Various L^2-signatures and a topological L^2-signature theorem

Wolfgang Lück*
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Thomas Schick†
Fakultät für Mathematik
Universität Göttingen
Bunsenstrasse 3
37073 Göttingen
Germany

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Abstract

For a normal covering over a closed oriented topological manifold we give a proof of the L^2-signature theorem with twisted coefficients, using Lipschitz structures and the Lipschitz signature operator introduced by Teleman. We also prove that the L-theory isomorphism conjecture as well as the $C^*_{\max}$-version of the Baum-Connes conjecture imply the L^2-signature theorem for a normal covering over a Poincaré space, provided that the group of deck transformations is torsion-free.

We discuss the various possible definitions of L^2-signatures (using the signature operator, using the cap product of differential forms, using a cap product in cellular L^2-cohomology, ...) in this situation, and prove that they all coincide.

Key words: L^2-signature, signature, Lipschitz manifolds, L^2-signature theorem.
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0 Introduction

Atiyah's celebrated L^2-index theorem [2] implies that the index of the signature operator of a closed oriented smooth manifold M with Riemannian metric coincides with the L^2-index of the signature operator on any normal covering

*email: lueck@math.uni-muenster.de
www: http://www.math.uni-muenster.de/u/lueck/org/staff/lueck/
†e-mail: schick@uni-math.gwdg.de
www: http://www.uni-math.gwdg.de/schick

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space of M. In particular, the signature and the L^2-signature for closed oriented smooth manifolds coincide. The (various) definitions of L^2-signatures are explained in Section 3.

The signature is of course also defined for closed oriented topological manifolds and, as long as there is a Lipschitz structure, there is even a signature operator whose index is the signature. In the first part of this paper, we address the question how to generalize the L^2-signature theorem to closed oriented topological manifolds.

Such an L^2-signature theorem for closed oriented topological manifolds does not seem to be in the literature. We give a proof along the lines of Atiyah's proof [2] of the smooth L^2-index theorem.

0.1 Theorem. *Let M be a closed connected oriented $4n$-dimensional Lipschitz manifold with normal covering $\overline{M} \to M$. Let D_V be the Lipschitz signature operator twisted with a Lipschitz bundle V and $\overline{D_V}$ its lift to $\overline{M}$. Then*

$$\operatorname{ind}(D_V) = \operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D_V}).$$

An immediate consequence is (using Sullivan's theorem that a Lipschitz structure exists on every topological manifold of dimension $\neq 4$)

0.2 Theorem. *Let M be a closed connected oriented topological manifold of dimension $4n$ with normal covering $\overline{M} \to M$. Then*

$$\operatorname{sign}^{(2)}(\overline{M}) = \operatorname{sign}(M).$$

Theorem 0.2 also follows from the L^2-signature theorem for closed oriented smooth manifolds and the fact that the forgetful map $\Omega_*(B\Gamma) \to \Omega_*^{\mathrm{top}}(B\Gamma)$ from the smooth bordism group over $B\Gamma$ to the topological one is rationally an isomorphism (compare Remark 1.7 and the discussion after [41, Theorem 1.6]), as was pointed out to us by Shmuel Weinberger. Note that the L^2-signature theorem implies in particular that the signature is multiplicative under finite coverings. This multiplicativity was proved for closed oriented topological manifolds in [35, Theorem 8].

For more general Poincaré duality spaces, which are not manifolds, such a multiplicativity result does not hold [29, Example 22.28], [45, Corollary 5.4.1]). It fails also if M is a compact oriented smooth manifold with nonempty boundary (compare [4, Proposition 2.12] together with the Atiyah-Patodi-Singer index theorem [3, Theorem 4.14]).

This implies in particular that the L^2-index theorem can not hold in the stated form in the greatest imaginable generality. In Section 2, we discuss to which extent the L^2-signature theorem does extend to Poincaré spaces $X = (X, \emptyset)$, and show that it is implied by the L-theory isomorphism conjecture or by the $C^*_{\max}$-version of the Baum-Connes conjecture, provided the covering group Γ is torsion-free. More precisely, we prove the following theorem in 2.3:

0.3 Theorem. *Let X be a $4n$-dimensional Poincaré space over $\mathbb{Q}$ (see Definition 2.2). Let $\overline{X} \to X$ be a normal covering with* torsion-free *covering group* Γ.

Assume that the (Baum-Connes) index map for the maximal group C^*-algebra

$$\mathrm{ind}\colon K_0(B\Gamma) \to K_0(C^*_{\max}\Gamma)$$

or the L-theory assembly map

$$A\colon H_{4n}(B\Gamma; \mathbb{L}_\bullet^{\langle -\infty\rangle}) \to L_0^{\langle -\infty\rangle}(\mathbb{Z}\Gamma)$$

is rationally surjective. Then

$$\mathrm{sign}^{(2)}(\overline{X}) = \mathrm{sign}(X).$$

In a companion paper [20] we show that, without any further assumption, multiplicativity of L^2-signatures under coverings holds "approximately" in the following sense:

0.4 Theorem. *[20, Theorem 0.1] Let (X, Y) be a $4n$-dimensional Poincaré pair over $\mathbb{Q}$. Suppose that there is a nested sequence of normal subgroups of finite index $\Gamma \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ such that the intersection of the Γ_k-s is trivial. Let $(X_k, Y_k) \to (X, Y)$ be the finite covering of X associated to $\Gamma_K \subseteq \Gamma$. Then the sequence $(\mathrm{sign}(X_k, Y_k)/[\Gamma : \Gamma_k])_{k\geq 1}$ converges and*

$$\lim_{k\to\infty} \frac{\mathrm{sign}(X_k, Y_k)}{[\Gamma : \Gamma_k]} = \mathrm{sign}^{(2)}(\overline{X}, \overline{Y}).$$

(In [20], we also prove a similar approximation result for amenable exhaustions).

In the last part of the present paper, we check that the various versions of L^2-signatures, e.g. given in terms of intersection pairings, the index of the signature operator, or as trace of an index element in the K-theory of certain C^*-algebras, all coincide whenever the definitions make sense. In the rest of the paper, and also in [20], we already freely jump between the different interpretations.

This comparison is even interesting for smooth manifolds, in particular for smooth manifolds with boundary. In this case, the L^2-signature is defined in terms of the intersection pairing on L^2-homology. In Theorem 3.10 we give a proof that this coincides with the answer predicted by the L^2-index theorem [25, Theorem 1.1]. Note that we deliberately write "answer predicted by the L^2-index theorem" and not "index of the signature operator", because before adding a certain well defined correction term (compare [5]) one can not expect to obtain the signature. The paper [25] only deals with the L^2-index of certain operators. The homological interpretation does not seem to have been checked in the literature.

Organization of the paper: In Section 1 we prove the L^2-signature theorem for closed topological manifolds.

In Section 2 we address the question, for which Poincaré spaces an L^2-signature theorem holds.

In Section 3, we compare the different definitions of L^2-signatures, and show that they all coincide.

1 L^2-signature theorem for topological manifolds

We prove the L^2-signature theorem for closed oriented Lipschitz manifolds. This does prove the theorem for arbitrary oriented topological manifolds because Sullivan constructs in dimensions ≥ 5 a (unique) Lipschitz structure on every topological manifold [39], and taking the product with $\mathbb{C}P^8$ if necessary (which does change neither the signature nor the L^2-signature (compare Proposition 3.36)), we may assume that the dimension of our manifold is sufficiently high. Note that we need only the existence, but not the uniqueness of this Lipschitz structure.

Now suppose that M is a closed connected oriented Lipschitz manifold of dimension $4n$ with a Lipschitz metric g and with fundamental group Γ. Let V be a finite dimensional Lipschitz Hermitian vector bundle over M with a (not necessarily flat) Lipschitz connection. Teleman [40] constructs then a twisted signature operator D_V (whose index is the topological signature of M if V is a trivial flat line bundle). For basics about Lipschitz manifolds, Lipschitz bundles and Lipschitz operators compare [40, Section 1–6], [14, Section 2], [15, Section 1]. The Lipschitz structure, the metric, the bundle, and the signature operator all can be lifted to $\overline{M}$ and then in particular $\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D}) = \operatorname{sign}^{(2)}(\overline{M})$ (if again V is a trivial line bundle). The task is now to compare $\operatorname{ind}(D_V)$ and $\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D_V})$, which in the smooth case is done in Atiyah's L^2-index theorem [2, (1.1)].

