) = \sigma_k(\Delta)[i]$ if i is sufficiently large, moreover, $\kappa(\sigma_k(\Delta)[i]) \leq \kappa(\sigma_k(\Delta))$ by Lemma 3.28, again for i sufficiently large. The inequality (3.41) follows.

It remains to treat the case 3.10(3). Here,

$$\ln \det_i(\Delta[i]) = \frac{1}{N_i} \ln \det_U(\Delta[i]). \quad (3.42)$$

On the other hand, $\Delta[i]$ is to be considered not as an element of $M(d \times d, \mathbb{C}U)$, but as an element of $M(N_i d \times N_i d, \mathbb{C}U)$. Consequently, by the induction hypothesis,

$$\ln \det_U(\Delta[i]) \geq -N_i d \sum_{k=2}^r \ln(\kappa(\sigma_k(\Delta[i])))$$

Dividing this inequality by $N_i > 0$, and using the fact that by Lemma 3.28 $\kappa(\sigma_k(\Delta)) \geq \kappa(\sigma_k(\Delta[i]))$, again we conclude that the inequality (3.41) is true. \square

Because of Theorem 3.36,

$$\ln \det_G(\Delta) \geq -d \sum_{k=2}^r \ln(\kappa(\sigma_k(\Delta))). \quad (3.43)$$

The induction principle of Proposition 3.37 therefore implies that, if $G \in \mathcal{G}$, then G has the determinant bound property.

Note, moreover, that we can now prove Theorem 3.12 inductively.

To carry out the induction step, we note that

$$\ker(\sigma_k B) = \ker((\sigma_k B)^* \sigma_k B) \quad k = 1, \dots, r.$$

The approximation result we have just proved implies (using Lemma 3.17) that

$$\dim_G(\ker((\sigma_k B)^* \sigma_k B)) = \lim_{i \in I} \dim_i(\ker((\sigma_k B)[i]^* \sigma_k B[i])).$$

Again we know that for each i $\ker((\sigma_k B)[i]^* \sigma_k B[i]) = \ker(\sigma_k B[i])$, and the induction hypothesis implies that $\dim_i \ker(\sigma_k B[i]) = \dim_i \ker(B[i])$ for each $k = 1, \dots, r$ and for each i . Therefore, we obtain

$$\dim_G(\ker(B)) = \dim_G(\ker(\sigma_k(B))).$$

By induction, if $G \in \mathcal{G}$ then G has the algebraic continuity property.

This finishes the proof of Theorems 3.4 and 3.12.

3.7 Proof of the stability of the Atiyah conjecture under limits

To conclude this section, we note that Theorem 3.12 immediately implies Proposition 1.9 because the L^2 -dimension of the kernel we have to compute is the limit of L^2 -dimensions which by assumption are integers. Since \mathbb{Z} is discrete, the limit has to be an integer, too.

4 Zero divisors: from algebraic to arbitrary

In the introduction, we claimed that it suffices to study $\overline{\mathbb{Q}}G$ to decide whether $\mathbb{C}G$ satisfies the zero divisor conjecture. For the readers convenience, we give a proof of this well known fact here.

4.1 Proposition. *Assume that there are $0 \neq a, b \in \mathbb{C}G$ with $ab = 0$. Then we can find $0 \neq A, B \in \overline{\mathbb{Q}}G$ with $AB = 0$.*

Proof. Write $a = \sum_{g \in G} \alpha_g g$, $b = \sum_{g \in G} \beta_g g$. Since only finitely many of the α_g and β_g are non-zero, they are contained in a finitely generated subfield $L \subset \mathbb{C}$. We may write $L = \overline{\mathbb{Q}}(x_1, \dots, x_n)[v]$ with x_1, \dots, x_n algebraically independent over \mathbb{Q} , and v algebraic over $\overline{\mathbb{Q}}(x_1, \dots, x_n)$ (because of the theorem about the primitive element, one such v will do). Upon multiplication by suitable non-zero elements of $\overline{\mathbb{Q}}(x_1, \dots, x_n)$, we may assume that v is integral over $\overline{\mathbb{Q}}[x_1, \dots, x_n]$, i.e. that there is a polynomial $0 \neq p(z) = z^n + p_{n-1}z^{n-1} + \dots + p_0$ with $p_j \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$, irreducible in $\overline{\mathbb{Q}}(x_1, \dots, x_n)[z]$, and with $p(v) = 0$. This can be achieved because the quotient field of the ring of integral elements of L is L itself.

