The topology of T-duality for T^n-bundles

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1 Introduction

1.1 The notion of T-duality has its origin in string theory. It asserts that a certain type of string theory on a target F partially compactified on a torus is equivalent to another type of string theory on a dual target $\hat{F}$. In the presence of an H-flux the topological aspects of T-duality are of particular interest.

In the present paper we concentrate on the topological aspects of T-duality, and we neglect geometric questions and physical interpretations.

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1.2 T-duality as studied in the present paper has a two sides, a constructive one and a descriptive one. The constructive side is the main focus of the present paper and described below. The descriptive picture relies on the notion of a T-duality triple tr13 5.5.

Let $T^n := U(1)^n$ denote the n-dimensional torus. Let B be some base space. We consider a pair (F, h) over B consisting of a T^n-principal bundle $F \to B$ and a class $h \in H^3(F, \mathbb{Z})$. In Definition drt 4.7 we introduce the notion of a T-dual pair using the following construction. In rt3 2.1 we construct a space $\mathbf{R}_n$ and a homotopy involution $T : \mathbf{R}_n \to \mathbf{R}_n$. This space classifies a set-valued functor $B \mapsto P_n(B) := [B, \mathbf{R}_n]$ with an automorphism $T_B : P_n(B) \to P_n(B)$ of order two induced by T ($[B, \mathbf{R}_n]$ denotes the homotopy classes of maps from B to $\mathbf{R}_n$). Let $B \to \tilde{P}_{(0)}(B)$ denote the functor which associates to a space B the set of isomorphism classes of pairs over B. Then we obtain a (forgetful) transformation $v_B : P_n(B) \to \tilde{P}_{(0)}(B)$. In order to define a T-dual of a pair (F, h) we must choose an element $x \in v_B^{-1}([F, h])$. Then we define the isomorphism class of the T-dual pair as the underlying class of pairs
filtex of $T_B(x)$, i.e. $[\hat{F}, \hat{h}] := v_B(T_B(x))$.

1.3 In this approach, by definition a T-dual pair exists precisely for each pair in the image of v_B. The cohomology of the total space of the fibre bundle $\pi : F \to B$ has a decreasing filtration

$$0 \subset \mathcal{F}^n H^n(F, \mathbb{Z}) \subset \mathcal{F}^{n-1} H^n(F, \mathbb{Z}) \subset \cdots \subset \mathcal{F}^1 H^n(F, \mathbb{Z}) \subset \mathcal{F}^0 H^n(F, \mathbb{Z}) = H^n(F, \mathbb{Z})$$

such that $\mathcal{F}^n H^n(F, \mathbb{Z}) = \pi^*(H^n(B, \mathbb{Z}))$. Choose a CW-decomposition $B^{(0)} \subset B^{(1)} \subset \cdots$ of B. By definition, $x \in \mathcal{F}^k H^n(F, \mathbb{Z})$ if its restriction to $\pi^{-1}(B^{(k-1)})$ is zero. The associated graded group is calculated by the Leray-Serre spectral sequence. For this reason we call this filtration the Leray-Serre filtration.

th1 **Theorem 1.1 (Corollary exex 4.8)** *The pair (F, h) admits a T-dual pair if and only if $h \in \mathcal{F}^2 H^3(F, \mathbb{Z})$.*

1.4 The approach to T-duality in the present paper is similar to (and inspired by) the point of view of bhm [1]. In that paper the authors work on smooth T^n-bundles equipped with real valued cohomology classes $h_{\mathbb{R}}$. The T-dual pair is constructed geometrically using differential forms.

1.5 An important aspect of (topological) T-duality is an (expected) isomorphism between suitable twisted cohomology groups of the bundle and its dual. In order to formulate precise statements about T-duality isomorphisms between twisted cohomology groups one must fix the isomorphism classes of the twists precisely and not only up to torsion. We refer to [3], Sec. 4.3, for an explicit example which shows that the torsion part plays a significant role. The present paper represents an improvement over [1] since it introduces and studies T-duality of pairs (F, h) with an integral cohomology class h.

Moreover, we make precise the role of the identification of twists u introduced in 5.2. This point is neglected in [1].

1.6 Note that a T-dual pair of (F, h) is not unique. This has already been observed in [3], Sec. 4.4. One of the results of the present paper is the detailed description of set of possible choices.

In order to define a T-dual of a pair (F, h) we must choose an element $x \in v_B^{-1}([F, h])$. In general the isomorphism class $v_B(T_B(x))$ of the T-dual pair will depend non-trivially on the choice of $x \in v_B^{-1}([F, h])$.

1.7 In order to study the fibers of v_B we construct a non-canonical isomorphism (see Corollary 4.6) with another functor $B \mapsto \tilde{P}_n(B)$. The elements of $\tilde{P}_n(B)$ have a geometric meaning, and the fiber of the forgetful map $\tilde{v}_B : \tilde{P}_n(B) \to \tilde{P}_{(0)}(B)$ can be studied by obstruction theory (see 4.9 and 4.10). It seems to be difficult to formulate a general result. But let us mention the following opposite extremes.

(1) If $B = S^3$, then the transformation $\tilde{v}_{S^3} : \tilde{P}_n(S^3) \to P_0(S^3)$ is an inclusion of a subset.

(2) If $B = \mathbf{R}_n$ is the space introduced in 2.1, then the fibers of $\tilde{v}_{\mathbf{R}_n} : P_n(\mathbf{R}_n) \to \tilde{P}_{(0)}(\mathbf{R}_n)$ are torsors over the group $H^2(T^n, \mathbb{Z})$.

1.8 In [4] and [5] the authors follow a completely different point of view. The class $h \in H^3(F, \mathbb{Z})$ is the Dixmier-Douady class of a unique isomorphism class of a stable continuous trace algebra $\mathcal{A} := \mathcal{A}(F, h)$ with spectrum F. The authors study the question of lifting the T^n-action on F to an $\mathbb{R}^n$-action on $\mathcal{A}$ such that the Mackey invariant is trivial. In this case the crossed product $\hat{\mathcal{A}} := \mathcal{A} \rtimes \mathbb{R}$ is again a continuous trace algebra with a spectrum $\hat{F}$ which is a T^n-principal bundle. Let $\hat{h} \in H^3(\hat{F}, \mathbb{Z})$ denote the Dixmier-Douady class of $\hat{\mathcal{A}}$. By the point of view of [4], [5] the pair $(\hat{F}, \hat{h})$ is the T-dual of (F, h).

The two problems in this approach are to decide under which conditions the required $\mathbb{R}^n$-action on $\mathcal{A}$ exists, and to study the set of choices for such an action. A satisfactory picture can be obtained in the cases $n = 1$ and $n = 2$. The case $n = 1$ is easy and has been reviewed in [3]. The main results of [4] deal with the case $n = 2$. The necessary and sufficient condition for the existence of the $\mathbb{R}^n$ action is again that $h \in \mathcal{F}^2 H^3(F, \mathbb{Z})$. It is then claimed in [4], that the action is unique. This is not always true. In fact, it follows from the diagram given in [4], Theorem 4.3.3, and the observation that d_2'' factors over $p_! : H^2(F, \mathbb{Z}) \to H^0(B, \mathbb{Z})$ that the group $H^0(B, \mathbb{Z})/\mathtt{im}(p_!)$ acts freely on the set of $\mathbb{R}$-actions with trivial Mackey invariant lifting the T^n-action on F. One can check that the same group acts freely on the fibers of v_B.

For general $n \geq 3$ the relation between the the classification of the $\mathbb{R}^n$-actions on $\mathcal{A}(F, h)$ with trivial Mackey invariant and the classification of the elements in $v_B^{-1}([F, h])$ is not clear.

1.9 In [2] the point of view of [4, 5] is generalized even further by considering torus bundles with completely arbitrary H-flux differential 3-form with integral periods. It is then argued that the resulting dual should be a bundle of non-associative non-commutative tori, and a number of (expected) properties of this duality are detailed. Questions about choices involved in the construction of the dual object, or about a lift of the invariants to integral cohomology were not in the focus of these papers.

1.10 The present paper also gives the correct description of the topological invariants of a[1] T-dual, i.e. relations between the Chern classes and the H^3-classes. This way, we improves the results of [5]. The calculation in [5] is wrong since the inclusion $X_j \to X$ in (4) of [5] does not exist in general.

[1] It is not clear at the moment that the notions of a T-dual used in the present paper coincides with that of [5]

1.11 Let us consider two pairs (F, h) and $(\hat{F}, \hat{h})$ over B. The interest of T-duality in topology stems to a great deal from an (expected) isomorphism of certain twisted cohomology groups of the two spaces. Let $h(\dots, \dots)$ denote a twisted cohomology theory in the sense of [3, Section 3]. Let $\mathcal{H}$ be a twist over F in the isomorphism class determined by h, and similarly, let $\hat{\mathcal{H}}$ be a twist over $\hat{F}$ in isomorphism class $\hat{h}$. Let $p : F \times_B \hat{F} \to F$ and $\hat{p} : F \times_B \hat{F} \to \hat{F}$ be the projections. We assume that there exists an isomorphism of twists $u : \hat{p}^*\hat{\mathcal{H}} \to p^*\mathcal{H}$. The triple $((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ is a T-duality triple (see Definition 5.5), if u satisfies a condition called $\mathcal{P}(u)$ (see 5.4). If we assume that $(\hat{F}, \hat{h})$ is T-dual to (F, h) (in the constructive way), then by Corollary 5.6 there exists an u such that $((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ is a T-duality triple.

1.12 Let us assume that $((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ is a T-duality triple. In Definition 5.7 we define a T-duality transformation $T : h(F, \mathcal{H}) \to h(\hat{F}, \hat{\mathcal{H}})$ of degree $-n$ which is linear over $h(B)$. It is given by $T := \hat{p}_! \circ u^* \circ p^*$.

Theorem 1.2 (Theorem 5.8) *If the twisted cohomology theory h is T-admissible in the sense of [3, Definition 3.7], then the T-duality transformation is an isomorphism.*

Examples for T-admissible cohomology theories are twisted extensions of K-theory or cohomology with coefficients in the graded ring $\mathbb{R}[x, x^{-1}]$, where x has degree 2.

A corresponding result is also stated in [1]. In the approach of [4] the T-duality isomorphism for twisted K-theory is equivalent to an isomorphism $K(\mathcal{A}) \cong K(\hat{\mathcal{A}})$. In fact it is a by now classical result that these groups are isomorphic by Connes' Thom isomorphism for crossed products with $\mathbb{R}^n$.