The subscript $\mathcal{N}\Gamma$ refers to the group von Neumann algebra. Basics about $\mathcal{N}\Gamma$, Hilbert $\mathcal{N}(\Gamma)$-modules, the standard trace $\operatorname{tr}_{\mathcal{N}\Gamma}$ and the von Neumann dimension $\dim_{\mathcal{N}\Gamma}$ as used in this paper can be found e.g. in [17, Section 1 and 2], [18, Section 1.1].

On Lipschitz manifolds, no pseudo-differential calculus in the usual sense exists. However, one has the following properties of the twisted signature operator which are essential for Atiyah's proof:

1.1 Theorem. *(1) (The closure of) D_V is an unbounded selfadjoint operator. The same is true for $\overline{D_V}$.*

(2) D_V and $\overline{D_V}$ have unit propagation speed, i.e.

$$\operatorname{supp}(e^{itD_V}) \subseteq \{(x,y);\ (x,y) \in M \times M \text{ and } d(x,y) \leq t\}$$

and correspondingly for $\overline{D_V}$.

(3) $(i+D)^{-1}$ is $(\dim(M)+1)$-summable.

Proof. These results are established by Hilsum for the untwisted Lipschitz-signature operator on a complete oriented Lipschitz manifold. Hilsum uses specific properties of the untwisted operator so that his proofs can not directly be applied. We reduce the twisted case to the untwisted case in the following way: we embed V as Hermitian vector bundle in an N-dimensional trivial bundle $\underline{N}$ with complement W. Choose a connection on W. We then have on $\underline{N}$ the trivial connection and the direct sum of these two connections. Correspondingly, we get two twisted signature operators D_N (which of course is the N-fold direct sum of the untwisted signature operator) and $D_V \oplus D_W$. A calculation in local coordinates shows that

$$D_V \oplus D_W = D_N + A$$

where A is a bundle homomorphism with bounded measurable and selfadjoint coefficients, therefore is a bounded selfadjoint operator on $L^2(\Omega^*(M, \underline{N}))$ (for all this compare [40, Section 6 and 7]. In fact, in [40] this is used as the definition of the twisted signature operator). Lifting this gives the corresponding splitting

$$\overline{D_V} \oplus \overline{D_W} = \overline{D_N} + \overline{A}.$$

By [15, Corollaire 1.8] the untwisted operators D_N and $\overline{D_N}$ and then also $D_N + A$ and $\overline{D_N} + \overline{A}$ are selfadjoint, therefore the same is true for the direct summands D_V and $\overline{D_V}$.

For summability we have to find a relation between $(D_N + i)^{-1}$ (which is $(\dim(M) + 1)$-summable by [14, Proposition 5.6]) and $(D_N + A + i)^{-1}$ (these are bounded operators because of self-adjointness). We compute

$$(D_N + A + i)^{-1} = (D_N + i)^{-1}(1 + A(D_N + i)^{-1})^{-1}.$$

Since the space of $(\dim(M) + 1)$-summable operators is an ideal in the space of bounded operators, we have to show that $(1 + A(D_N + i)^{-1})^{-1}$ is bounded. We know in particular that $(D_N + i)^{-1}$ and therefore also $A(D_N + i)^{-1}$ is compact. Therefore $1 + A(D_N + i)^{-1}$ is Fredholm of index 0. Consequently, it is invertible if and only if its kernel is trivial. Now

$$\begin{aligned} (1 + A(D_N + i)^{-1})f = 0 \quad &\iff A(D_N + i)^{-1} f = -(D_N + i)(D_N + i)^{-1} f \\ &\overset{g:=(D_N+i)^{-1}f}{\iff} (A + D_N)g = -ig. \end{aligned}$$

Since $D_N + A$ is selfadjoint, its spectrum does not contain $-i$ so that $g = 0$ and therefore $\ker(1 + A(D_N + i)^{-1}) = \{0\}$. Hence $(i + D_N + A)^{-1}$ and its summand $(i + D_V)^{-1}$ are $(2m + 1)$-summable, too.

For finite propagation speed we use the proof of [15, Corollaire 1.11]. There, certain properties of the commutator $[D, h]$ with a Lipschitz function h on M are used. Observe that the bundle homomorphism A commutes with the multiplication operator h, therefore $[D_N, h] = [D_N + A, h]$ so that the proof for D_N also applies to $D_N + A$. Since D_V is a direct summand in $D_N + A$, finite propagation speed follows also for D_V. Exactly the same argument applies to $\overline{D_V}$. □

1.2 Lemma. *Let R be a bounded trace class operator on $L^2(M, E)$ for some Lipschitz bundle E. Suppose*

$$\operatorname{supp} R(s) \subseteq U_\epsilon(\operatorname{supp} s) := \{x \in M \mid d(x, \operatorname{supp}(s)) < \epsilon\}$$

for every section $s \in L^2(M, E)$, and suppose the covering $\overline{M} \to M$ and the bundle E are trivial over balls of radius 3ϵ. Then R can be canonically lifted to a bounded operator $\overline{R}$ on $L^2(\overline{M}, \overline{E})$ and $\overline{R}$ is of Γ-trace class with

$$\operatorname{tr}_{\mathcal{N}\Gamma}(\overline{R}) = \operatorname{tr}(R).$$

Proof. Decompose $M := \amalg_{i=1}^n V_i$ with measurable subset V_i each of which has diameter less than ϵ. Choose a lift $\overline{V_i}$ for each V_i. Then $\overline{M} = \bigcup_{i=1}^n \bigcup_{\gamma\in\Gamma} \gamma(\overline{V_i})$ and the union is disjoint up to sets of measure zero. Let ϕ_i^γ be the characteristic function of $\gamma(\overline{V_i})$. Every $\overline{s} \in L^2(M,E)$ is a sum $\sum \phi_i^\gamma \overline{s}$. By linearity we only have to define $\overline{R}(\phi_i^\gamma \overline{s})\ \forall i, \gamma$. We can identify the 2ϵ-neighborhood of $\gamma(\overline{V_i})$ with a corresponding neighborhood of V_i in M, and since R has only propagation ϵ, in this way $\overline{R}(\phi_i^\gamma \overline{s}) := R(\phi_i^\gamma \overline{s})$ is well defined. Since $|\overline{s}|_{L^2(\overline{M},\overline{E})} = \sum |\phi_i^\gamma \overline{s}|_{L^2(\overline{M},\overline{E})}$ and R is bounded, this makes sense also for the infinite sum $\sum \phi_i^\gamma \overline{s}$. In addition this show $\|\overline{R}\| \le \|R\|$.

Let ϕ_i be the characteristic function of V_i. Multiplication with ϕ_i is a bounded operator on $L^2(M,E)$, therefore $R\phi_i$ is of trace class for each i. For fixed i, choose a fundamental domain of the covering which contains the 2ϵ-neighborhood of $\overline{V_i}$. This induces an obvious identification

$$L^2(\overline{M},\overline{E}) \cong L^2(M,E) \otimes l^2(\Gamma).$$

Moreover, under this identification the operator $\overline{R_i} = \sum_{\gamma\in\Gamma} \overline{R}\phi_i^\gamma$ becomes $R\phi \otimes \mathrm{id}_{l^2(\Gamma)}$. By standard properties of the Γ-trace (compare e.g. [38, Theorem 2.3(6)]) $\overline{R_i}$ is of Γ-trace class and $\mathrm{tr}_{\mathcal{N}\Gamma}(\overline{R_i}) = \mathrm{tr}(R\phi_i)$ (note that $\mathrm{id}\colon l^2(\Gamma) \to l^2(\Gamma)$ is of Γ-trace class with $\mathrm{tr}_{\mathcal{N}\Gamma}(\mathrm{id}) = 1$). But $\overline{R} = \sum_{i=1}^n \overline{R_i}$ and $R = \sum_{i=1}^n R\phi_i$. By linearity, $\overline{R}$ is of Γ-trace class with

$$\mathrm{tr}_{\mathcal{N}\Gamma}(\overline{R}) = \mathrm{tr}(R). \quad \square$$

Using the properties established in Theorem 1.1 we can essentially use Atiyah's proof to show:

1.3 Theorem. *In the situation described above*

$$\mathrm{ind}_{\mathcal{N}\Gamma}(\overline{D_V}) = \mathrm{ind}(D_V).$$

In particular, $\mathrm{sign}(M) = \mathrm{sign}^{(2)}(M)$.