Moreover, by multiplying a and b by appropriate non-zero elements of $\overline{\mathbb{Q}}[x_1, \dots, x_n]$ (the common denominator of the α_g or β_g , respectively), we may assume that $0 \neq a, b \in \overline{\mathbb{Q}}[x_1, \dots, x_n][v]G$, with $ab = 0$.

We now proceed to construct a ring homomorphism

$$\phi: \overline{\mathbb{Q}}[x_1, \dots, x_n][v] \rightarrow \overline{\mathbb{Q}}$$

with $0 \neq A := \phi(a)$ and $0 \neq B := \phi(b)$. Obviously, $AB = \phi(ab) = 0$, and this proves the claim.

To construct ϕ , observe that substitution of algebraic numbers for x_1, \dots, x_n defines a well defined homomorphism $\phi_0: \overline{\mathbb{Q}}[x_1, \dots, x_n] \rightarrow \overline{\mathbb{Q}}$. Each such homomorphism can be extended to $\overline{\mathbb{Q}}[x_1, \dots, x_n][v]$, provided $\phi_0(p(z))$ has a solution in $\overline{\mathbb{Q}}$. Since the highest coefficient of p is 1, $\phi(p)$ is not a constant polynomial. Consequently, $\overline{\mathbb{Q}}$ being algebraically closed, the required solution of $\phi_0(p(z))$ and therefore the extension of the ring homomorphism ϕ_0 to $\overline{\mathbb{Q}}[x_1, \dots, x_n][v]$ always exists.

Choose now $g, g' \in G$ with $\alpha_g \neq 0 \neq \beta_{g'}$. Observe that $\alpha_g \in \overline{\mathbb{Q}}[x_1, \dots, x_n][v]$ is integral over $\overline{\mathbb{Q}}[x_1, \dots, x_n]$ since v is integral. The same holds for $\beta_{g'}$. In particular, there exist irreducible

polynomials

$$q(z) = z^m + q_{m-1}z^{m-1} + \cdots + q_0, \quad r(z) = z^{m'} + \cdots + r_0 \in \overline{\mathbb{Q}}[x_1, \dots, x_n][z]$$

with $q(\alpha_g) = 0 = r(\beta_{g'})$. Irreducibility implies in particular that

$$q_0 = q_0(x_1, \dots, x_n) \neq 0 \neq r_0(x_1, \dots, x_n).$$

By induction on n , using the fact that $\overline{\mathbb{Q}}$ is infinite and that any non-zero polynomial over a field has only finitely many zeros, we show that there are $d_1, \dots, d_n \in \overline{\mathbb{Q}}$ with $q_0(d_1, \dots, d_n) \neq 0 \neq r_0(d_1, \dots, d_n)$.

Consequently, if ϕ_0 is the corresponding substitution homomorphism

$$\phi_0: \overline{\mathbb{Q}}[x_1, \dots, x_n] \rightarrow \overline{\mathbb{Q}},$$

0 is not one of the zeros of the polynomials $\phi_0(r)$ or $\phi_0(q)$. Let

$$\phi: \overline{\mathbb{Q}}[x_1, \dots, x_n][v] \rightarrow \overline{\mathbb{Q}}$$

be an extension of ϕ_0 . Then $A_g := \phi(\alpha_g)$ is a zero of $\phi(q) = \phi_0(q)$, and $B_{g'} := \phi(\beta_{g'})$ is a zero of $\phi(r)$, i.e. $A_g \neq 0 \neq B_{g'}$.