1.13 Assume that $F = B \times T^n$ is the trivial T^n-bundle and that we consider the trivial twist $h = 0$. Then the T-dual bundle is again the trivial bundle, $\hat{F} = B \times T^n$, and the dual twist vanishes: $\hat{h} = 0$. In this situation the T-duality transformation $T : K(B \times T^n) \to K(B \times T^n)$ is a K-theory version of the Fourier-Mukai transformation (see e.g. [6]).

Note that the algebraic geometric analog is more precise. In this case F and $\hat{F}$ are bundles of dual abelian varieties. On $F \times_B \hat{F}$ one has the so-called Poincaré sheaf $\mathcal{P}$. Its first

Chern class c_1 (considered as an automorphism of the trivial twist) satisfies the condition $\mathcal{P}(c_1)$. The Fourier-Mukai transformation is a functor between bounded derived categories of coherent sheaves $T : D^b(F) \to D^b(\hat{F})$ given on objects by $T(X) = R\hat{p}_*(\mathcal{P} \overset{L}{\otimes} p^*X)$. Thus the T-duality transformation considered in the present paper is a coarsification since it takes in a certain sense only the isomorphism classes of objects into account. But note that the tensor product with the Poincaré sheaf plays the role of an automorphism of the trivial twist.

A bundle of abelian varieties has a section. Therefore this case corresponds to the case of trivial T^n-bundles in the present paper. Non-trivial bundles can be interpreted as bundles of torsors. In this case in general a good analog of the Poincaré bundle such that $\mathcal{P}(c_1)$ (see condp 5.4) is satisfied may not exist. We argue below that in the topological situation we must replace the Poincaré bundle by an isomorphisms u of non-trivial twists in order to satisfy $\mathcal{P}(u)$, and to have a T-duality *isomorphism*. In algebraic geometry a similar observation is known, where twists are represented by Azumaya algebras.

1.14 As a specific example in the topological case, let $F \to B$ be some possibly non-trivial T^n-bundle, and let the twist on F be trivial (i.e. $h = 0$). Then the pair $(F, 0)$ admits a T-dual pair $(\hat{F}, \hat{h})$. This is not unique in general (see the example in bunkeschickt [3], Sec. 4.4), but one choice is the trivial bundle $\hat{F} := B \times T^n$. In order to describe the class $\hat{h} \in H^3(\hat{F}, \mathbb{Z})$, we consider $\hat{F}$ as the fiber product of n-copies of a trivial T^1-bundle $F_1 := B \times T^1$. Let $\pi_i : \hat{F} \to F_1$ denote the corresponding projections. Then one possible choice of $\hat{h}$ is $\hat{h} = \sum_{i=1}^n \mathtt{pr}_i^* h_1$, where $h_i = \mathbf{c}_i \times \mathrm{or}_{T^1} \in H^3(B \times T^1, \mathbb{Z})$, $\mathbf{c}_i \in H^2(B, \mathbb{Z})$ denotes the Chern classes of F, and $\mathrm{or}_{T^1} \in H^1(T^1, \mathbb{Z})$ is the orientation class. This example shows that it is necessary to introduce twists in order to generalize the Fourier-Mukai transformation to non-trivial torus bundle.

2 The space $\mathbf{R}_n$

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expla *2.1* Throughout this chapter (and the whole paper) we fix $n \geq 1$.

2.2 If G is an abelian group and $k \in \mathbb{N}$, then we consider the homotopy type $K(G,n)$ of the Eilenberg MacLane space. It is characterized by $\pi_i(K(G,k)) \cong 0$ for $i \neq k$, and $\pi_k(K(G,k)) = G$. We will denote a CW-complex representing this homotopy type by the same symbol.

The Eilenberg-MacLane space $K(G,k)$ classifies the cohomology functor $H^k(\dots,G)$. In fact, there is a universal class $z \in H^k(K(G,k),G)$ such that $f \mapsto f^*(z)$ induces a natural isomorphism $[B,K(G,k)] \to H^k(B,G)$, where $[B,K(G,k)]$ denotes homotopy classes of maps.

Occasionally, we will interpret $K(\mathbb{Z},2)$ also as the classifying space of $T^1 := U(1)$. An explicit model is U/T^1, where U is the unitary group of a separable infinite dimensional Hilbert space. The bundle $U \to U/T^1$ is the universal T^1-principal bundle. Note further that $K(\mathbb{Z}^n,2) \cong K(\mathbb{Z},2)^n$ has the homotopy type of BT^n, and this space carries a universal T^n-bundle $U^n \to K(\mathbb{Z},2)^n$.

2.3 We consider the Eilenberg-MacLane space $K(\mathbb{Z}^{2n},2) \cong K(\mathbb{Z},2)^{2n} \cong K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2)$. Let $x_i, \hat{x}_i$, $i = 1,\dots,n$ be the canonical generators of $H^2(K(\mathbb{Z},2)^{2n},\mathbb{Z})$. Let $q : K(\mathbb{Z},2)^{2n} \to K(\mathbb{Z},4)$ be the map classifying the class

$$x_1 \cup \hat{x}_1 + \dots + x_n \cup \hat{x}_n \in H^4(K(\mathbb{Z},2)^{2n},\mathbb{Z}) \ .$$

rt3 **Definition 2.1** *We define the homotopy type $\mathbf{R}_n$ by the homotopy pull-back diagram*

$$\begin{array}{ccc} \mathbf{R}_n & \stackrel{(\mathbf{c},\hat{\mathbf{c}})}{\to} & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) \\ \downarrow & & q \downarrow \\ * & \to & K(\mathbb{Z},4) \end{array} .$$

2.4 For later use we determine the homotopy groups of $\mathbf{R}_n$.

hh **Lemma 2.2** *The homotopy groups of $\mathbf{R}_n$ are given by*

i	0	1	2	3	≥ 4
$\pi_i(\mathbf{R}_n)$	$*$	0	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

Proof. The homotopy fiber of $(\mathbf{c}, \hat{\mathbf{c}})$ is homotopy equivalent to the one of $* \to K(\mathbb{Z}, 4)$, i.e to $K(\mathbb{Z}, 3)$. The assertion follows immediately from the long exact sequence of homotopy groups. $\square$

tz1

2.5 Write $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ and $\hat{\mathbf{c}} = (\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_n)$. We denote $\mathbf{c}_i$ and $\hat{\mathbf{c}}_i$ toe be the components of $\mathbf{c}$ or $\hat{\mathbf{c}}$, respectively.

gh **Lemma 2.3** *We have*

i	0	1	2	3	4
$H^i(\mathbf{R}_n, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$

.

Here $H^2(\mathbf{R}^n, \mathbb{Z})$ is freely generated by the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$, and $H^4(\mathbf{R}_n, \mathbb{Z})$ is generated by all possible products of the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$ subject to one relation

$$0 = \mathbf{c}_1 \cup \hat{\mathbf{c}}_1 + \cdots + \mathbf{c}_n \cup \hat{\mathbf{c}}_n \ .$$

Proof. Recall from the proof of Lemma hh 2.2 that the homotopy fiber of $\mathbf{R}_n \to K(\mathbb{Z}, 2)^{2n}$ is $K(\mathbb{Z}, 3)$. The relevant part of the second page of the Leray-Serre spectral sequence $E_2^{p,q} \cong H^p(K(\mathbb{Z}^{2n}, 2), H^q(K(\mathbb{Z}, 3)))$ therefore becomes

3	$\mathbb{Z}$	0	$*$	0	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{\frac{2n(2n+1)}{2}}$
q/p	0	1	2	3	4

.

We read off that $H^2(\mathbf{R}^n, \mathbb{Z}) \cong \mathbb{Z}^{2n}$ is generated by the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$. The group $E_2^{4,0} \cong H^4(K(\mathbb{Z}, 2)^{2n}, \mathbb{Z})$ is freely generated by all possible products of the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$.

Let $z_3 \in H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \cong E_2^{0,3}$ be the canonical generator. It also generates the group $\bar{E}_2^{0,3}$ of the Leray-Serre spectral sequence of the homotopy fibration $* \to K(\mathbb{Z}, 4)$. A part of its second page is

3	$\mathbb{Z}$	0	0	0	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
q/p	0	1	2	3	4

.

Since $H^3(*,\mathbb{Z}) \cong 0 \cong H^4(*,\mathbb{Z})$ we conclude that $\bar{d}_2^{0,3}(z_3) = z_4 \in H^4(K(\mathbb{Z},4),\mathbb{Z}) \cong \bar{E}_2^{4,0}$ is the generator. Now by construction $q^* z_4 = x_1 \cup \hat{x}_1 + \dots + x_n \cup \hat{x}_n$, and by naturality of the spectral sequences $d_2^{0,3}(z) = q^* z_4$. This implies the assertion about $H^4(\mathbf{R}_n,\mathbb{Z})$. □

2.6 Recall that we consider $K(\mathbb{Z}^n,2) \cong BT^n$ (see expla 2.2).

Definition 2.4 *We define $\pi_n : \mathbf{F}_n \to \mathbf{R}_n$ to be the T^n-bundle which is classified by $\mathbf{c} : \mathbf{R} \to K(\mathbb{Z}^n,2)$, i.e. the pullback of the universal bundle $U^n n$.*

Let $U \to K(\mathbb{Z},2)$ be the universal T^1-bundle. The n-fold product $U^n \to K(\mathbb{Z},2)^n$ is the universal T^n-bundle. By definition we get a pull-back diagram

$$\begin{array}{ccc} \mathbf{F}_n & \to & U^n \\ \pi_n \downarrow & & \downarrow \\ \mathbf{R}_n & \stackrel{\mathbf{c}}{\to} & K(\mathbb{Z},2)^n \end{array} .$$

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2.7 Let us fix canonical generators $y_1,\dots,y_n \in H^1(T^n,\mathbb{Z})$. Let $\mathcal{F}^k H^*(\mathbf{F},\mathbb{Z})$, $k = 0,1,\dots$ be the decreasing Leray-Serre filtration and $E_2^{p,q}$ be the second page of the Leray-Serre spectral sequence for the bundle $\mathbf{F}_n \to \mathbf{R}_n$.

re1 **Lemma 2.5** *We have*

i	0	1	2	3
$H^i(\mathbf{F}_n,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^n$	$\mathbb{Z}$

Here the group $H^2(\mathbf{F}_n,\mathbb{Z})$ is freely generated by the components of $\pi_n^* \hat{\mathbf{c}}$. In particular, restriction to the fiber of $\mathbf{F}_n \to \mathbf{R}_n$ induces the zero homomorphism on $H^2(\mathbf{F}_n,\mathbb{Z})$. Furthermore, $H^3(\mathbf{F}_n,\mathbb{Z})$ is generated by a class $\mathbf{h}_n \in \mathcal{F}^2 H^3(\mathbf{F}_n,\mathbb{Z})$ which is represented by the cocycle $y_1 \otimes \hat{\mathbf{c}}_1 + \dots + y_n \otimes \hat{\mathbf{c}}_n \in E_2^{2,1}$.