We proceed with an outline of the proof. For details, we refer to Atiyah's article [2]. Assume throughout that $\dim M = 4n$ is divisible by four.

D_V is an unbounded operator on $L^2\Omega^*(M,V)$, $\overline{D_V}$ is its lift to $L^2\Omega^*(\overline{M},\overline{V})$. The Hodge-$*$ operator induces a $\mathbb{Z}/2$-grading on $L^2\Omega^*$, and D_V is an odd operator with respect to this grading. What we are really interested in is the graded index of D_V, i.e. the index of D_V^+ which maps the $+1$-eigenspace of the grading operator

$$\tau := i^{p(p-1)+2n} * \qquad \text{(on } p\text{-forms)} \tag{1.4}$$

to the -1-eigenspace. Note that (using a fundamental domain) $L^2\Omega^*(\overline{M},\overline{V}) \cong L^2(\Omega^*(M,V)) \otimes l^2(\Gamma)$. The problem is that kernel and cokernel of D_V and $\overline{D_V}$ can not be related to each other using this product decomposition, because the corresponding projection operators are highly nonlocal.

First step: construct a specific almost local parametrix for D_V (the same one is already used in [24, Lemma 5]). To do this fix ϵ such that the locally trivial covering $\overline{M} \to M$ is trivial over balls of radius 3ϵ (this is possible since M is compact). Choose a function $u \in C^\infty(\mathbb{R})$ such that

(1) u is odd: $u(-x) = -u(x)\ \forall x \in \mathbb{R}$,

(2) the function $v(x) = 1 - xu(x)$ is rapidly decreasing,

(3) the Fourier transforms of u and v are compactly supported with supports contained inside the interval $(-\epsilon, \epsilon)$.

By Theorem 1.1 (1) D_V is selfadjoint. Using functional calculus, we can construct $Q = u(D_V)$ and $R = v(D_V)$ and conclude

$$D_V Q = 1 - R = Q D_V.$$

Moreover, unit propagation speed (see Theorem 1.1 (2)) implies that Q and R are supported in an ϵ-neighborhood of the diagonal, i.e. $\mathrm{supp}(Qf) \subseteq U_\epsilon(\mathrm{supp}(f))$ for any $f \in L^2\Omega^*(M)$. By Lemma 1.2 we can lift Q and R to operators $\overline{Q}$ and $\overline{R}$. Hence we lift the whole equation to

$$\overline{D_V Q} = 1 - \overline{R} = \overline{Q D_V}$$

(to check the that the domains coincide use that $\overline{D_V}$ is a closed operator).

Second step: the parametrix property. We required that v is rapidly decreasing. This implies that $(i+x)^N v(x)$ is bounded for every $N \in \mathbb{N}$ and therefore that $v(D_V) = (i + D_V)^{-2m-1}\left((i+D_V)^{2m+1} v(D_V)\right)$ is of trace class, since by Theorem 1.1 (3) $(i + D_V)^{-1}$ is $(\dim(M)+1)$-summable, therefore its $(\dim(M)+1)$st power is 1-summable, i.e. of trace class.

Now remember that D_V was anti-commuting with the grading operator τ (i.e. $\tau D_V = -D_V \tau$). Since $u(x)$ is odd the same is true for $Q = u(D_V)$ by Lemma 1.6 below. Since $v(x) = 1 - xu(x)$ is even, $R = v(D_V)$ commutes with τ. We therefore get a splitting

$$D_V^- Q^+ = 1 - R^+ = Q^- D_V^+; \qquad D_V^+ Q^- = 1 - R^- = Q^+ D_V^- \tag{1.5}$$

where $R^\pm$ is the restriction of R to the ± 1-eigenspace of τ. Since τ is a local operator, the operators $R^\pm$ are ϵ-local and their lifts are $\overline{R}^\pm$. By Lemma 1.2 $\overline{R}^\pm$ are of Γ-trace class and

$$\mathrm{tr}_{\mathcal{N}\Gamma}(\overline{R}^\pm) = \mathrm{tr}(R^\pm).$$

Step 3: Computing the index. The main point now is that all the conditions are fulfilled to apply Atiyah's principle of computing the index in terms of an arbitrary parametrix. This is formalized in [38, Proposition 2.6]: Let H_0 be the projection onto the kernel of D_V^+ and H_1 the projection onto the cokernel of D_V^+ (which is the kernel of D_V^- since D_V^- is the adjoint of D_V^+). Define $T_0 = (1-H_0)R^+(1-H_0)$ and $T_1 = (1-H_1)R^-(1-H_1)$. Multiplication of (1.5) with H_0 or H_1, respectively, yields $H_0 = R^+ H_0$ and $H_1 = H_1 R^-$. This implies

$$\mathrm{tr}(T_0) = \mathrm{tr}(R^+) - \mathrm{tr}(H_0);$$
$$\mathrm{tr}(T_1) = \mathrm{tr}(R^-) - \mathrm{tr}(H_1).$$

We want to show that $\operatorname{ind}(D_V^+) = \operatorname{tr}(H_0) - \operatorname{tr}(H_1)$ coincides with $\operatorname{tr}(R^+) - \operatorname{tr}(R^-)$. To do this, it therefore suffices to show that $\operatorname{tr}(T_0) = \operatorname{tr}(T_1)$. If H is the projection onto $\ker(D_V)$ then $D(1-H)R(1-H) = (1-H)R(1-H)D$ since all of these are functions of D. Restriction to the positive subspace yields $T_1 D_V^+ = D_V^+ T_0$. Since $\ker(D_V^+) \subset \ker(T_0)$ and $\ker(D_V^-) = \ker((D_V^+)^*) \subseteq \ker(T_1)$, $\operatorname{tr}(T_0) = \operatorname{tr}(T_1)$ is the conclusion of [38, Proposition 2.6] for the ordinary trace (where the group Γ is trivial).

Exactly the same reasoning applies on the universal covering $\overline{M}$ when computing the Γ-trace, to the effect that

$$\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D_V}^+) = \operatorname{tr}_{\mathcal{N}\Gamma}(\overline{R}^+) - \operatorname{tr}_{\mathcal{N}\Gamma}(\overline{R}^-) = \operatorname{tr}(R^+) - \operatorname{tr}(R^-) = \operatorname{ind}(D_V^+).$$

In the above proof we used:

1.6 Lemma. *Let H be a $\mathbb{Z}/2$-graded Hilbert space with grading operator τ. Let D be a selfadjoint (not necessarily bounded) odd operator on H (i.e. $\tau D = -D\tau$). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function. If f is odd or even then $f(D)$ is an odd or even operator, respectively.*

Proof. The grading operator is a unitary idempotent, i.e. $\tau = \tau^* = \tau^{-1}$. Therefore $\tau^{-1} D\tau = -D$. Uniqueness of the spectral calculus implies $\tau^{-1} f(D)\tau = f(\tau^{-1} D\tau)$ for every function f. But f even implies $f(-D) = f(D)$, and f odd implies $f(-D) = -f(D)$ which concludes the proof. □

1.7 Remark. Shmuel Weinberger pointed out to us that one can also use a bordism argument to reduce the L^2-signature theorem for closed oriented topological manifolds to Atiyah's L^2-index theorem for closed oriented smooth manifolds.

Indeed, every topological vector bundle V over a topological manifold M has a multiple which is topologically bordant to a smooth vector bundle over a smooth manifold (compare [41, Theorem 1.6 and the following discussion]).

It then remains to prove that the topological twisted signature is a bordism invariant. This is not clear from the classical proof of bordism invariance of the signature, which relies on the homological interpretation of the signature, and this is not available for twisted signature. However, Teleman [41, Theorem 1.2] proves the bordism invariance for the ordinary twisted signature, and we expect that a proof for the bordism invariance of twisted L^2-signature is possible along similar lines.

When looking at manifolds with boundary, equality of signature and L^2-signature fails as badly as possible. This follows from the fact that essentially arbitrary intersection forms can be constructed, if the boundary is non-empty. This is an easy consequence of Wall's non-simply connected generalization of Milnor's plumbing construction (compare [46, Proof of Theorem 5.8]). Since we are not aware of a reference of this fact in the literature, and since this is quite interesting a result, we prove it here in reasonable detail.