However, with $A = \phi(a)$ and $B = \phi(b)$, A_g is the coefficient of g in A , and $B_{g'}$ is the coefficient of g' in B . It follows that $A \neq 0 \neq B$, as desired, but $AB = 0$. \square

5 Approximation of L^2 -Betti numbers over the complex group ring

It is an interesting question, in the situation of Theorem 3.12, whether the convergence result also holds for matrices over the complex group ring. We don't know about any counterexample. Here we will give some positive results. Let R be a domain. Recall that R satisfies the *Ore condition* or equivalently, R is an *Ore domain*, means that given $r, s \in R$ with $s \neq 0$, then we can find $r_1, r_2, s_1, s_2 \in R$ with $s_1, s_2 \neq 0$ such that $r_1s = s_1r$ and $sr_2 = rs_2$ (some authors call this the left and right Ore condition). In this situation we can form the division ring of fractions whose elements are of the form $s^{-1}r$; these elements will also be of the form rs^{-1} .

5.1 Proposition. *Assume G is torsion free, $\mathbb{C}G$ is an Ore domain and whenever $0 \neq \alpha \in \mathbb{C}G$, $0 \neq f \in l^2(G)$ then $\alpha f \neq 0$. Let G be a subgroup of the direct or inverse limit of a directed system of groups G_i , $i \in I$. If $B \in M(d \times d, \mathbb{C}G)$ and $B[i] \in M(d \times d, \mathbb{C}G_i)$ are constructed as in 3.10, then*

$$\dim_G(\ker(B)) = \lim_{i \in I} \dim_{G_i}(\ker(B[i])). \quad (5.2)$$

5.3 Corollary. *The statement of Proposition 5.1 is true if G is torsion free elementary amenable or if G is amenable and left-orderable.*

Proof. If G is elementary amenable and torsion free then the strong Atiyah conjecture over $\mathbb{C}G$ is true for G and therefore $\dim_G \ker(\alpha) = 0$ if $0 \neq \alpha \in \mathbb{C}G$, hence α has trivial kernel on $l^2(G)$. If G is right orderable then $0 \neq \alpha \in \mathbb{C}G$ has trivial kernel on $l^2(G)$ by [10]. In particular $\mathbb{C}G$ has no non-trivial zero divisors. Since G is amenable, by an argument of Tamari [19] given in Theorem 5.4 $\mathbb{C}G$ fulfills the Ore condition. Therefore the assumptions of Proposition 5.1 are fulfilled. \square

For the convenience of the reader we repeat Tamari's result:

5.4 Theorem. *Suppose G is amenable, R is a division ring (e.g. a field), and $R * G$ is a crossed product (e.g. the group ring RG). Assume $R * G$ has no non-trivial zero divisors. Then $R * G$ fulfills the Ore condition.*

Proof. Suppose we are given $\alpha, \sigma \in R * G$ with $\sigma \neq 0$. Without loss of generality, we need to find $\beta, \tau \in R * G$ with $\tau \neq 0$ such that $\beta\sigma = \tau\alpha$. Write $\alpha = \sum_{g \in G} a_g$ and $\sigma = \sum_{g \in G} s_g$, where $a_g, s_g \in R$ for all $g \in G$. Let $Z = \text{supp } \alpha \cup \text{supp } \sigma$ (where $\text{supp } \alpha$ denotes $\{g \in G \mid a_g \neq 0\}$, the support of α). Using the Følner condition, we obtain a finite subset X of G such that $\sum_{g \in Z} |Xg \setminus X| < |X|$. Let us write $\beta = \sum_{x \in X} b_x x$ and $\tau = \sum_{x \in X} t_x x$ where $b_x, t_x \in R$ are to be determined. We want to solve the equation $\beta\sigma = \tau\alpha$, which when written out in full becomes

$$\sum_{x \in X, g \in G} b_x (x s_g x^{-1}) x g = \sum_{x \in X, g \in G} t_x (x a_g x^{-1}) x g.$$

By equating coefficients, this yields at most $2|X| - 1$ homogeneous equations in the $2|X|$ unknowns b_x, t_x . We deduce that there exist β, τ , not both zero, such that $\beta\sigma = \tau\alpha$. Since $R * G$ is a domain and $\sigma \neq 0$, we see that $\tau \neq 0$ is not a zero divisor and the result is proved. \square

Proposition 1.10 is a direct consequence of the corollary because there G is a subgroup of the inverse limit of the quotients G/G_i .