Proof. We write out the second page $E_2^{p,q}$ (isomorphic to $H^q(T^n,\mathbb{Z})\otimes H^p(\mathbf{R}_n,\mathbb{Z})$).

3	$\mathbb{Z}^{\frac{n(n-1)(n-2)}{6}}$				
2	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	0	$\mathbb{Z}^{n^2(n-1)}$	0	
1	$\mathbb{Z}^n$	0	$\mathbb{Z}^{2n^2}$	0	
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$
q/p	0	1	2	3	4

.

We first calculate $d_2^{0,1}(y_i)\in E_2^{2,0}\cong H^2(\mathbf{R}_n,\mathbb{Z})$. To this end we consider the universal T^1-bundle $U\to K(\mathbb{Z},2)$. Let $y\in H^1(T^1,\mathbb{Z})$ be the canonical generator. It also generates the group $\bar E_2^{0,1}$ of the associated Leray-Serre spectral sequence

1	$\mathbb{Z}$	0	$*$
0	$\mathbb{Z}$	0	$\mathbb{Z}$
q/p	0	1	2

.

Since U is contractible $\bar d_2^{0,1}(y)\in\bar E_2^{2,0}\cong H^2(K(\mathbb{Z},2),\mathbb{Z})$ is a generator. The bundle $\pi_n:\mathbf{F}_n\to\mathbf{R}_n$ is obtained as a pull-back of an n-fold product of the bundle $U\to K(\mathbb{Z},2)$ via the components of $\mathbf{c}$. We conclude by the naturality of the spectral sequences that $d_2^{0,1}(y_i)\in\{\mathbf{c}_i,-\mathbf{c}_i\}$.

We see that $d_2^{0,1}$ is an isomorphism onto the subgroup of $E_2^{0,2}$ generated by $\mathbf{c}$ so that $E_3^{0,1}\cong 0$ and $E_3^{2,0}$ is freely generated by the components of $\hat{\mathbf{c}}$. We see already that $H^1(\mathbf{F}_n,\mathbb{Z})\cong 0$.

The group $E_2^{0,2}\cong H^2(T^n,\mathbb{Z})$ is freely generated by all products $y_i\cup y_j$, $i<j$. We now use the multiplicativity of the Leray-Serre spectral sequence in order see that $d_2^{0,2}(y_i\cup y_j)=\pm(y_j\otimes\mathbf{c}_i-y_i\otimes\mathbf{c}_j)$. We conclude that $d_2^{0,2}$ is injective. This implies that $E_3^{0,2}\cong 0$, and it follows that $H^2(\mathbf{F}_n,\mathbb{Z})$ is freely generated by the components of $\pi_n^*\hat{\mathbf{c}}$.

We have $E_2^{4,0}\cong H^4(\mathbf{R}_n,\mathbb{Z})$ and $d_2^{1,2}(y_i\otimes\mathbf{c}_j)=\mathbf{c}_i\cup\mathbf{c}_j$ and $d_2^{1,2}(y_i\otimes\hat{\mathbf{c}}_j)=\mathbf{c}_i\cup\hat{\mathbf{c}}_j$.

In order to calculate $\ker(d_2^{1,2})$ recall the relation $\mathbf{c}_1\cup\hat{\mathbf{c}}_1+\cdots+\mathbf{c}_n\cup\hat{\mathbf{c}}_n=0$. Let $h:=y_1\otimes\hat{\mathbf{c}}_1+\cdots+y_n\cup\hat{\mathbf{c}}_n$. Then we have $d_2^{1,2}(h)=0$. We claim that $\ker(d_2^{1,2})\cong\mathbb{Z}h\oplus\mathtt{im}(d_2^{0,2})$. Let $t:=\sum_{i,j=1}^n a_{i,j}y_i\otimes\mathbf{c}_j+b_{i,j}y_i\otimes\hat{\mathbf{c}}_j$ for $a_{i,j},b_{i,j}\in\mathbb{Z}$ and assume that $d_2^{1,2}(t)=0$. Then

$\sum_{i,j=1}^n a_{i,j}\mathbf{c}_i \cup \mathbf{c}_j + b_{i,j}\mathbf{c}_i \cup \hat{\mathbf{c}}_j = 0$. This implies that $a_{i,j} + a_{j,i} = 0$, $b_{i,j} = 0$ for $i \neq j$, and that there exists $b \in \mathbb{Z}$ such that $b_{i,i} = b$ for all $i = 1, \dots, n$. But then we can write $t = \sum_{i<j} a_{i,j} d_2^{0,2}(y_j \cup y_i) + bh$. It follows that $E_3^{2,1} \cong \mathbb{Z}$ is generated by the class of h.

The group $E_2^{3,0} \cong H^3(T^n, \mathbb{Z})$ is freely generated by the products $y_i \cup y_j \cup y_k$, $i < j < k$. Furthermore $E_2^{2,2} \cong H^2(T^n, \mathbb{Z}) \otimes H^2(\mathbf{R}_n, \mathbb{Z})$ and $d_2^{0,3}(y_i \cup y_j \cup y_k) = y_j \cup y_k \otimes \mathbf{c}_i - y_i \cup y_k \otimes \mathbf{c}_j + y_i \cup y_j \otimes \mathbf{c}_k$. We thus see by a simple calculation that $d_2^{0,3}$ is injective. We conclude that $H^3(\mathbf{F}_n, \mathbb{Z}) \cong \mathcal{F}^2 H^3(\mathbf{F}_n, \mathbb{Z})$ is generated by the class $\mathbf{h}_n$ represented by $h \in E_2^{1,2}$. $\square$

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2.8 In the proof of Lemma 2.2 we have found a homotopy Cartesian square

$$\begin{array}{ccc} K(\mathbb{Z},3) & \stackrel{i}{\to} & \mathbf{R}_n \\ \downarrow & & (\mathbf{c},\hat{\mathbf{c}}) \downarrow \\ * & \to & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) \end{array} .$$

We form the pull-back of this square by the map $\psi : K(\mathbb{Z}^n, 2) \cong U^n \times K(\mathbb{Z}^n, 2) \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ and obtain a cube of homotopy Cartesian squares

$$\begin{array}{ccccccc} i^*\mathbf{F}_n & & & \stackrel{I}{\longrightarrow} & & & \mathbf{F}_n \\ & \mathtt{pr} \searrow & & & & \pi_n \swarrow & \\ & & K(\mathbb{Z},3) & \stackrel{i}{\to} & \mathbf{R}_n & & \\ \lambda \downarrow & & \downarrow & & (\mathbf{c},\hat{\mathbf{c}}) \downarrow & & \kappa \downarrow \\ & & * & \to & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) & & \\ & \nearrow & & & & \psi \nwarrow & \\ K(\mathbb{Z},1)^n & & & \longrightarrow & & & K(\mathbb{Z}^n,2) \end{array} .$$

g16 **Lemma 2.6** *We have $I^*\mathbf{h}_n = \pm \mathtt{pr}^* z_3$, where $z_3 \in H^3(K(\mathbb{Z},3), \mathbb{Z})$ is the canonical generator.*

Proof. The fiber of κ is equivalent to $K(\mathbb{Z},3)$. The second page E_2 of the corresponding Leray-Serre spectral sequence has the form

3	$\mathbb{Z}$	0	$*$	$*$	$*$
2	0	0	0	0	$*$
1	0	0	0	0	$*$
0	$\mathbb{Z}$	0	$\mathbb{Z}^n$	0	$\mathbb{Z}^{\frac{n(n+1)}{2}}$
q/p	0	1	2	3	4

.

We know that $H^3(\mathbf{F}_n,\mathbb{Z}) \cong \mathbb{Z}$ is freely generated by $\mathbf{h}_n$. We see that $d_2^{0,3} = 0$ and $E_2^{0,3} \cong H^3(\mathbf{R}_n,\mathbb{Z})$ is generated by $\mathbf{h}_n$. On the other hand the group $E_2^{0,3} \cong H^3(K(\mathbb{Z},3),\mathbb{Z})$ is freely generated by z_3. Therefore $\mathbf{h}_n = \pm z_3$.

Note that $i^*\mathbf{F}_n \cong K(\mathbb{Z},3) \times K(\mathbb{Z},1)^n$ so that the Leray-Serre spectral sequence $\bar{E}$ of λ degenerates. Let $I^* : E_2 \to \bar{E}_2$ be the induced map of the second pages and note that $\bar{E}_2$ has the form

3	$\mathbb{Z}$	$\mathbb{Z}^n$	$*$	$*$	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^n$	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{\frac{n(n-1)(n-2)}{6}}$	$\mathbb{Z}^{\frac{n(n-1)(n-2)(n-3)}{24}}$
q/p	0	1	2	3	4

,

where $\bar{E}_2^{0,3}$ is freely generated by $z_3 \in H^3(K(\mathbb{Z},3),\mathbb{Z})$. The map I induces an equivalence of the fibers of λ and κ. In particular, it induces an isomorphism $I^* : E_2^{0,3} \to \bar{E}_2^{0,3}$ identifying the generators above. This implies that $I^*\mathbf{h}_n = \pm \mathtt{pr}^* z_3$. $\square$

3 Universal T-duality

tz

3.1 Let $(e_1,\dots,e_n,\hat{e}_1,\dots,\hat{e}_n)$ be the standard basis of $\mathbb{Z}^{2n}$. Let $G_n \subset GL(2n,\mathbb{Z})$ be the subgroup of transformations which fix the form $q : \mathbb{Z}^{2n} \to \mathbb{Z}$ given by $q(\sum_{i=1}^n a_i e_i + b_i \hat{e}_i) := \sum_{i=1}^n a_i b_i$. In the usual notation $G_n = O(n,n,\mathbb{Z})$. In the present paper it will be called the group of T-duality transformations.