1.8 Proposition. *Fix a dimension $2k \geq 6$ and a finitely presented group π. Let X be a closed $(2k-1)$-dimensional manifold with fundamental group π and with Morse decomposition without a k-handle. Let $V \cong (\mathbb{Z}\pi)^l$ be a free finitely generated $\mathbb{Z}\pi$-module with (possible singular) $(-1)^k$-self dual map $\sigma\colon V \to V^* := \mathrm{Hom}_{\mathbb{Z}\pi}(V, \mathbb{Z}\pi)$ of the form $\sigma = \psi + (-1)^k\psi^*$ (i.e. σ has a quadratic refinement).*

Then there is a compact manifold with boundary $(W; X, Y)$ of dimension $2k$ with boundary $\partial W = X \amalg Y$ and with fundamental group π, such that the Morse chain complex $C_*(\tilde{W})$ of the universal covering $\tilde{W}$ is isomorphic to $C_*(\tilde{X}) \oplus V$, where V is considered as trivial chain complex concentrated in the middle dimension k, and with inverse Poincaré duality homomorphism

$$C_{2k-*}(\tilde{W}) \to C_{2k-*}(\tilde{W}, \partial\tilde{W}) \xrightarrow{PD^{-1}} C^*(\tilde{W})$$

which in the middle dimension is exactly σ. Here PD^{-1} is a chain homotopy inverse to the cup product with $[W, \partial W]$.

Proof. We use Wall's extension of Milnor's plumbing construction, as described in [46, Proof of Theorem 5.8].

More precisely, start with $X \times [0,1]$. Choose l disjoint embedded discs $D_i^{2k-1} \subset X$. Let $i\colon S^{k-1} \times D^k \to D^{2k-1}$ be the standard embedding. By composition we obtain r disjoint embeddings $f_i\colon S^{k-1} \times D_i^k \hookrightarrow X$. Choose lifts to the universal covering $\tilde{X}$. We now simultaneously deform the f_i to new embeddings f_i^1 using regular homotopies η_i. The η_i can be regarded as framed immersions of $S^{k-1} \times [0,1]$ to $X \times [0,1]$ (with boundary embedded). One can now count intersections and self-intersections as in [46, (5.2)] (taking the fundamental group into account using the chosen lifts). By [44, p. 247] the intersections and self-intersections can be chosen arbitrarily and independently.

Now attach k-handles to $X \times [0,1]$ with attaching maps $f_i^1 \times 1$. Let W be the resulting manifold. Evidently, $\partial W = X \amalg Y$, where Y is obtained from X by certain surgeries. Since the attaching maps are by construction homotopic to trivial embeddings, the statement about the cellular chain complex follows.

It remains to adjust the intersection form. Choose the η_i in such a way that the intersection of η_i with η_j is $\sigma(e_i)(e_j)$ where $\{e_i\}$ is the preferred bases of $(\mathbb{Z}\pi)^r$ and where we use the canonical isomorphism $V \cong V^{**}$. Moreover, choose η_i such that the self-intersection of η_i is $\psi(e_i)$. Then the intersection of η_i with itself is $\psi(e_i) + (-1)^k\psi(e_i)^*$, since our normal bundles are trivial.

A canonical basis $\{S_i\}$ for the middle degree chain complex is given by the cores of the attached handles, completed to spheres using the images of the η_i in $X \times [0,1]$ and the discs in D_i^{2k-1} spanning the images of the f_i (and with corners rounded). Then $S_i \cap S_j = \eta_i \cap \eta_j$, and the statement about the intersection form follows from the usual calculation of the Poincaré duality homomophism using intersection numbers. □

1.9 Remark. Note that we could also prove a version of Proposition 1.8 for manifolds with middle dimensional handles in a Morse decomposition, with an additional summand in the middle degree chain complex.

Observe that we use the usual translation of Poincaré duality to homology, which, because of the use of intersection numbers is more convenient to deal with in the case of smooth manifolds than the cohomological version. Proposition 1.8 implies, together with Lemma 3.25 the following corollary.

1.10 Corollary. *If, in Proposition 1.8, X has a Morse decomposition without any k-cells and $V \cong (\mathbb{Z}\pi)^l$, then the manifold W has for an arbitrary $\mathbb{Z}\pi$-module K "intersection form" for homology twisted with K*

$$H_k(W;K) = K^l \xrightarrow{\mathrm{id}_K \otimes_{\mathbb{Z}\pi}\sigma} K^l \cong H^k(W;K).$$

In particular, for π the augmentation module $\epsilon\colon \mathbb{Z}\pi \to \mathbb{R}$ we get the ordinary intersection form

$$H_k(W;\mathbb{R}) = \mathbb{R}^l \xrightarrow{\epsilon(\sigma)} \mathbb{R}^l \cong H^k(W;\mathbb{R}) = H_k(W;\mathbb{R})$$

where we use the canonical identification $H_k(W;\mathbb{R}) = H^k(W;\mathbb{R})$ coming from cellular Hodge decomposition. Note that the (ordinary) signature is the signature of this self adjoint map (i.e. the difference of the dimensions of positive and negative eigenspaces).

Similarly, if $K = l^2(\pi)$ we get the "L^2-intersection form"

$$H_k^{(2)}(W) = (l^2(\pi))^l \xrightarrow{\sigma} (l^2(\pi))^l \cong H^k_{(2)}(W) = H_k^{(2)}(W)$$

where we use the canonical identification $H_k^{(2)}(W) = H^k_{(2)}(W)$ coming from cellular Hodge decomposition. Note that the L^2-signature is the L^2-signature of this self adjoint map (i.e. the difference of L^2-dimensions of positive and negative spectral parts). Compare also (2.5) *and* (2.6) *and Section 3.4.*

Note that, if $2k-1 \geq 7$, for any finitely presented group π one can construct a closed manifold X with fundamental group π and with a CW-structure without cells in dimension k.

1.11 Theorem. *Given any non-trivial finitely presented group π and any dimension $4k \geq 8$, there is a manifold W with boundary and with fundamental group π, such that*

$$\mathrm{sign}^{(2)}(\tilde{W}) \neq \mathrm{sign}(W).$$

Proof. This follows immediately from Corollary 1.10, if we can produce appropriate (singular) intersection forms over $\mathbb{Z}\pi$. We use the fact that the signature and the L^2-signature can be computed in therms of the homology intersection form as well as the cohomological one, compare 3.25.

Any non-trivial group π contains a non-trivial cyclic group Γ. Any finitely generated free $\mathbb{Z}\Gamma$ module with a given (possibly degenerate) intersection form can be induced up to a finitely generated free $\mathbb{Z}\pi$ module with induced intersection form, and the ordinary signature as well as the L^2-signature of the induced intersection form coincides with the original ones (compare also the proof of Remark 2.7). Therefore, it suffices to treat the case π cyclic.

Using the canonical basis, we identify $(\mathbb{Z}\pi)^l$ with its dual. It suffices to consider the case $l = 1$. In the case $\pi = \mathbb{Z}$ take A to be the $(1,1)$-matrix $(1 - z)$ for $z \in \mathbb{Z}$ a generator and let $\sigma\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ be given by multiplication with $A^* + A$. Then the augmentation $\epsilon\colon \mathbb{Z}\pi \to \mathbb{R}$ gives $\epsilon(A + A^*) = 0$ and yields zero as ordinary signature. The map $A + A^*\colon l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a *positive* weak isomorphism and yields therefore the L^2-signature 1 (the spectrum is contained in $[0, \infty)$, but there is no kernel). Notice that $A + A^*$ is not invertible over $\mathbb{Z}[\mathbb{Z}]$ so that we get no contradiction to the conjecture that for torsion-free Γ the maps $\mathrm{sign}^{(2)}$ and sign defined in (2.5) and (2.6) agree.