Proof of Proposition 5.1. By [18, 7.3] (5.2) holds if $d = 1$. We use the Ore condition to reduce the case of matrices to the case $d = 1$. Since $\mathbb{C}G \subset l^2(G)$ the assumptions imply that $\mathbb{C}G$ does not have zero divisors. Therefore the Ore localization DG of $\mathbb{C}G$ is a skew field. Moreover, since $0 \neq \alpha \in \mathbb{C}G$ has no kernel on $l^2(G)$ it becomes invertible in the ring UG of operators affiliated to the group von Neumann algebra of G [12, §8] and [16, Definition 7, p. 741]. Therefore DG embeds into UG . (By [16, 4.4] G fulfills the strong Atiyah conjecture over $\mathbb{C}G$.)

Fix now $B \in M(d \times d, \mathbb{C}G)$. By linear algebra, we find invertible matrices $X, Y \in M(d \times d, DG)$ such that $XYB = \text{diag}(v_1, \dots, v_d)$ with $v_i \in DG$. The Ore condition implies that we can find $a \in \mathbb{C}G - \{0\}$ such that $v_i = \alpha_i a^{-1}$ with $\alpha_i \in \mathbb{C}G$, for $i = 1, \dots, d$. Therefore $XYBa = \text{diag}(\alpha_1, \dots, \alpha_d)$. Applying the same principle to the entries of X and Ya , we can find $x, y \in \mathbb{C}G - \{0\}$ such that $xX = U$ and $Yay = V$ with $U, V \in M(d \times d, \mathbb{C}G)$. Altogether we arrive at

$$UBV = \text{diag}(a_1, \dots, a_d) \tag{5.5}$$

where all objects are defined over $\mathbb{C}G$ ($a_k = x\alpha_k y$). Moreover, U and V are invertible over DG and therefore also over UG and hence have trivial kernel as operators on $l^2(G)^d$. (5.5) translates to

$$U[i]B[i]V[i] = \text{diag}((a_1)_i, \dots, (a_d)_i)$$

(if G is the direct limit of the groups G_i , then the images of the left and right hand sides of the above in $M(d \times d, \mathbb{C}G)$ are equal, consequently for all sufficiently large i the above equality will be true).

By [18, 7.2] $\dim_{G_i} \ker(U[i]) \xrightarrow{i \in I} 0$ and $\dim_{G_i} \ker(V[i]) \xrightarrow{i \in I} 0$. The one dimensional case immediately implies

$$\dim_{G_i} \ker(\text{diag}(a_{1i}, \dots, a_{di})) \rightarrow \dim_G \ker(\text{diag}(a_1, \dots, a_d)).$$

We have the exact sequences

$$0 \rightarrow \ker(V[i]) \hookrightarrow \ker(U[i]B[i]V[i]) \xrightarrow{V[i]} \ker(U[i]B[i]) \quad (5.6)$$

$$0 \rightarrow \ker(B[i]) \hookrightarrow \ker(U[i]B[i]) \xrightarrow{B[i]} \ker(U[i]). \quad (5.7)$$

Because of additivity of the L^2 -dimension [14, Lemma 1.4(4)]

$$\begin{aligned} \dim_{G_i} \ker(U[i]B[i]V[i]) - \dim_{G_i} \ker(U[i]) - \dim_{G_i} \ker(V[i]) \\ \leq \dim_{G_i} \ker(B[i]) \leq \dim_{G_i} \ker(U[i]B[i]). \end{aligned}$$