3.2 Each $g \in G_n$ induces an equivalence $g : K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \to K(\mathbb{Z}^n, 2)) \times K(\mathbb{Z}^n, 2)$.

re **Lemma 3.1** *There exist a unique lift $\tilde{g}$ in the diagram*

$$\begin{array}{ccc} \mathbf{R}_n & \xrightarrow{\tilde{g}} & \mathbf{R}_n \\ \downarrow{\scriptstyle (\mathbf{c},\hat{\mathbf{c}})} & & \downarrow{\scriptstyle (\mathbf{c},\hat{\mathbf{c}})} \\ K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) & \xrightarrow{g} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} .$$

Proof. We apply obstruction theory to the problem of existence and classification of lifts $\tilde{g}$. In fact, the obstruction is the class $(\mathbf{c}, \hat{\mathbf{c}})^* g^* (\sum_{i=1}^n x_i \cup \hat{x}_i) \in H^4(\mathbf{R}_n, \mathbb{Z})$ which vanishes since g preserves q. Therefore lifts $\tilde{g}$ exist. The set of lifts is a torsor over $H^3(\mathbf{R}_n, \mathbb{Z})$. Since $H^3(\mathbf{R}_n, \mathbb{Z}) = \{0\}$, the lift $\tilde{g}$ is unique. $\square$

3.3 We let $t \in G_n$ be the transformation given by $t(e_i) = \hat{e}_i$ and $t(\hat{e}_i) = e_i$.

tt1 **Definition 3.2** *The universal T-duality is the lift $T := \tilde{t} : \mathbf{R}_n \to \mathbf{R}_n$ of t according to* re *3.1.*

Note that $T \circ T = \mathtt{id}_{\mathbf{R}_n}$ since $t \circ t = 1 \in G_n$.

3.4

rtz **Definition 3.3** *The dual T^n-bundle is defined by the pull-back*

$$\begin{array}{ccc} \hat{\mathbf{F}}_n & \stackrel{\tilde{T}}{\to} & \mathbf{F}_n \\ \downarrow & & \downarrow \\ \mathbf{R}_n & \stackrel{T}{\to} & \mathbf{R}_n \end{array} .$$

We further define $\hat{\mathbf{h}}_n := \tilde{T}^* \mathbf{h}_n \in H^3(\hat{\mathbf{F}}_n, \mathbb{Z})$.

adef

3.5 We consider the pull-back diagram

$$
\begin{tikzcd}
 & \mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n \arrow[dl, "a_n"'] \arrow[dr, "\hat{a}_n"] & \\
\mathbf{F}_n \arrow[dr, "\pi_n"'] & & \hat{\mathbf{F}}_n \arrow[dl, "\hat{\pi}_n"] \\
 & \mathbf{R}_n &
\end{tikzcd} .
$$

aeq **Lemma 3.4** *We have*

i	0	1	2	3
$H^i(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$

Moreover, $a_n^* \mathbf{h}_n = \hat{a}_n^* \hat{\mathbf{h}}_n$, *and this element generates* $H^3(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z})$.

Proof. We use the Leray-Serre spectral sequence. The second page is $E_2^{p,q} \cong H^q(T^n \times T^n, \mathbb{Z}) \otimes H^p(\mathbf{R}_n, \mathbb{Z})$. Let $y_i, \hat{y}_i$, $i = 1, \dots, n$ denote the canonical generators of $H^1(T^n \times T^n, \mathbb{Z})$ (so that the elements decorated with $\hat{}$ come from the right copy). The relevant part of the second page has the form

3	$\mathbb{Z}^{\frac{2n(2n-1)(2n-2)}{6}}$				
2	$\mathbb{Z}^{\frac{2n(2n-1)}{2}}$	0	$\mathbb{Z}^{2n^2(2n-1)}$	0	
1	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{4n^2}$	0	
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$
q/p	0	1	2	3	4

Now $E_2^{0,1}$ is freely generated by $y_i, \hat{y}_i$, and $E_2^{2,0}$ is freely generated by $\mathbf{c}_i, \hat{\mathbf{c}}_i$. A similar argument as in the proof of Lemma re1 2.5 shows that $d_2^{0,1}(y_i) = \mathbf{c}_i$ and $d_2^{0,1}(\hat{y}_i) = \hat{\mathbf{c}}_i$. We conclude that $d_2^{0,1}$ is injective and $H^1(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z}) \cong 0$.

The group $E_2^{0,2}$ is freely generated by the products $y_i \cup y_j$, $\hat{y}_i \cup \hat{y}_j$, $i < j$ and $y_i \cup \hat{y}_j$. We have $d_2^{0,2}(y_i \cup y_j) = y_j \otimes \mathbf{c}_i - y_i \otimes \mathbf{c}_j$, $d_2^{0,2}(\hat{y}_i \cup \hat{y}_j) = \hat{y}_j \otimes \hat{\mathbf{c}}_i - \hat{y}_i \otimes \hat{\mathbf{c}}_j$, and $d_2^{0,2}(y_i \cup \hat{y}_j) = \hat{y}_j \otimes \mathbf{c}_i - y_i \otimes \hat{\mathbf{c}}_j$. It follows that $d_2^{0,2}$ is injective.

In a similar way we see that $d_2^{0,3}$ is injective.

We now calculate $E_3^{2,1}$. We claim that this group is freely generated by one class which can be represented by $\sum_{i=1}^n y_i \otimes \hat{\mathbf{c}}_i$ or alternatively by $\sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i$. In view of the construction of $\mathbf{h}_n$ and $\hat{\mathbf{h}}_n$, and by naturality of the Leray-Serre spectral sequence this would imply the assertion of the lemma about the third cohomology.

The group $E_2^{4,0}$ is generated by the products $\mathbf{c}_i \cup \mathbf{c}_j, \hat{\mathbf{c}}_i \cup \hat{\mathbf{c}}_j$, $i \leq j$, and all products $\mathbf{c}_i \cup \hat{\mathbf{c}}_j$ subject to one relation $\sum_{i=1}^n \mathbf{c}_i \cup \hat{\mathbf{c}}_i = 0$. Let $t = \sum_{i,j} a_{i,j} y_i \otimes \mathbf{c}_j + \sum_{i,j} b_{i,j} y_i \otimes \hat{\mathbf{c}}_j + \sum_{i,j} c_{i,j} \hat{y}_i \otimes \mathbf{c}_j + \sum_{i,j} d_{i,j} \hat{y}_i \otimes \hat{\mathbf{c}}_j$ and assume that $d_2^{2,1}(t) = 0$. Then we have $\sum_{i,j} a_{i,j} \mathbf{c}_i \otimes \mathbf{c}_j + \sum_{i,j} b_{i,j} \mathbf{c}_i \otimes \hat{\mathbf{c}}_j + \sum_{i,j} c_{i,j} \hat{\mathbf{c}}_i \otimes \mathbf{c}_j + \sum_{i,j} d_{i,j} \hat{\mathbf{c}}_i \otimes \hat{\mathbf{c}}_j = 0$. This implies that $a_{i,j} + a_{j,i} = 0$, $d_{i,j} + d_{j,i} = 0$, for all i,j, $b_{i,j} + c_{i,j} = 0$ for all $i \neq j$, and that there exists a unique $e \in \mathbb{Z}$ such that $b_{i,i} + c_{i,i} = e$ for all $i = 1, \ldots, n$. We can now write $t = \sum_{i<j} a_{i,j} d_2^{0,2}(y_j \cup y_i) + \sum_{i<j} d_{i,j} d_2^{0,2}(\hat{y}_j \cup \hat{y}_i) - \sum_{i \neq j} b_{i,j} d_2^{0,2}(y_i \cup \hat{y}_j) - \sum_{i=1}^n b_{i,i} d_2^{0,2}(y_i \cup \hat{y}_i) + e \sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i$. This already shows that $E_3^{2,1}$ is freely generated by the class of $\sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i$. Finally note that $\sum_{i=1}^n d_2^{0,2}(y_i \cup \hat{y}_i) = \sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i - \sum_{i=1}^n y_i \otimes \hat{\mathbf{c}}_i$. This finishes the proof of the claim. □

4 $\mathbf{R}_n$ as a classifying space

fff

4.1 The homotopy type $\mathbf{R}_n$ classifies a contravariant set-valued functor P_n on the category of topological spaces defined on objects by $P_n(X) := [X, \mathbf{R}_n]$. In tz 3.1 we have introduced the group G_n of T-duality transformations which by acts by automorphisms on P_n. Of particular interest is the T-duality transformation T given in tt1 3.2.

In the present section we want to describe the geometric meaning of the objects classified by $\mathbf{R}_n$ and the action of T. To this end we give another construction of the homotopy type $\mathbf{R}_n$. We closely follow Section 2 of bunkeschickt [3], which deals with the special case $n = 1$.

4.2 We start with the universal T^n-bundle $U^n \to K(\mathbb{Z},2)^n$ and the T^n-space $\mathtt{Map}(T^n, K(\mathbb{Z},3))$, where T^n acts by reparametrization. In a first step we form the associated bundle

$$p : \tilde{\mathbf{R}}_n(0) := U^n \times_{T^n} \mathtt{Map}(T^n, K(\mathbb{Z},3)) \to K(\mathbb{Z},2)^n \ .$$

We define a T^n-bundle via pull-back

$$\begin{array}{ccc} \tilde{\mathbf{F}}_n(0) & \to & U^n \\ \downarrow & & \downarrow \\ \tilde{\mathbf{R}}_n(0) & \stackrel{p}{\to} & K(\mathbb{Z},2)^n \end{array} \ .$$

There is a canonical map

$$\tilde{\mathbf{h}}_n(0) \colon \tilde{\mathbf{F}}_n(0) \to K(\mathbb{Z},3)^n \ .$$

It is given by $\tilde{\mathbf{h}}_n(0)([v,\phi],u) := \phi(s)$, where $s \in T^n$ is the unique element such that $sv = u$. Here $u, v \in U^n$, $\phi \in \mathtt{Map}(T^n, K(\mathbb{Z},3))$, $[v,\phi] \in \tilde{\mathbf{R}}_n(0)$, and $([v,\phi],u) \in \tilde{\mathbf{F}}_n(0)$.

4.3 A pair (F, h) over a space B consists of a T^n-bundle $F \to B$ and a class $h \in H^3(F,\mathbb{Z})$. An isomorphism between pairs (F,h) and (F',h') is given by a diagram

$$\begin{array}{ccc} F & \stackrel{\Phi}{\to} & F' \\ \downarrow & & \downarrow \\ B & = & B \end{array} \ ,$$

where Φ is a T^n-bundle isomorphism such that $\Phi^* h' = h$.

Given a map $f : B' \to B$ of spaces we can form the pull-back

$$\begin{array}{ccc} F' & \stackrel{\tilde{f}}{\to} & F \\ \downarrow & & \downarrow \\ B' & \stackrel{f}{\to} & B \end{array} \ .$$

We define the pair $f^*(F,h) := (F', \tilde{f}^* h)$ over B'. Pull-back preserves isomorphism classes of pairs.