If π is a finite cyclic group of order $p > 1$, we let $A = (1-z)$ and $\sigma\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ again be given by multiplication with $(1 - z) + (1 - z^{-1})$ where z is a generator of π. The augmentation yields the operator zero with ordinary signature 0. On the other hand, on $l^2(\pi) = \mathbb{C}\pi$ the operator $A + A^*$ is non-negative with one-dimensional kernel (it diagonalizes with eigenvalues the values of $(1 - z) + (1 - z^{-1})$ at all p-th roots of unity). Therefore its signature (over $\mathbb{C}$) is $\dim_{\mathbb{C}} \mathbb{C}\pi - 1 = p - 1$. The L^2-signature is obtained by division by $\dim_{\mathbb{C}} \mathbb{C}\pi = p$ and therefore is $1 - 1/p \neq 0$. □

2 L^2-index theorem for Poincaré spaces

In this section we want to discuss special cases where the L^2-signature theorem for closed Poincaré duality spaces is true. For finite fundamental groups, there are the counterexamples mentioned in the introduction. For torsion-free fundamental groups, however, the L^2-signature theorem follows from the $C^*_{\max}$-version of the Baum-Connes conjecture or from the L-theory isomorphism conjecture.

Recall that there are symmetric L-groups $L^n_\epsilon(R)$ and quadratic L-groups $L_n^\epsilon(R)$ for certain decorations $\epsilon = p, h, s$ and $\langle -\infty \rangle$ and that there are symmetrization maps $L_n^\epsilon(R) \to L^n_\epsilon(R)$, where in our context the ring with involution and unit R is $\mathbb{Z}\Gamma$, $\mathbb{Q}\Gamma$ or $\mathbb{C}\Gamma$ or the maximal group C^*-algebra $C^*_{\max}\Gamma$. If one inverts 2, then the decoration ϵ does not matter and the symmetrization map is bijective. If we omit the decoration, we usually think of $\epsilon = p$, i.e. the L-theory based on finitely generated projective modules. A reference for these definitions and facts is for instance [28, page 19, Section 1.10]. Note that for C^*-algebras A there is a natural isomorphism between L-theory and topological K-theory [34, Theorem 1.6]

$$L^n(A) \xrightarrow{\cong} K_n(A) \tag{2.1}$$

which will be used in the sequel without mentioning it. In dimension $n = 0$ it sends the class of a non-degenerate sesquilinear form on a finitely generated projective module P to the difference of the classes given by the positive part P_+ and by the negative part P_-.

The next definition is due to Wall [45]:

2.2 Definition. *A d-dimensional* Poincaré pair (X, Y) over $\mathbb{Q}$ *is a pair of finite CW-complexes (X, Y) such that X is connected, together with a so called* fundamental class $[X, Y] \in H_d(X, Y; \mathbb{Q})$ *such that for the universal covering and* *hence for any Γ-covering $p\colon \overline{X} \to X$ the Poincaré $\mathbb{Q}\Gamma$-chain map induced by the cap product with (a representative of) the fundamental class*

$$- \cap [X,Y]\colon C^{d-*}(\overline{X},\overline{Y};\mathbb{Q}) \to C_*(\overline{X};\mathbb{Q})$$

*is a $\mathbb{Q}\Gamma$-chain homotopy equivalence. If $Y = \emptyset$, we abbreviate $X = (X,\emptyset)$ and call it a d-*dimensional Poincaré space.

Here $C_*(\overline{X};\mathbb{Q})$ is the cellular $\mathbb{Q}\Gamma$-chain complex and $C^{d-*}(\overline{X},\overline{Y};\mathbb{Q})$ is the dual $\mathbb{Q}\Gamma$-chain complex $\hom_{\mathbb{Q}\Gamma}(C_{d-*}(\overline{X},\overline{Y};\mathbb{Q}),\mathbb{Q}\Gamma)$. Examples of Poincaré pairs are given by a compact connected topological oriented manifold X with boundary Y or merely by a connected closed oriented rational homology manifold.

2.3 Theorem. *Let X be a $4n$-dimensional Poincaré space over $\mathbb{Q}$. Let $\overline{X} \to X$ be a normal covering with* torsion-free *covering group Γ. Assume that the (Baum-Connes) index map for the maximal group C^*-algebra*

$$\mathrm{ind}\colon K_0(B\Gamma) \to K_0(C^*_{\max}\Gamma)$$

or the L-theory assembly map

$$A\colon H_{4n}(B\Gamma;\mathbb{L}_\bullet^{\langle-\infty\rangle}) \to L_0^{\langle-\infty\rangle}(\mathbb{Z}\Gamma)$$

is rationally surjective. Then

$$\mathrm{sign}^{(2)}(\overline{X}) = \mathrm{sign}(X).$$

Proof. Since X has no boundary, its symmetric signature

$$\sigma(\overline{X}) \;\in\; L^0(\mathbb{Q}\Gamma) \tag{2.4}$$

as an element in the symmetric projective L-group $L^0(\mathbb{Q}\Gamma)$ is defined (for the definitions compare e.g. [22], [27, Proposition 2.1], [28, page 26]).

The L^2-signature $\mathrm{sign}^{(2)}(\overline{X})$ is the image of $\sigma(\overline{X})$ under the canonical map

$$\mathrm{sign}^{(2)}\colon L^0(\mathbb{Q}\Gamma) \;\to\; \mathbb{R} \tag{2.5}$$

which is the composition of change of rings homomorphism $L^0(\mathbb{Q}\Gamma) \to L^0(\mathcal{N}\Gamma)$, the isomorphism $L^0(\mathcal{N}\Gamma) = K_0(\mathcal{N}\Gamma)$ and the map induced by the standard trace $\mathrm{tr}_{\mathcal{N}\Gamma}\colon K_0(\mathcal{N}\Gamma) \to \mathbb{R}$. The signature $\mathrm{sign}(X)$ is the image of $\sigma(\overline{X})$ under the canonical map

$$\mathrm{sign}\colon L^0(\mathbb{Q}\Gamma) \;\to\; \mathbb{Z} \tag{2.6}$$

which is the composition

$$L^0(\mathbb{Q}\Gamma) \to L^0(\mathbb{Q}) \to L^0(\mathbb{C}) = K_0(\mathbb{C}) = \mathbb{Z}.$$

Hence it suffices to show that the maps $\mathrm{sign}^{(2)}$ and sign defined in (2.5) and (2.6) agree.

We begin with the case where the Baum-Connes index map is assumed to be rationally surjective. By the Baum-Douglas description of K-homology, every element of $K_0(B\Gamma)$ is given by a map of a closed oriented smooth manifold $M \to B\Gamma$ and an elliptic operator D on M. Its index in $K_0(C^*_{\max}\Gamma)$ is obtained by twisting D with the pull back of the canonical $C^*_{\max}\Gamma$-bundle on $B\Gamma$. The image of this index element under the composition

$$t^{(2)}\colon K_0(C^*_{\max}\Gamma) \to K_0(\mathcal{N}\Gamma) \xrightarrow{\operatorname{tr}_{\mathcal{N}\Gamma}} \mathbb{R}$$

can be read off directly as the L^2-index in the sense of Atiyah of the operator $\overline{D}$ lifted to the Γ-covering of M which is the pull back of $E\Gamma$ via the map $M \to B\Gamma$. On the other hand, the image of this element under the composition

$$t\colon K_0(C^*_{\max}\Gamma) \to K_0(C^*_{\max}\{1\}) = K_0(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z}$$

is just the index of D. (Here we need to deal with the maximal group C^*-algebra, because the reduced group C^*-algebra is not functorial under group homomorphisms such as $\Gamma \to \{1\}$.) Atiyah's L^2-index theorem [2, (1.1)] now states that these two numbers coincide. Hence the two maps $t^{(2)}$ and t above coincide since we assume that the index map $K_0(B\Gamma) \to K_0(C^*_{\max}\Gamma)$ is rationally surjective. This implies that the maps $\operatorname{sign}^{(2)}$ and sign defined in (2.5) and (2.6) above coincide since $\operatorname{sign}^{(2)}$ and sign, respectively, are given by the composition of $t^{(2)}$ and t, respectively, with the map

$$L^0(\mathbb{Q}\Gamma) \to L^0(C^*_{\max}\Gamma) \xrightarrow{\cong} K_0(C^*_{\max}\Gamma).$$