Since all these operators are endomorphism of the same finite Hilbert $\mathcal{N}G_i$ -module $l^2(G_i)^d$,

$$\begin{aligned} \dim_{G_i} \ker(U[i]B[i]) &= \dim_{G_i} \ker(B[i]^*U[i]^*) \\ &\leq \dim_{G_i} \ker(V[i]^*B[i]^*U[i]^*) = \dim_{G_i} \ker(U[i]B[i]V[i]), \end{aligned}$$

and similarly $\dim_G \ker(UBV) = \dim_G \ker(B)$. Everything together implies (5.2). \square

6 Absence of transcendental eigenvalues

6.1 Definition. We say that a discrete group G has the *algebraic eigenvalue property* (or the *algebraic eigenvalue property for algebraic matrices*), if for every matrix $A \in M(d \times d, \overline{\mathbb{Q}}G)$ the eigenvalues of A , acting on $l^2(G)^d$ are algebraic. We say G has the *algebraic eigenvalue property for rational matrices*, if the same is true for every matrix $A \in M(d \times d, \mathbb{Q}G)$.

Observe that for a group G with the algebraic eigenvalue property, and a matrix A over $\overline{\mathbb{Q}}G$, a transcendental number is necessarily a point of continuity for the spectral density function of A .

6.2 Corollary. *Assume that we are in the situation described in 3.10, and that G has the algebraic eigenvalue property. If $B \in M(d \times d, \overline{\mathbb{Q}}G)$, $A = B^*B$, and $\lambda \in \mathbb{R}$ is a transcendental number, then*

$$\dim_G(\ker(A - \lambda)) = \lim_{i \in I} \dim_i(\ker(A[i] - \lambda)). \quad (6.3)$$

Moreover, if $G_i \in \mathcal{G}$, then (6.3) holds for every number $\lambda \in \mathbb{R}$.

Proof. By [18, Lemma 7.2] (6.3) is true whenever λ is not an eigenvalue of A . But this is the case for transcendental numbers by the very definition of the algebraic eigenvalue property. All other real numbers are algebraic, so that the second statement follows from Theorem 3.12. \square

6.4 Example. The trivial group has the algebraic eigenvalue property, since the eigenvalues are the zeros of the characteristic polynomial. The same is true for every finite group. More generally, if G contains a subgroup H of finite index, and H has the algebraic eigenvalue property, then the same is true for G . And if G has the algebraic eigenvalue property and H is a subgroup of G , then H has the algebraic eigenvalue property, too.

Proof. This follows immediately from [18, Proposition 3.1] and from [12, Lemma 8.6]. \square

The following theorem is due to Roman Sauer [15].

6.5 Theorem. *Free groups have the algebraic eigenvalue property.*

Proof. Assume that G is a free group and $A \in M(d \times d, \overline{\mathbb{Q}}G)$. Sauer proves a much stronger result that the series

$$f(z) := \sum_{n=0}^{\infty} \operatorname{tr}_G(A^n) z^{-n-1}$$

is an algebraic function over $\overline{\mathbb{Q}}(z)$. Every eigenvalue of A is a pole of $f(z)$, consequently, as Sauer points out, it is an algebraic number. \square

6.6 Theorem. *Assume H has the algebraic eigenvalue property, or the algebraic eigenvalue property for rational matrices. Let G be a generalized amenable extension of H . Then G also has the algebraic eigenvalue property, or the algebraic eigenvalue property for rational matrices, respectively.*

Proof. This is a direct consequence of Theorem 3.11 and the definition of the algebraic eigenvalue property of H . \square

6.7 Corollary. *Every amenable group, and every group in the class \mathcal{C} of Linnell has the algebraic eigenvalue property.*

Proof. Since the trivial group has the algebraic eigenvalue property, Theorem 6.6 implies the statement for amenable groups.

The proof of the algebraic eigenvalue property for groups in \mathcal{C} is done by transfinite induction. The start of the induction is the case of free groups, which is covered by Sauer's result. For the induction step, we have to control extensions with elementary amenable quotient, which is done using Theorem 6.6, and directed unions. The algebraic eigenvalue property for a group holds if and only if it holds for all its finitely generated subgroups. This observation takes care of directed unions.