4.4 Let $\tilde{P}_{(0)}$ be the contravariant set-valued functor which associates to each space B the set $\tilde{P}_{(0)}(B)$ of equivalence classes of pairs.

cl1 **Lemma 4.1** *The space $\tilde{\mathbf{R}}_n(0)$ is the classifying space for $\tilde{P}_{(0)}$. More precisely, the pair $[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{h}}_n(0)] \in P_{(0)}(\tilde{\mathbf{R}}_n(0))$ induces a natural isomorphism $\tilde{v}_B : [B, \tilde{\mathbf{R}}_n(0)] \to \tilde{P}_{(0)}(B)$ such that $\tilde{v}_B(f) = f^*[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{h}}_n(0)]$.*

Proof. This is completely analogous to the proof of bunkeschickt [3, Proposition 2.6]. We therefore refrain from repeating the proof here. □

4.5 By Lemma cl1 4.1 we have an isomorphism $\pi_0(\tilde{\mathbf{R}}_n(0)) \stackrel{def}{=} \tilde{P}_{(0)}(*)$. Furthermore, note that $P_{(0)}(*) \cong H^3(T^n, \mathbb{Z})$ canonically. We define $\tilde{\mathbf{R}}_n(1) \subset \tilde{\mathbf{R}}_n(0)$ to be the component which corresponds to $0 \in H^3(T^n, \mathbb{Z})$. Restricting the pair $[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{H}}_n(0)]$ gives the pair $[\tilde{\mathbf{F}}_n(1), \tilde{\mathbf{h}}_n(1)]$ over $\tilde{\mathbf{R}}_n(1)$. We let $\tilde{P}_{(1)}$ be the functor classified by $\tilde{\mathbf{R}}_n(1)$. Observe that $\tilde{P}_{(1)}(B) \subset \tilde{P}_{(0)}(B)$ is the set of isomorphism classes of pairs $[F, h]$ such that the restriction
cl2 of h to the fibers of F vanishes.

4.6 By Lemma cl1 4.1 we have $\tilde{P}_{(0)}(S^1) \cong H^3(S^1 \times T^n)$. By the Künneth formula $H^3(S^1 \times T^n, \mathbb{Z}) \cong H^3(T^n, \mathbb{Z}) \oplus H^2(T^n, \mathbb{Z})$, and $\pi_1(\tilde{\mathbf{R}}_n(1)) \stackrel{def}{=} \tilde{P}_{(1)}(S^1 \times T^n) \cong H^2(T^n, \mathbb{Z})$ corresponds to the second summand.

We consider the isomorphism $\phi : \pi_1(\tilde{\mathbf{R}}_n(1)) \to H^2(T^n, \mathbb{Z})$ as a cohomology class $\phi \in H^1(\tilde{\mathbf{R}}_n(1), H^2(T^n, \mathbb{Z})))$, i.e. as a homotopy class of maps $\phi : \tilde{\mathbf{R}}_n(1) \to K(H^2(T^n, \mathbb{Z}), 1)$. We define $\tilde{\mathbf{R}}_n$ as the homotopy pull-back

$$\begin{array}{ccc} \tilde{\mathbf{R}}_n & \to & \tilde{\mathbf{R}}_n(1) \\ \downarrow & & \phi \downarrow \\ * & \to & K(H^2(T^n), 1) \end{array} .$$

Furthermore, we consider the pull-back

$$\begin{array}{ccc} \tilde{\mathbf{F}}_n & \to & \tilde{\mathbf{F}}_n(1) \\ \downarrow & & \downarrow \\ \tilde{\mathbf{R}}_n & \to & \tilde{\mathbf{R}}_n(1) \end{array} ,$$

and let $\tilde{\mathbf{h}}_n \in H^3(\tilde{\mathbf{F}}_n, \mathbb{Z})$ be the pullback of $\tilde{\mathbf{h}}_n(1)$. Note that by construction and naturality $\tilde{\mathbf{h}}_n$ pulls back to zero on the fiber of $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. Since $\tilde{\mathbf{R}}_n$ is simply connected, it then even belongs to the second step $\mathcal{F}^2 H^3(\tilde{\mathbf{F}}_n, \mathbb{Z})$ of the Leray-Serre filtration associated with $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$.

4.7

ght **Lemma 4.2** *The homotopy groups of* $\tilde{\mathbf{R}}_n$ *are given by*

i	0	1	2	3	≥ 4
$\pi_i(\tilde{\mathbf{R}}_n)$	$*$	0	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

Proof. By construction, $\tilde{\mathbf{R}}_n$ is connected and simply connected. The homotopy fiber of $\tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$ is equivalent to the homotopy fiber of $* \to K(H^2(T^n,\mathbb{Z}),1)$, i.e. to $K(H^2(T^n,\mathbb{Z}),0)$. Hence this map induces an isomorphism $\pi_i(\tilde{\mathbf{R}}_n) \cong \pi_i(\tilde{\mathbf{R}}_n(1))$ for $i \geq 2$.

Since $K(\mathbb{Z},k)$ is an h-space for each k we have an equivalence

$$\mathtt{Map}(T^1, K(\mathbb{Z},3)) \simeq K(\mathbb{Z},3) \times \Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},3) \times K(\mathbb{Z},2)\ .$$

We use the exponential law to write $\mathtt{Map}(T^n, K(\mathbb{Z},3))$ as an iterated mapping space, and obtain in the same way an equivalence

$$\mathtt{Map}(T^n, K(\mathbb{Z},3)) \simeq K(\mathbb{Z},3) \times K(\mathbb{Z},2)^n \times K(\mathbb{Z},1)^{\frac{n(n-1)}{2}} \times K(\mathbb{Z},0)^{\frac{n(n-1)(n-2)}{6}}\ .$$

The long exact sequence of homotopy groups for

$$\mathtt{Map}(T^n, K(\mathbb{Z},3)) \to \tilde{\mathbf{R}}_n(0) \to K(\mathbb{Z},2)^n$$

and the fact that $\pi_3(K(\mathbb{Z},2)^n) \cong 0 \cong \pi_1(K(\mathbb{Z},2)^n)$ and $\pi_2(K(\mathbb{Z},2)^n) \cong \mathbb{Z}^n$ yields the exact sequence

$$0 \to \mathbb{Z}^n \to \pi_2(\tilde{\mathbf{R}}_n(1)) \to \mathbb{Z}^n \xrightarrow{\delta} \mathbb{Z}^{\frac{n(n-1)}{2}} \xrightarrow{\alpha} \pi_1(\tilde{\mathbf{R}}_n(1)) \to 0\ .$$

Furthermore, we observe that $\pi_3(\tilde{\mathbf{R}}(1)) \cong \mathbb{Z}$ and $\pi_i(\tilde{\mathbf{R}}_n(1)) = 0$ for $i \geq 4$.

We have seen in Subsection cl2 4.6 that $\pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^2(T^2,\mathbb{Z}) \cong \mathbb{Z}^{\frac{n(n-1)}{2}}$. We conclude that α must be surjective. Consequently it is injective and $\delta = 0$.

We therefore have an exact sequence

$$0 \to \mathbb{Z}^n \to \pi_2(\tilde{\mathbf{R}}_n(1)) \to \mathbb{Z}^n \to 0\ ,$$

and this implies that $\pi_2(\tilde{\mathbf{R}}_n(1)) \cong \pi_2(\tilde{\mathbf{R}}_n) \cong \mathbb{Z}^{2n}$. □

bc1

4.8 Since the fiber of $p\colon \tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$ is equivalent to $K(H^2(T^n,\mathbb{Z}),0) \cong \pi_1(\tilde{\mathbf{R}}_n(1))$ we can consider this map as the universal covering of $\tilde{\mathbf{R}}_n(1)$. We now consider the problem of existence and classification of lifts in the diagram

$$\begin{array}{ccccc} & & \tilde{\mathbf{R}}_n & \to & * \\ & \tilde{f} \nearrow & p\downarrow & & \downarrow \\ B & \xrightarrow{f} & \tilde{\mathbf{R}}_n(1) & \xrightarrow{\phi} & K(H^2(T^n,\mathbb{Z}),1) \end{array} .$$

It follows from the construction of $\tilde{\mathbf{R}}_n$ that a lift $\tilde{f}$ exists if and only if $\phi \circ f$ is homotopic to a constant map. The lift itself depends on the choice of an explicit homotopy. If a lift exists, the set of homotopy classes of lifts is a torsor over $H^0(B, H^2(T^n,\mathbb{Z}))$.

clh

4.9 The classification of homotopy classes $\tilde{f}$ (considered just as maps, not as lifts) lifting a homotopy class f is more subtle. In order to study this problem we assume that B is path connected and equipped with a base point $b \in B$. Let $\tilde{f}_0$ be a lift of f and consider $x \in \pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^0(B, H^2(T^n,\mathbb{Z}))$. Then we consider the lift $\tilde{f}_1 = x\tilde{f}_0$, i.e. the composition of $\tilde{f}_0$ with the deck transformation associated to x.

Assume that $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic. Let $H : I \times B \to \tilde{\mathbf{R}}_n$ be a homotopy. Then $p \circ H : S^1 \times B \to \tilde{\mathbf{R}}_n(1)$ can be considered as a map $h : B \to \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1))$. We have the following diagram

$$\begin{array}{ccc} \{b\} & \xrightarrow{x} & \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1)) \\ \downarrow & h \nearrow & \mathtt{ev}_1 \downarrow \\ B & \xrightarrow{f} & \tilde{\mathbf{R}}_n(1) \end{array} , \qquad (4.3)$$

de9

where $\mathtt{ev}_1 : \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1)) \to \tilde{\mathbf{R}}_n(1)$ is the evaluation at $1 \in S^1$.

Vice versa, if $x \in \pi_1(\tilde{\mathbf{R}}_n(1))$ is such that the diagram above admits a lift h, then $\tilde{f}_0$ and $x\tilde{f}_0$ are homotopic.

clh1

4.10 The existence problem for a lift h can be studied using obstruction theory. The fiber of the map $\mathtt{ev}_1$ is $\Omega\tilde{\mathbf{R}}_n(1))$. In the proof of Lemma 4.2 (ght) we have seen that the homotopy groups of $\tilde{\mathbf{R}}_n(1)$ are given by

i	0	1	2	3	≥ 4
$\pi_i(\tilde{\mathbf{R}}_n(1))$	$*$	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

It follows that the homotopy groups of $\Omega\tilde{\mathbf{R}}_n(1)$ are given by

i	0	1	2	≥ 3
$\pi_i(\Omega\tilde{\mathbf{R}}_n)$	$Z^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

We therefore have obstructions in $H^1(B,\mathbb{Z}^{\frac{n(n-1)}{2}})$, $H^2(B,\mathbb{Z}^{2n})$ and $H^3(B,\mathbb{Z})$, and a general
p2 discussion seems to be complicated.