Now suppose that the L-theoretic assembly map is rationally surjective. The symmetric signature defines for any CW-complex Y a natural homomorphism

$$\sigma\colon \Omega_*(Y) \to L^*(\mathbb{Z}\pi_1(Y)).$$

The change of ring and decoration map $L_*^{\langle -\infty\rangle}(\mathbb{Z}\pi_1(Y)) \to L_*(\mathbb{Q}\pi_1(Y))$ and the symmetrization map $L_*(\mathbb{Q}\pi_1(Y)) \to L^*(\mathbb{Q}\pi_1(Y))$ are bijective after inverting 2 [28, pages 19, 104 and 376] and [26, Proposition 8.2 or 3.3]. By the universal properties of assembly maps, $\sigma \otimes_{\mathbb{Z}} \mathbb{Q}$ can be factorized as

$$\begin{aligned}\sigma \otimes_{\mathbb{Z}} \mathbb{Q}\colon \Omega_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_*(Y; \mathbb{L}_\bullet^{\langle -\infty\rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{A \otimes_{\mathbb{Z}} \mathbb{Q}} L_*^{\langle -\infty\rangle}(\mathbb{Z}\pi_1(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}\\ \xrightarrow{\cong} L^*(\mathbb{Q}\pi_1(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}.\end{aligned}$$

where the first map is a transformation of homology theories with values in $\mathbb{Q}$-vector spaces. The first map is surjective for $Y = \{Pt.\}$. Recall that every homology theory with values in rational vector spaces which vanishes in negative degrees is a direct sum of copies of shifted ordinary homology with rational coefficients (i.e. the corresponding spectrum is a wedge of rational Eilenberg-Mac Lane spectra) (compare [7]). It follows that the first map is surjective for all Y. This could also be concluded using homological Chern characters. The second map is surjective for $Y = B\Gamma$ by assumption and the third map is always bijective. Hence

$$\sigma\colon \Omega_*(B\Gamma) \to L^*(\mathbb{Q}\Gamma)$$

is rationally surjective. This implies that rationally every element in $L^0(\mathbb{Q}\Gamma)$ is a combination of elements of the form $\sigma(\overline{M})$ for Γ-coverings $\overline{M} \to M$ with closed oriented smooth manifolds M of dimension divisible by four as basis. This follows also from the geometric interpretation of the assembly map in terms of the surgery sequence (see for instance [29, Proposition 18.3] for the topological category). For coverings $\overline{M} \to M$ as above we know already $\operatorname{sign}^{(2)}(\overline{M}) = \operatorname{sign}(M)$. Hence the maps $\operatorname{sign}^{(2)}$ and sign defined in (2.5) and (2.6) above coincide. In [47], a similar argument is used to prove homotopy invariance of ρ-invariants under the same assumptions we are making. □

The "max"-Baum-Connes conjecture used in Theorem 2.3 is true for K-amenable torsion-free groups for which the Baum-Connes conjecture is true, e.g. torsion free amenable groups or torsion-free discrete subgroups of $SU(n,1)$ or $SO(n,1)$. For more information about the Baum-Connes Conjecture see for instance [13], [23], [43].

Examples of groups for which the L-theory isomorphism conjecture is known are torsion-free poly-finite-or-cyclic groups [9], fundamental groups of closed non-positively curved manifolds [10], or knot groups [1].

2.7 Remark. We have seen in the proof of Theorem 2.3 that for a given finitely presented group Γ the L^2-index formula $\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X)$ holds for each Γ-covering $\overline{X} \to X$ with a $4n$-dimensional Poincaré space X as base if the maps $\operatorname{sign}^{(2)}$ and sign defined in (2.5) and (2.6) agree. It turns out that this is an if and only if statement. Namely, rationally any element in $L^0(\mathbb{Q}\Gamma)$ can be realized as $\sigma(\overline{X})$ by the following argument. Fix a closed manifold N of dimension $4n-1 \geq 7$ with $\pi_1(N) = \Gamma$ and $\eta \in L_0^s(\mathbb{Z}\Gamma)$. By Wall's realization theorem [46, Theorem 5.8] there is a normal map of degree one with underlying map $(f, \partial f)\colon (M, \partial M) \to (N \times [0,1], N \times \{0,1\})$ such that ∂f is a homotopy equivalence and the associated surgery obstruction is η. The symmetrization map $L_0^s(\mathbb{Z}\Gamma) \to L_s^0(\mathbb{Z}\Gamma)$ sends the surgery obstruction to the symmetric signature $\sigma(\overline{X})$ of the obvious Γ-covering of the $4n$-dimensional Poincaré space X which is obtained by glueing M and $N \times [0,1]$ together along their boundary with the homotopy equivalence ∂f [30, Proposition 6.4]. Since the composition

$$L_0^s(\mathbb{Z}\Gamma) \to L_s^0(\mathbb{Z}\Gamma) \to L^0(\mathbb{Z}\Gamma) \to L^0(\mathbb{Q}\Gamma)$$

is bijective after inverting two, the claim follows.

It is not hard to check that the maps $\operatorname{sign}^{(2)}$ and sign defined in (2.5) and (2.6) are different for Γ a finite cyclic group of prime order (see for instance [29, Example 22.28]). Since for an inclusion $i\colon \Gamma \to \Gamma'$ of groups the composition of the map $\operatorname{sign}^{(2)}$ for Γ' with the induction homomorphisms $i_*\colon L^0(\mathbb{Q}\Gamma) \to L^0(\mathbb{Q}\Gamma')$ is the map $\operatorname{sign}^{(2)}$ for Γ and similar for sign, the maps $\operatorname{sign}^{(2)}$ and sign for Γ can only agree if and only if Γ is torsion-free. In particular the conclusion in Theorem 2.3 that $\mathrm{sign}^{(2)}(\overline{X}) = \mathrm{sign}(X)$ holds for Γ-coverings $\overline{X} \to X$ over $4n$-dimensional Poincaré spaces X can only be true if Γ is torsion-free.

2.8 Question. To which extend does Theorem 2.3 hold for arbitrary torsion-free groups?

Note that a negative answer would give rise to interesting elements in the (quite mysterious and not well understood) $K_0(C^*_{max}\Gamma)$ arising as (higher) signatures for closed Poincaré duality spaces which, if the Baum-Connes conjecture for Γ is true, lie in the kernel of the map $K_0(C^*_{max}\Gamma) \to K_0(C^*_r\Gamma)$.

3 Different definitions of L^2-signatures

Throughout this section we consider a compact connected oriented $d = 4n$-dimensional Riemannian manifold M, possibly with boundary ∂M, together with a Γ-covering $\overline{M} \to M$. We denote by $\overline{\partial M}$ the preimage of ∂M. More generally, we consider a $d = 4n$-dimensional Poincaré pair (X, Y) over $\mathbb{Q}$ with a Γ-covering $(\overline{X}, \overline{X}) \to (X, Y)$. We will denote by $u\colon M \to B\Gamma$ and $u\colon X \to B\Gamma$ the classifying maps of the Γ-coverings.

We present several different ways to define the L^2-signature and show that they in fact coincide. One can use the L^2-index of the signature operator to define $\mathrm{sign}^{(2)}_{\mathrm{an}}(\overline{M})$ provided $\partial M = \emptyset$. Using K-theory and L-theory respectively one can define $\mathrm{sign}^{(2)}_K(\overline{M})$ and $\mathrm{sign}^{(2)}_L(\overline{M})$ respectively if $\partial M = \emptyset$. We will define signature pairings on L^2-de Rham cohomology, and also on combinatorial L^2-cohomology and take the von Neumann signature of these. This will yield $\mathrm{sign}^{(2)}_{\mathrm{forms}}(\overline{M}, \overline{\partial M})$ and $\mathrm{sign}^{(2)}_{\mathrm{chain}}(\overline{X}, \overline{Y})$.

3.1 Analytic L^2-signatures

3.1 Definition. *Assume $\partial M = \emptyset$. The* analytic L^2-signature *is the L^2-index (in the graded sense) of its signature operator, i.e. if $\overline{D} = d + \delta$ is the signature operator on $\overline{M}$ and if $\overline{D}^{\pm}$ is its positive/negative part with respect to the signature splitting on $L^2\Omega^*(\overline{M})$ (i.e. the restriction to the ± 1-eigenspace of $\tau = \pm *$ (compare* (1.4)*) where $*$ is the Hodge-$*$-operator) then*

$$\mathrm{sign}^{(2)}_{\mathrm{an}}(\overline{M}) := \mathrm{ind}_{\mathcal{N}\Gamma}(\overline{D}^+) := \dim_{\mathcal{N}\Gamma}(\ker \overline{D}^+) - \dim_{\mathcal{N}\Gamma}(\ker(\overline{D}^+)^*). \tag{3.2}$$

Note that $(\overline{D}^+)^* = \overline{D}^-$. This works not only for smooth Riemannian manifolds, but also for Lipschitz manifolds with Lipschitz Riemannian metrics and the corresponding Lipschitz signature operator.