The details of the transfinite induction are left to the reader. They follow the pattern of [11]. \square

In view of these results, we make the following conjecture:

6.8 Conjecture. *Every discrete group G has the algebraic eigenvalue property.*

We can give further evidence for this conjecture in the case of G belonging to the class \mathcal{G} , using the knowledge of the spectrum that we have in this case.

6.9 Theorem. *Assume $G \in \mathcal{G}$ and $A = A^* \in M(d \times d, \mathbb{Z}Q)$. Assume that $\lambda \in \mathbb{R}$ is a transcendental number, but that for every $n \in \mathbb{N}$ there is a rational number p_n/q_n with $q_n \geq 2$ such that*

$$0 < \left| \lambda - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^n}. \quad (6.10)$$

Then λ is not an eigenvalue of A acting on $l^2(G)^d$.

Observe that it follows from Liouville's theorem [9, Satz 191] that a number λ satisfying the second set of assumptions of Theorem 6.9 is automatically transcendental.

Proof of Theorem 6.9. Since $G \in \mathcal{G}$, we can apply (3.9) to the operator

$$V_n := (q_n A - p_n)^*(q_n A - p_n).$$

Observe that

$$\|V_n\| \leq (q_n \|A\| + |p_n|)^2 \leq q_n^2 (\|A\| + \lambda + 1)^2, \quad (6.11)$$

since by (6.10) $|p_n| \leq \lambda q_n + q_n$. If λ is an eigenvalue of A , then $s_n := (q_n \lambda - p_n)^2$ will be an eigenvalue of V_n (since $A = A^*$). Observe that

$$s_n \leq q_n^{2-2n} < 1. \quad (6.12)$$

If

$$0 < \alpha := \dim_G(\ker(A - \lambda))$$

is the normalized dimension of the eigenspace to λ , then

$$F_{V_n}(s_n) - F_{V_n}(0) = F_{V_n}((q_n \lambda - p_n)^2) - F_{V_n}(0) \geq \alpha. \quad (6.13)$$

Since V_n has integer coefficients, $\|V_n\| \geq 1$. Thus, using (3.9),

$$\begin{aligned} F_{V_n}(s_n) - F_{V_n}(0) &\leq \frac{d \ln \|V_n\|}{-\ln(s_n / \|V_n\|)} \leq \frac{d \ln \|V_n\|}{-\ln(s_n)} \\ &\leq \frac{d \cdot 2 \cdot (\ln(q_n) + \ln(\|A\| + \lambda + 1))}{(2n - 2) \ln(q_n)}, \end{aligned}$$

using $\|V_n\| \geq 1$, $s_n < 1$, (6.12), and (6.11). The right hand side becomes arbitrarily small as $n \rightarrow \infty$. Consequently, $\alpha = 0$ in (6.13), i.e. λ is not an eigenvalue of A . \square

6.14 Remark. It is obvious that Theorem 6.9 immediately extends to transcendental numbers $\lambda \in \mathbb{R}$ which have very good approximations by not very “complex” algebraic numbers. Here, the complexity of an algebraic number ξ would be measured in terms of its denominator (i.e. how big is $k \in \mathbb{N}$ such that $k\xi$ is an algebraic integer), in terms of the degree of its minimal polynomial, and in terms of the absolute value of the other zeros of the minimal polynomial. The precise definition of “good approximation” has to be made in such a way that, in the proof, the inequality (3.9) used in Theorem 6.9 can be replaced by the inequality (3.8). We can extend Theorem 6.9 in the same way to the case where $A \in M(d \times d, \overline{\mathbb{Q}}G)$. The details are left to the reader.

Unfortunately, the set of real numbers which admit such approximations appears to be of measure zero. At least, this is true by [9, Satz 198] for numbers covered by Theorem 6.9, and the proof of Hardy and Wright seems to carry over without difficulty to the larger set described above.

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