4.11 Let $\tilde{P}_n$ be the set-valued functor classified by $\tilde{\mathbf{R}}_n$. For each space B we have a natural transformation $u_B : \tilde{P}_n(B) \to \tilde{P}_{(1)}(B)$ induced by composition with the map $\tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$.

We conclude from Subsection 4.8 that the fibers of u_B are homogeneous spaces over $H^0(B,H^2(T^n,\mathbb{Z}))$.

Consider a class $[F,h] \in \tilde{P}_{(1)}(B)$. By Subsection 4.8, it belongs to the image of u_B if and only the restriction $[F,h]_{|B^{(1)}}$ to a 1-skeleton $B^{(1)} \subset B$ is trivial. Since every principal torus bundle on a 1-dimensional complex is trivial, this is equivalent to the condition that the restriction of h to $F_{|B^{(1)}}$ is trivial, or equivalently $h \in \mathcal{F}^2H^3(F,\mathbb{Z})$. Let us fix a map $f : B \to \tilde{\mathbf{R}}_n(1)$ representing a pair $[F,h]$ with this property. If we choose a homotopy from $\phi \circ f$ to the constant map, then we distinguish an element in the fiber $u_B^{-1}([F,h])$.

4.12 In Section 2 we have introduced a pair $(\mathbf{F}_n,\mathbf{h}_n)$ over $\mathbf{R}_n$ such that $\mathbf{h}_n \in \mathcal{F}^2H^3(\mathbf{F}_n,\mathbb{Z})$. This isomorphism class of pairs gives rise to a classifying map $f(1) : \mathbf{R}_n \to \tilde{\mathbf{R}}_n(1)$.

var **Lemma 4.4** *The set of homotopy classes of maps f which are lifts of $f(1)$ in the diagram*

$$\begin{tikzcd} & \tilde{\mathbf{R}}_n \arrow[d, "p"] \\ \mathbf{R}_n \arrow[ur, "f"] \arrow[r, "f(1)"] & \tilde{\mathbf{R}}_n(1) \end{tikzcd}$$

is a torsor over $H^2(T^n,\mathbb{Z})$.

Proof.

Since $\mathbf{R}_n$ is simply connected we know the existence of lifts. Let now $x \in \pi_1(\tilde{\mathbf{R}}_n(1))$. We choose a base point $b \in \mathbf{R}_n$ and identify $\pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^2(\mathbf{F}_{n,b}, \mathbb{Z})$, where $\mathbf{F}_{n,b}$ denotes the fiber of $\mathbf{F}_n$ over b. In particular we view $x \in H^2(\mathbf{F}_{n,b}, \mathbb{Z})$.

We must show that the existence of a lift h in the diagram 4.3 (with B replaced by $\mathbf{R}_n$ and f replaced by $f(1)$) implies that $x = 0$. Assume that a lift h exists, adjoint to a map $H\colon S^1 \times \mathbf{R}_n \to \tilde{\mathbf{R}}_n(1)$. This corresponds to a T^n-bundle $F \to S^1 \times \mathbf{R}_n$ and a class $h \in H^3(F, \mathbb{Z})$. Let $\mathtt{pr} : S^1 \times \mathbf{R}_n \to \mathbf{R}_n$ be the projection. Since $\mathbf{R}_n$ is simply connected $\mathtt{pr}$ induces in isomorphism in second cohomology and the bundle F is the pull-back via $\mathtt{pr}$ of a T^n-bundle from $\mathbf{R}_n$. Since H restricts to f on $\{1\} \times \mathbf{R}_n$ and the corresponding T^n-bundle is $\mathbf{F}_n$, necessarily $F \cong \mathtt{pr}^*\mathbf{F}_n = S^1 \times \mathbf{F}_n$. By the Künneth formula $H^3(F, \mathbb{Z}) \cong H^3(\mathbf{F}_n, \mathbb{Z}) \oplus H^2(\mathbf{F}_n, \mathbb{Z})$, with corresponding decomposition $h = \mathbf{h}_n \oplus u$. By the definition of h and the calculation of $\pi_1(\tilde{\mathbf{R}}_n(1))$ in Subsection 4.6, the restriction of u to $\mathbf{F}_{n,b}$ is x. Since the restriction $H^2(\mathbf{F}_n, \mathbb{Z}) \to H^2(\mathbf{F}_{n,b}, \mathbb{Z})$ is trivial by the description of $H^2(\mathbf{F}_n, \mathbb{Z})$ given in Lemma 2.5 it follows that $x = 0$. □

4.13 We now fix one choice of f in Lemma 4.4.

var1 **Proposition 4.5** *The map $f : \mathbf{R}_n \to \tilde{\mathbf{R}}_n$ is a weak homotopy equivalence.*

Proof. By Lemma 2.2 and Lemma 4.2 it suffices to show that f induces isomorphisms on π_2 and π_3.

Note that $H^2(\tilde{\mathbf{R}}_n, \mathbb{Z}) \cong \mathtt{Hom}_{\mathbb{Z}}(\pi_2(\tilde{\mathbf{R}}_n), \mathbb{Z}) \cong \mathbb{Z}^{2n}$. We have a natural map $x : \tilde{\mathbf{R}}_n \to K(\mathbb{Z}, 2)^n$ which classifies the T^n-principal bundle $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. We consider the components $x_i \in H^2(\tilde{\mathbf{R}}_n, \mathbb{Z})$.

Let $y_1, \ldots, y_n$ be the generators of $H^1(T^n, \mathbb{Z})$ as before. We consider the second page of the Leray-Serre spectral sequence $\tilde{E}_2^{p,q}$ of the fibration $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. Since $\mathbf{h}_n \in \mathcal{F}^2 H^3(\tilde{\mathbf{F}}_n, \mathbb{Z})$, there are elements $\hat{x}_1, \ldots, \hat{x}_n \in H^2(\tilde{\mathbf{R}}_n, \mathbb{Z})$ and $a_{i,j} \in \mathbb{Z}$ such that $\tilde{\mathbf{h}}_n$ is represented by $\tilde{h} = \sum_{i,j} a_{i,j} y_i \otimes x_j + \sum_i y_i \otimes \hat{x}_i \in \tilde{E}_2^{2,1} \cong H^1(T^n, \mathbb{Z}) \otimes H^2(\tilde{\mathbf{R}}_n, \mathbb{Z})$.

Under the pull-back induced by f the spectral sequence $\tilde{E}$ is mapped to the spectral sequence E considered in cd1 2.7. In particular $\tilde{h}$ is mapped to $h = \sum_{i=1}^{n} y_i \otimes \hat{\mathbf{c}}_i$. Since $f^* x_i = \mathbf{c}_i$ we see that $a_{i,j} = 0$. Furthermore we see that $f^* \hat{x}_i = \hat{\mathbf{c}}_i$. It follows that $f^* : H^2(\tilde{\mathbf{R}}_n, \mathbb{Z}) \to H^2(\mathbf{R}_n, \mathbb{Z})$ maps $(x_1, \dots, x_n, \hat{x}_1, \dots, \hat{x}_n)$ to a basis and is therefore surjective and injective. This implies that $f_* : \pi_2(\mathbf{R}_n) \to \pi_2(\tilde{\mathbf{R}}_n)$ is an isomorphism.

It now suffices to show that $f : \pi_3(\mathbf{R}_n) \to \pi_3(\tilde{\mathbf{R}}_n)$ is surjective. A generator $g \in \pi_3(\tilde{\mathbf{R}}_n)$ is represented by a map $g : S^3 \to \tilde{\mathbf{R}}_n$. The corresponding pair is the trivial torus bundle $\mathtt{pr}_1 : S^3 \times T^n \to S^3$ with the cohomology class of the form $h = \mathtt{pr}_1^* z$ for some $z \in H^3(S^3, \mathbb{Z})$ which is a generator.

It suffices to show that the isomorphism class of pairs $[S^3 \times T^n, \mathtt{pr}_1^* z]$ is the pull back of $[\mathbf{F}_n, \mathbf{h}_n]$ on $\mathbf{R}_n$. We consider the composition $g : S^3 \to K(\mathbb{Z}, 3) \xrightarrow{i} \mathbf{R}_n$, where the map i was defined in f16 2.8 and the first map realizes a generator of $\pi_3(K(\mathbb{Z}, 3))$. It then follows immediately from Lemma g16 2.6 that $g^*[\mathbf{F}_n, \mathbf{h}_n] = [S^3 \times T^n, \pm \mathtt{pr}_1^* z]$. Choosing the opposite generator of $\pi_3(K(\mathbb{Z}, 3))$, if necessary, the assertion follows. □

epsd

4.14

hhll **Corollary 4.6** *The functors P_n classified by $\mathbf{R}_n$ and $\tilde{P}_n$ classified by $\tilde{\mathbf{R}}_n$ are isomorphic. The group $H^2(T^n, \mathbb{Z})$ acts freely on the set of such isomorphisms, and it acts transitively if we fix the composition with $p_* : \tilde{P}_n \to \tilde{P}_{(1)}$.*

Let B be a space. Note that the group G_n of T-duality transformations acts by automorphisms on $P(B)$. In particular each element $x \in P_n(B)$ has a canonical T-dual $\hat{x} := T(x)$ (see tt1 3.2).

Assume that we have fixed an isomorphism $\epsilon : P_n \cong \tilde{P}_n$. Then we can transfer the T-duality transformations to $\tilde{P}_n$. Assume that (F, h) is a pair over B.

drt **Definition 4.7** *We say that (F, h) has a T-dual pair $(\hat{F}, \hat{h})$ if there exists a lift $x \in \tilde{P}_n(B)$ of $[F, h] \in \tilde{P}_{(0)}(B)$ such that $(\hat{F}, \hat{h})$ is the underlying pair of $\epsilon \circ T \circ \epsilon^{-1}(x)$.*

Recall that $\mathcal{F}^2H^3(F,\mathbb{Z}) \subset H^3(F,\mathbb{Z})$ denotes the second step of the Leray-Serre filtration of the cohomology associated to the bundle $F \to B$ (see filtex 1.3).

exex **Corollary 4.8** *A pair (F,h) admits a T-dual pair if and only if $h \in \mathcal{F}^2H^3(F,\mathbb{Z})$.*

In bunkeschickt [3, Section 4.4], we have given an example where for $n \geq 2$ even the topology of the underlying T^n-bundle of a T-dual pair $(\hat{F},\hat{h})$ depends on the choice of the lift of (F,h) to an element of $\tilde{P}_n(B)$.

5 T-duality transformations in twisted cohomology

w2w2w

5.1 In bunkeschickt [3, Section 3.1] we have introduced a set of axioms which describe the basic properties of twists and twisted cohomology theories which are used in connection with T-duality considerations. Since in the present paper the main focus lies on the category of twists and since simple realizations can be given, we shall recall here two explicite models.