If $\partial M \neq \emptyset$ one still can use the signature operator. However, one has to supply it with the non-local Atiyah-Patodi-Singer boundary conditions. Moreover, to get the signature, one has to subtract a certain correction term (corresponding to "extended L^2-solutions on the cylinder) from the index (compare [3, (4.7)–(4.14)]). To avoid this we directly *define* the analytic index as the "corrected cohomological" expression of the index formula, namely, we put in the case $\partial M \neq \emptyset$

$$\operatorname{sign}^{(2)}_{\mathrm{an}}(\overline{M}, \overline{\partial M}) \quad := \quad \int_M L(M) - \eta^{(2)}(\overline{\partial M}) + \int_{\partial M} \Pi_L(\partial M). \qquad (3.3)$$

This coincides with the above definition if $\partial M = \emptyset$, and by [25, Theorem 1.1] it also is the L^2-index of the signature operator (minus the standard correction term) if $\partial M \neq \emptyset$.

3.2 The K-theoretic L^2-signature

Suppose $\partial M = \emptyset$. Form the flat twisted von Neumann algebra bundle $\mathcal{N} := \mathcal{N}\Gamma \times_\Gamma \overline{M}$ with fiber the group von Neumann algebra $\mathcal{N}\Gamma$. Given any elliptic differential operator $D\colon C^\infty(E) \to C^\infty(F)$ of order d on M, one can twist this operator with the bundle $\mathcal{N}$ to obtain an elliptic C^*-operator on C^*-vector bundles $\mathcal{E}$, $\mathcal{F}$. An overview over this construction (for general C^*-bundles) can be found in [33, Section 1].

One can define Sobolev spaces $H^s(\mathcal{E})$ of sections of $\mathcal{E}$, and similarly for $\mathcal{F}$. These are Hilbert $\mathcal{N}\Gamma$-modules, in particular, they have an inner product with values in $\mathcal{N}\Gamma$. The twisted operator then is a bounded operator

$$D_{\mathcal{N}}\colon H^s(\mathcal{E}) \to H^{s-d}(\mathcal{F}),$$

with a parametrix $Q\colon H^{s-d}(\mathcal{F}) \to H^s(\mathcal{E})$.

Then we define

$$\operatorname{ind}_{K_0(\mathcal{N}\Gamma)}(D_{\mathcal{N}}) := [\ker(D_{\mathcal{N}} + K)] - [\operatorname{coker}(D_{\mathcal{N}} + K)] \in K_0(\mathcal{N}\Gamma),$$

where we have to perturb by a C^*-compact operator K to assure that kernel and cokernel are indeed finitely generated projective modules over $\mathcal{N}\Gamma$.

The standard trace $\operatorname{tr}_{\mathcal{N}\Gamma}$ defines (being a positive trace) a homomorphism

$$\operatorname{tr}_{\mathcal{N}\Gamma}\colon K_0(\mathcal{N}\Gamma) \to \mathbb{R}.$$

3.4 Definition. *If $\partial M = \emptyset$, we define the* K-theoretic L^2-index

$$\operatorname{ind}^{(2)}_K(D_{\mathcal{N}}) := \operatorname{tr}_{\mathcal{N}\Gamma}(\operatorname{ind}_{K_0(\mathcal{N}\Gamma)}(D_{\mathcal{N}})) \in \mathbb{R},$$

and the K-theoretic L^2-signature *as the corresponding index of the signature operator D^+:*

$$\operatorname{sign}^{(2)}_K(\overline{M}) := \operatorname{ind}^{(2)}_K(D^+_{\mathcal{N}}).$$

3.5 Theorem. *Suppose $\partial M = \emptyset$. For any elliptic differential operator D on M we have*

$$\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D}) = \operatorname{ind}^{(2)}_K(D_{\mathcal{N}}),$$

where $\overline{D}$ is the lift of D to the Γ-covering, considered as unbounded operator on L^2-sections, and

$$\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D}) := \dim_{\mathcal{N}\Gamma}(\ker(\overline{D})) - \dim_{\mathcal{N}\Gamma}(\ker(\overline{D}^*))$$

is defined as in (3.2) *for the special case of the signature operator.*

In particular we get

$$\operatorname{sign}^{(2)}_{\mathrm{an}}(\overline{M}) \;=\; \operatorname{sign}^{(2)}_K(\overline{M}).$$

A proof for this well known result can be found in [36].

For the signature and the signature operator, the only operators we are interested in here, we can actually rely on a different set of results (already discussed at length in the literature) which relate the higher signatures to surgery obstructions in L-theory groups. This is discussed in Subsection 3.5.

3.3 The de Rham L^2-signature

Now we allow from the start that $\partial M \neq \emptyset$.

Let V be a Hilbert space and let $s\colon V \times V \to \mathbb{C}$ be a sesquilinear pairing which is bounded. For us, sesquilinear also means $s(v,w) = \overline{s(w,v)}$. We can associate to it a selfadjoint bounded operator

$$A\colon V \to V \tag{3.6}$$

which is uniquely determined by the property that $s(v_1,v_2) = \langle v_1, A(v_2)\rangle$ holds for all $v_1, v_2 \in V$. From A we obtain an orthogonal splitting $V = V_- \oplus V_- \oplus V_0$ of Hilbert spaces, where V_+ is the image of $\chi_{(0,\infty)}(A)$, V_0 is the kernel of A and V_- is the image of $\chi_{(-\infty,0)}(A)$. The pairing s is non-degenerate if and only if V_0 is trivial. (One might want to require that 0 is not contained in the spectrum of A as an ever stronger version of non-degeneracy). If V is a Hilbert module over the von Neumann algebra $\mathcal{N}\Gamma$ and s is Γ-invariant, then A is Γ-equivariant and the splitting above is a splitting of Hilbert $\mathcal{N}\Gamma$-modules. The *L^2-signature* of s is in this case defined as

$$\operatorname{sign}^{(2)}(s) \;=\; \dim_{\mathcal{N}\Gamma}(V_+) - \dim_{\mathcal{N}\Gamma}(V_-). \tag{3.7}$$

The cup-product of two L^2-forms is an L^1-form. If this product form is of the dimension of the manifold, we can integrate. In this way we get a pairing

$$\langle\cdot,\cdot\rangle\colon L^2\Omega^p(\overline{M},\overline{\partial M}) \times L^2\Omega^{4n-p}(\overline{M},\overline{\partial M}) \to \mathbb{C}$$

which passes to L^2-cohomology as in the compact case. One should remark that this pairing factorizes through $\operatorname{im}(H^p_{(2)}(\overline{M},\overline{\partial M}) \to H^p_{(2)}(\overline{M}))$. The restriction of the pairing to the middle dimension

$$s_{\mathrm{forms}}\colon H^{2n}_{(2)}(\overline{M},\overline{\partial M}) \times H^{2n}_{(2)}(\overline{M},\overline{\partial M}) \to \mathbb{C} \tag{3.8}$$

is a sesquilinear, bounded and Γ-invariant pairing.

3.9 Definition. *Define the* de Rham L^2-signature

$$\operatorname{sign}^{(2)}_{\mathrm{forms}}(\overline{M},\overline{\partial M}) := \operatorname{sign}^{(2)}(s)$$

to be the L^2-signature $\operatorname{sign}^{(2)}(s_{\mathrm{forms}})$ *defined in* (3.7) *for the pairing* s_{forms} *introduced in* (3.8).

Note that this does work for Lipschitz Riemannian manifolds as well as for smooth Riemannian manifolds.