For the first model we fix a space $K(\mathbb{Z},3)$ representing the homotopy type of the Eilenberg-MacLane space. It determines a model of twists as follows. To each space B we associate the category of twists $T(B)$. Its objects are the maps $\mathcal{H} : B \to K(\mathbb{Z},3)$. The morphisms $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ are homotopy classes of homotopies from f to f'. Note that $T(B)$ is a groupoid. Its isomorphism classes are classified by $H^3(B,\mathbb{Z})$, and the sets $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ are torsors over $H^2(B,\mathbb{Z})$ in a natural way. Note that $B \mapsto T(B)$ extends to a groupoid-valued functor. In order to define the monoidal structure on $T(B)$ we use the h-space structure of $K(\mathbb{Z},3)$.

Our second model is as follows. It associates to each space locally trivial bundles $\mathcal{H} \to B$ of C^*-algebras with fibre K, the compact operators of a separable infinite-dimensional Hilbert space. The structure group of such a bundle is PU, the projective unitary group of the Hilbert space with the strong topology. The morphisms $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ are homotopy classes of bundle isomorphisms. We again obtain a groupoid-valued functor $B \mapsto T(B)$. Since we have an isomorphism of homotopy types $BPU \simeq K(\mathbb{Z},3)$ we see again that the isomorphism classes of objects in $T(B)$ are classified by $H^3(B,\mathbb{Z})$ (this class is usually called the Dixmir-Douady invariant). And furthermore, $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ is again a torsor over $H^2(B,\mathbb{Z})$. The monoidal structure on $T(B)$ is induced by the fiberwise tensor product.

The second picture is very suitable to give a short definition of twisted K-theory at least for locally compact B. If $\mathcal{H}\in T(B)$ is a twist, i.e. a bundle of C^*-algebras $\mathcal{H}\to B$, then we can form the C^*-algebra $C_0(B,\mathcal{H})$ of continuous sections of $\mathcal{H}$ vanishing at infinity. Then we define

$$K(B,\mathcal{H}):=K(C_0(B,\mathcal{H}))\ .$$

Below we will only use the axioms fixed in [3, Section 3.1] and not rely on any concrete realization.

5.2 For a space B we consider a category $T(B)$ of twists as in [3, Section 3.1]. Let $\mathcal{H}\in T(\mathbf{F}_n)$ be a twist with isomorphism class $[\mathcal{H}]=\mathbf{h}_n\in H^3(\mathbf{F}_n,\mathbb{Z})$. Set further $\hat{\mathcal{H}}:=\tilde{T}^*\mathcal{H}\in T(\hat{\mathbf{F}}_n)$, where $\tilde{T}$ was defined in 3.3. Then $[\hat{\mathcal{H}}]=\hat{\mathbf{h}}_n$ (see again 3.3).

Recall the definition of a_n and $\hat{a}_n$ in 3.5. Since $a_n^*\mathbf{h}_n=\hat{a}_n^*\hat{\mathbf{h}}_n$ by Lemma 3.4, we conclude that there exists an isomorphism $u:\hat{a}_n^*\hat{\mathcal{H}}\to a_n^*\mathcal{H}$ in $T(\mathbf{F}_n\times_{\mathbf{R}_n}\hat{\mathbf{F}}_n)$. Since by Lemma 3.4 $H^2(\mathbf{F}_n\times_{\mathbf{R}_n}\hat{\mathbf{F}}_n,\mathbb{Z})=0$, this isomorphism is in fact unique.

5.3 We consider the pull-back diagrams

$$\begin{array}{ccc} T^n\times\hat{T}^n & \stackrel{v}{\to} & \mathbf{F}_n\times_{\mathbf{R}_n}\hat{\mathbf{F}}_n\\ \downarrow && \downarrow\\ * & \to & \mathbf{R}_n\end{array}\ ,\qquad \begin{array}{ccc} \hat{T}^n & \stackrel{\hat{w}}{\to} & \hat{\mathbf{F}}_n\\ \downarrow && \downarrow\\ * & \to & \mathbf{R}_n\end{array}\ ,\qquad \begin{array}{ccc} T^n & \stackrel{w}{\to} & \mathbf{F}_n\\ \downarrow && \downarrow\\ * & \to & \mathbf{R}_n\end{array}\ .$$

Let $y_1,\dots,y_n,\hat{y}_1,\dots\hat{y}_n$ be the canonical generators of $H^1(T^n\times T^n,\mathbb{Z})$ as in the proof of Lemma 3.4.

We choose trivializations of twists $\hat{t}:0\to\hat{w}^*\hat{\mathcal{H}}$ and $t:w^*\mathcal{H}\to 0$. Then we consider the automorphism

$$\phi:0\stackrel{\mathtt{pr}_2^*\hat{t}}{\to}\mathtt{pr}_2^*\hat{w}^*\hat{\mathcal{H}}\cong v^*\hat{a}_n^*\hat{\mathcal{H}}\stackrel{v^*u}{\to}v^*a_n^*\mathcal{H}\cong\mathtt{pr}_1^*w^*\mathcal{H}\stackrel{\mathtt{pr}_1^*t}{\to}0\ .$$

We can identify ϕ with a class $\phi \in H^2(T^n \times \hat{T}^n, \mathbb{Z})$. If we decompose

$$\phi = \phi_l \oplus \phi_m \oplus \phi_r \in H^2(T^n, \mathbb{Z}) \times 1 \oplus H^1(T^n, \mathbb{Z}) \otimes H^1(\hat{T}^n, \mathbb{Z}) \oplus 1 \times H^2(\hat{T}^n, \mathbb{Z})$$

according to the Künneth formula, then the component $\phi_m \in H^1(T^n, \mathbb{Z}) \otimes H^1(\hat{T}^n, \mathbb{Z})$ is well-defined, not depending on the choices of t and $\hat{t}$.

5.4

cond3 **Proposition 5.1** *We have*

$$\phi_m = \sum_{i=1}^n y_i \otimes \hat{y}_i \ .$$

Proof. Note that by Lemma 4.4 and Proposition 4.5 we have a canonical equivalence $\mathbf{R}_1 \cong \tilde{\mathbf{R}}_1$. In [3] we studied in detail the topology of $\mathbf{R}_1$ and the associated T-duality. The idea of the proof is to reduce the present task to the case $n = 1$.

We consider the n-fold product $F \to B$ of the T^1-bundle $\mathbf{F}_1 \to \mathbf{F}_1$ (i.e. $F = (\mathbf{F}_1)^n$, $B = (\mathbf{R}_1)^n$). Let $p_i : B \to \mathbf{R}_1$ denote the projections. Let $(z, \hat{z}) : B \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ be the map with components $z_i := p_i^* \mathbf{c}$ and $\hat{z}_i := p_i^* \hat{\mathbf{c}}$. We now apply obstruction theory to the lifting problem

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & f \nearrow & (\mathbf{c}, \hat{\mathbf{c}}) \downarrow \\ B & \stackrel{(z,\hat{z})}{\to} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} .$$

It follows from $z_i \cup \hat{z}_i = 0$ that $\sum_{i=1}^n z_i \cup \hat{z}_i = 0$ so that this diagram admits a lift. Since $H^3(B, \mathbb{Z}) \cong 0$ such a lift is in fact unique.

We therefore have a pull-back diagram of principal T^n-bundles

$$\begin{array}{ccc} F & \stackrel{\tilde{f}}{\to} & \mathbf{F}_n \\ \downarrow & & \downarrow \\ B & \stackrel{f}{\to} & \mathbf{R}_n \end{array} .$$

We define $h := \tilde{f}^* \mathbf{h}_n$. In this way we obtain a pair (F, h) over B. We further have natural projections $\mathtt{pr}_i : F \to \mathbf{F}_1$, $i = 1, \ldots, n$. Using the characterizations of $\mathbf{h}_n$ and $\mathbf{h}_1$ in

Lemma 2.5, the naturality of Leray-Serre spectral sequences, and $H^3(B,\mathbb{Z})\cong 0$, we see that

$$h=\sum_{i=1}^{n}\mathtt{pr}_n\mathbf{h}_1\ . \tag{5.2}$$

Let $T_B : B\to B$ be the product of the T-duality transformations $T:\mathbf{R}_1\to\mathbf{R}_1$ on each factor. Since T is a homotopy equivalence, there is a unique lift α in

$$\begin{array}{ccc} & & \mathbf{R}_n\\ & \alpha\nearrow & T\downarrow\\ B & \stackrel{f\circ T_B}{\to} & \mathbf{R}_n\end{array}\ .$$

Since the composition $B\stackrel{\alpha}{\to}\mathbf{R}_n\to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$ coincides with $(z,\hat{z})$ we have $\alpha=f$, and the following diagram commutes upto homotopy

$$\begin{array}{ccc} B & \stackrel{f}{\to} & \mathbf{R}_n\\ T_B\downarrow & & T\downarrow\\ B & \stackrel{f}{\to} & \mathbf{R}_n\end{array}\ .$$

This shows that we have a pull-back diagram

$$\begin{array}{ccc} \hat{F} & \stackrel{\hat{f}}{\to} & \hat{\mathbf{F}}_n\\ \downarrow & & \downarrow\\ B & \stackrel{f}{\to} & \mathbf{R}_n\end{array}\ ,$$

where $\hat{F}\to B$ is the n-fold product of $\hat{\mathbf{F}}_1\to\mathbf{R}_1$. We get the diagram

$$\begin{array}{ccccc} T^n\times T^n & \to & F\times_B\hat{F} & \stackrel{\tilde{f}\times_B\hat{f}}{\to} & \mathbf{F}_n\times_{\mathbf{R}_n}\hat{\mathbf{F}}_n\\ \downarrow & & \downarrow & & \downarrow\\ * & \to & B & \stackrel{f}{\to} & \mathbf{R}_n\end{array}\ .$$

It follows from Lemma 3.4 (in the case $n=1$) and the Künneth formula that $H^2(F\times_B\hat{F},\mathbb{Z})\cong 0$. Therefore $(\tilde{f}\times_B\hat{f})^*u:(\tilde{f}\times_B\hat{f})^*\hat{a}_n^*\hat{\mathcal{H}}\to(\tilde{f}\times_B\hat{f})^*a_n^*\mathcal{H}$ is the unique isomorphism.

Let $\mathcal{V}\in T(\mathbf{F}_1)$ and $\hat{\mathcal{V}}\in T(\hat{\mathbf{F}}_1)$ be twists in the classes $\mathbf{h}_1$ and $\hat{\mathbf{h}}_1$. Then by (5.2) there exists isomorphisms $\kappa:\tilde{f}^*\mathcal{H}\cong\sum_{i=1}^n\mathtt{pr}_i^*\mathcal{V}$ and $\hat{\kappa}:\hat{f}^*\hat{\mathcal{H}}\cong\sum_{i=1}^n\mathtt{pr}_i^*\hat{\mathcal{V}}$. The choice of these isomorphisms is not unique. However, their restrictions to the fiber over $* \to B$ is unique. This follows from the structure of $H^2(F,\mathbb{Z})$ (and of $H^2(\hat{F},\mathbb{Z})$) implied by Lemma re1 2.5 in the case $n=1$ and the Künneth-formula.

At this moment we fix some choices of κ and $\hat{\kappa}$.

Let $q_i : F \times_B \hat{F} \to \mathbf{F}_1 \times_{\mathbf{R}_1} \hat{\mathbf{F}}_1$ be the projection onto the ith component, and let $r : F \times_B \hat{F} \to F$, $\hat{r} : F \times_B \hat{F} \to \hat{F}$ denote the projections onto the factors. Note that $\tilde{f} \circ r = a_n \circ (\tilde{f} \times_B \hat{f})$, $\hat{f} \circ \hat{r} = \hat{a}_n \circ (\tilde{f} \times_B \hat{f})$, $a_1 \circ q_i = \mathtt{pr}_i \circ r$ and $\hat{a}_1 \circ q_i = \mathtt{pr}_i \circ \hat{r}$. We now have fixed isomorphisms

$$r^*\kappa \colon (\tilde{f} \times_B \hat{f})^* a_n^* \mathcal{H} \cong \sum_{i=1}^n q_i^* a_1^* \mathcal{V} \qquad , \qquad \hat{r}^*\hat{\kappa} : (\tilde{f} \times_B \hat{f})^* \hat{a}_n^* \hat{\mathcal{H}} \cong \sum_{i=1}^n q_i^* \hat{a}_1^* \hat{\mathcal{V}}.$$

Note that there is a unique isomorphism $\psi : \hat{a}_1^* \hat{\mathcal{V}} \to a_1^* \mathcal{V}$. This induces another isomorphism

$$\Phi : (\tilde{f} \times_B \hat{f})^* \hat{a}_n^* \hat{\mathcal{H}} \overset{\hat{r}^*\hat{\kappa}}{\cong} \sum_{i=1}^n q_i^* \hat{a}_1^* \hat{\mathcal{V}} \overset{\sum_{i=1}^n q_i^*\psi}{\cong} \sum_{i=1}^n q_i^* a_1^* \mathcal{V} \overset{(r^*\kappa)^{-1}}{\cong} (\tilde{f} \times_B \hat{f})^* a_n^* \mathcal{H} \ .$$

It follows that $(\tilde{f} \times_B \hat{f})^* u = \Phi$. We can now restrict Φ to the fiber $T^n \times \hat{T}^n$.

It was shown in bunkeschickt [3, 3.2.4] that the restriction of ψ to a fiber $T^1 \times \hat{T}^1$ is classified by a generator of $H^2(T^1 \times \hat{T}^1, \mathbb{Z})$, namely by $y \cup \hat{y}$ in the canonical basis of $H^1(T^1 \times \hat{T}^1, \mathbb{Z})$. If we restrict the whole composition defining Φ to the fiber and use the definition of ϕ_m, together with the uniqueness of the restrictions κ and $\hat{\kappa}$ and our freedom in the choice of t and $\hat{t}$, we obtain the result. $\square$

5.5 Let us consider a diagram

$$\begin{array}{ccccc} & & F \times_B \hat{F} & & \\ & p \swarrow & & \hat{p} \searrow & \\ F & & & & \hat{F} \\ & \searrow & & \swarrow & \\ & & B & & \end{array} ,$$

where $F \to B$ and $\hat{F} \to B$ are some T^n-principle bundles. Let $\mathcal{H} \in T(F)$ and $\hat{\mathcal{H}} \in T(\hat{F})$ be twists. We assume that $(F, \mathcal{H})$ and $(\hat{F}, \hat{\mathcal{H}})$ are *intertwined* in the following sense:

if $c_1, \dots, c_n \in H^2(B,\mathbb{Z})$ are the Chern classes of F and $\hat{c}_1, \dots, \hat{c}_n \in H^2(B,\mathbb{Z})$ are the Chern classes of $\hat{F}$, then the isomorphism class $[\mathcal{H}] \in H^3(F,\mathbb{Z})$ shall be represented by $\sum y_i \otimes \hat{c}_i \in E_2^{1,2}$, the E_2-page of the Leray-Serre spectral sequence for F, where we use the canonical generators $y_i \in H^1(F_b,\mathbb{Z})$ of the (principal) fiber of F. We also require the corresponding statement for $[\hat{\mathcal{H}}]$. Note that this implies in particular that $[\mathcal{H}] \in \mathcal{F}^2 H^3(F,\mathbb{Z})$ for the Leray-Serre filtration.

Let us further assume that there exists an isomorphism $u : \hat{p}^*\hat{\mathcal{H}} \to p^*\mathcal{H}$.

Fix $b \in B$. Let F_b and $\hat{F}_b$ denote the fibers over b. The existence of u implies the existence of isomorphisms $v : \mathcal{H}_{|F_b} \to 0$ and $\hat{v} : 0 \to \hat{\mathcal{H}}_{|\hat{F}_b}$. We now consider the composition

$$u(b) : (0 \stackrel{\hat{p}^*_{|\hat{F}_b}\hat{v}}{\to} \hat{p}^*_{|\hat{F}_b}\hat{\mathcal{H}}_{|\hat{F}_b} \stackrel{u_{|F_b\times\hat{F}_b}}{\to} p^*_{|F_b}\mathcal{H}_{|F_b} \stackrel{p^*_{|F_b}v}{\to} 0) \in H^2(F_b \times \hat{F}_b, \mathbb{Z}) \ .$$

Definition 5.3 *We say that u satisfies the condition $\mathcal{P}(u)$, if*

$$u(b) \in \sum_{i=1}^n y_i \cup \hat{y}_i + \mathtt{span}(y_i \cup y_j, \hat{y}_i \cup \hat{y}_j) \qquad \forall b \in B \ . \tag{5.4}$$ condp

Of course, it suffices to check this for one point in each component of B.

tr13 **Definition 5.5** *An n-dimensional T-duality triple over B is an intertwined triple $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ as above such that u satisfies $\mathcal{P}(u)$.*

zt5

5.6 Here is our main construction of T-duality triples. Let $f : B \to \mathbf{R}_n$ classify an element $x \in P_n(B)$. Let (F,h) represent the underlying isomorphism class of pairs. Then we have a pull-back

$$\begin{array}{ccc} F & \stackrel{\tilde{f}}{\to} & \mathbf{F}_n \\ \downarrow & & \downarrow \\ B & \stackrel{f}{\to} & \mathbf{R}_n \end{array}$$

such that $h = \tilde{f}^*\mathbf{h}_n$. Let $\hat{x} := T(x) \in P_n(B)$ be the T-dual element and let $(\hat{F}, \hat{h})$ represent the underlying isomorphism class of the dual pair. We have a pull-back diagram

$$\begin{array}{ccc} \hat{F} & \stackrel{\hat{f}}{\to} & \hat{\mathbf{F}}_n \\ \downarrow & & \downarrow \\ B & \stackrel{f}{\to} & \mathbf{R}_n \end{array} .$$

such that $\hat{f}^*\mathbf{h}_n = \hat{h}$. We consider the diagram

$$\begin{tikzcd} & F \times_B \hat{F} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ F \arrow[dr] & & \hat{F} \arrow[dl] \\ & B & \end{tikzcd} .$$

Let $\tilde{f} \times_B \hat{f} \colon F \times_B \hat{F} \to \mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n$ be the canonical map. We then get a T-duality triple $((F, \tilde{f}^*\mathcal{H}), (\hat{F}, \hat{f}^*\hat{\mathcal{H}}), (\tilde{f} \times_B \hat{f})^*u)$. In fact, $\mathcal{P}((\tilde{f} \times_B \hat{f})^*u))$ is satisfied by Propostion 5.1, and the two triples are intertwined by Lemma 2.5 and naturality of the Leray-Serre spectral sequence.

5.7 Consider a T^n-bundle $F \to B$ and a twist $\mathcal{H} \in T(B)$. The following follows from Corollary 4.8 and Proposition 5.1.

Corollary 5.6 *The pair $(F, \mathcal{H})$ belongs to some n-dimensional T-duality triple $((F, \mathcal{H}), (\hat{\mathcal{F}}, \hat{\mathcal{H}}), u)$ if and only if $[\mathcal{H}] \in \mathcal{F}^2 H^3(F, \mathbb{Z})$.*

In general, the corresponding T-duality triple is not unique. It is obtained by lifting $[F, [\mathcal{H}]] \in \tilde{P}_{(0)}(B)$ to some element $[B \stackrel{f}{\to} \mathbf{R}_n] \in P_n(B)$ and pulling back the universal T-duality triple $((\mathbf{F}_n, \mathcal{H}), (\hat{\mathbf{F}}_n, \hat{\mathcal{H}}), u)$ over $\mathbf{R}_n$ using f.

5.8 There are natural notions of an isomorphism between T-duality triples and of pull-back along a map of base spaces. We thus can define the set-valued functor which associates to a space the set of isomorphisms classes $\mathtt{Triple}_n(B)$ of n-dimensional T-duality triples. After some calculations we expect that $\mathbf{R}_n$ is close but not equal to a classifying space for $\mathtt{Triple}_n$.

5.9 In bunkeschickt [3, Definition 3.7] we have introduced the notion of a T-admissible twisted cohomology theory $h(\dots,\dots)$. Examples are twisted K-theory (see w2w2w 5.1) and twisted two-periodized cohomology with real coefficients. Let $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ be a T-duality triple.

trd **Definition 5.7** *We define the T-duality transformation*

$$T := \hat{p}_! \circ u^* \circ p^* : h(F,\mathcal{H}) \to h(\hat{F},\hat{\mathcal{H}}) \ .$$

This transformation has degree $-n$.

5.10

isot **Theorem 5.8** *If $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}))$ is a T-duality triple, and if h is a T-admissible twisted cohomology theory, then the T-duality transformation* trd 5.7 *is an isomorphism.*

Proof. Exactly as in bunkeschickt [3, Proof of Theorem 3.13] one uses induction, the Mayer-Vietoris exact sequence and the 5-lemma to reduce this to the case $B=*$. For $B=*$ one observes that the n-dimensional T-duality transformation is an iterated 1-dimensional T-duality transformation which is an isomorphism by the definition of T-admissibility and bunkeschickt [3, Theorem 3.13]. □

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