If M is closed, the pairing is non-degenerate because to any $\omega \in L^2\Omega^{2n}(\overline{M})$ we can assign $*\omega \in L^2\Omega^{2n}(\overline{M})$ and $\int_{\overline{M}} \omega \wedge *\omega > 0$ if $\omega \neq 0$. Moreover, we see that the splitting in this case is given by the ± 1-eigenspaces of $*$: $H^+ = \ker(* - 1)$ and $H^- = \ker(* + 1)$ (this makes sense if we identify the homology with the L^2-harmonic forms as can be done by Hodge theory). Moreover, the classical arguments apply to show that

$$\operatorname{ind}_{\mathcal{N}\Gamma}(\overline{D}^+) = \dim_{\mathcal{N}\Gamma}(H^+) - \dim_{\mathcal{N}\Gamma}(H^-),$$

i.e. all signatures $\operatorname{sign}_{\mathrm{an}}(\overline{M})$, $\operatorname{sign}_K(\overline{M})$ and $\operatorname{sign}_{\mathrm{forms}}(\overline{M})$, defined so far, coincide. This also works for Lipschitz manifolds (compare [40, Theorem 5.3] for the compact case).

The proof that $\operatorname{sign}^{(2)}_{\mathrm{an}}(\overline{M}, \overline{\partial M}) = \operatorname{sign}^{(2)}_{\mathrm{forms}}(\overline{M}, \overline{\partial M})$ for manifolds with boundary (which (up to the usual error term) amounts to the fact that the index of the signature operator with APS-boundary conditions in fact gives the signature) is non-trivial even in the compact case, compare [3, (2.3)] and the discussion after [3, (4.5)]. Moreover, this argument can not directly be used in the L^2-case, since it makes use e.g. of a gap near zero in the spectrum of the signature operator on ∂M. To circumvent this requires considerable effort.

3.10 Theorem. *If M is a compact connected oriented $4n$-dimensional manifold with boundary ∂M and $\overline{M} \to M$ is Γ-covering as before, then*

$$\operatorname{sign}^{(2)}_{\mathrm{an}}(\overline{M}, \overline{\partial M}) = \operatorname{sign}^{(2)}_{\mathrm{forms}}(\overline{M}, \overline{\partial M}). \tag{3.11}$$

First assume that the metric on M has a product structure near the boundary. The proof in the classical case in [3] consists of two steps. In the first step they prove that the analytical index is the signature of the Poincaré duality pairing on the L^2-harmonic forms on M_∞. Here M_∞ is M with an infinite cylinder $\partial M \times [0, \infty)$ attached to the boundary (with the product metric, which gives a smooth metric on all of M_∞ because we started with a product metric near ∂M).

We can similarly form $\overline{M}_\infty$ by attaching a cylinder to $\overline{M}$ (this is a Γ-covering of M_∞). Let $\mathcal{H}^p_{(2)}(\overline{M}_\infty)$ be the L^2-harmonic p-forms on this manifold. Vaillant [42, 5.16] proves that the L^2-signature $\operatorname{sign}^{(2)}(s_{\mathrm{forms}})$ of the intersection pairing

$$s_\infty \colon \mathcal{H}^{2n}_{(2)}(\overline{M}_\infty) \times \mathcal{H}^{2n}_{(2)}(\overline{M}_\infty) \to \mathbb{C}.$$

is $\operatorname{sign}^{(2)}_{\mathrm{an}}(\overline{M})$. This is a non-trivial fact which we don't know a short and easy proof of. The L^2-version of the first step in the treatment in [3] follows. Hence it remains to prove

$$\operatorname{sign}^{(2)}_{\mathrm{forms}}(\overline{M}) \;=\; \operatorname{sign}^{(2)}(s_\infty).$$

We do this in the following sequence of lemmas.

Remember first that we can define the L^2-homology of $\overline{M}$ as the reduced homology of the chain complex of L^2-differential forms on $\overline{M}$ (with no boundary conditions: compare [19, Section 5] or [18, Sections 1.4.2, 1.5] where a short account of different competing definitions is given).

Hence, restriction gives a map

$$r^p \colon \mathcal{H}^p(\overline{M}_\infty) \to H^p_{(2)}(\overline{M}).$$

We also have the natural map

$$i^p \colon H^p_{(2)}(\overline{M}, \overline{\partial M}) \to H^p_{(2)}(\overline{M}).$$

We will show that the closures of the image of r^p and the image of i^p coincide and that the pairings on $\mathcal{H}^{2n}_{(2)}(\overline{M}_\infty)$ and on $i^{2n}(H^{2n}_{(2)}(\overline{M}, \overline{\partial M}))$ have the same L^2-signature. Observe that the pairing on $H^{2n}_{(2)}(\overline{M}, \overline{\partial M})$ is well defined by a standard integration by parts argument, and the same argument shows that it descends to $\operatorname{im}(i^{2n} \colon H^{2n}_{(2)}(\overline{M}, \overline{\partial M}) \to H^{2n}_{(2)}(\overline{M}))$.

We first prove:

3.12 Lemma. *The image of r^p lies in the closure of the image of i^p.*

Proof. Let

$$q^p \colon H^p_{(2)}(\overline{M}) \to H^p_{(2)}(\overline{\partial M})$$

be the map given by restriction. To prove the statement, because of the long weakly exact sequence for the L^2-cohomology of the pair $(\overline{M}, \overline{\partial M})$ we only have to check that $q^p \circ r^p$ vanishes. If $\omega \in \mathcal{H}(\overline{M}_\infty)$ then by definition ω is L^2-integrable. Because of elliptic regularity, it lies in $H^\infty := \bigcap_{s\geq 0} H^s$, i.e. all derivatives are in L^2. In particular, using the continuous restriction homomorphism to codimension 1 submanifolds $H^s(\overline{M}_\infty) \to L^2(\overline{\partial M} \times \{t\})$ $(s > 1/2)$, for $t \in [0, \infty)$ the pull back map indeed gives L^2-forms on $\overline{\partial M} \times \{t\} = \overline{\partial M}$, i.e.

$$q[t]^p \colon \mathcal{H}^p(\overline{M}_\infty) \to L^2\Omega^p(\overline{\partial M}).$$

Notice that $q[0]^p = q^p \circ r^p$. The maps $q[t]^p$ are continuous, and all the manifolds $\overline{\partial M} \times [r, \infty)$ are isometric. Given a form $\omega \in \mathcal{H}^p(\overline{M}_\infty)$, the sequence of its restrictions ω_t to $\overline{\partial M} \times [t, \infty)$ tends to zero in all Sobolev norms (where we use the isometry just described to compare the different ω_t). Therefore the sequence $q[t]^p(\omega)$ in $L^2\Omega^p(\overline{\partial M})$ tends to zero as $t \to \infty$.

Now all forms $q[t]^p(\omega)$ represent the same element in the reduced L^2-homology of $\overline{\partial M}$. This is true since, on the cylinder, we can write $\omega = \omega_1(u) + \omega_2(u) \wedge du$ (if u is the cylinder variable), with $\omega_{1,2}$ L^2-functions on $[0, \infty]$ with values in $L^2\Omega^*(\overline{\partial M})$. Observe that ω is closed. Therefore

$$0 = d\omega = d\omega_1(u) \pm \frac{\partial \omega_1(u)}{\partial u} \wedge du + d\omega_2(u) \wedge du$$

Since the summands with and without du are linearly independent, from this we get

$$\pm \frac{\partial \omega_1(u)}{\partial u} = (d\omega_2(u)).$$

Integrating this equation with respect to u we get

$$\omega_1(t) - \omega_1(0) = \pm d(\int_0^t \omega_2(u)\ du).$$

But $\omega_1(t)$ is the pullback of ω to the submanifold $\overline{\partial M} \times \{t\}$, and we conclude

$$q[t]^p(\omega) - q[0]^p(\omega) = \pm d(\int_0^t \omega_2(u)\ du). \tag{3.13}$$

We consider $\omega_{1,2}$ to be L^2-functions on $[0,\infty)$ with values in the Hilbert space $L^2\Omega^*(\overline{\partial M})$. To those, we can apply the Cauchy-Schwarz inequality: the inner product of $\omega_2(u)$ and the constant function with value 1 satisfies:

$$\begin{aligned} \left|\langle \omega_2(u), 1\rangle_{L^2([0,t];L^2\Omega^*(\overline{\widetilde{M}}))}\right|^2 &= \left|\int_0^t \omega_2(u)\ du\right|^2 \\ &\leq \int_0^t 1^2\ du \cdot \int_0^t |\omega_2(u)|^2\ du \leq t\int_0^t |\omega(u)|^2\ du, \end{aligned} \tag{3.14}$$

Therefore the difference on the left hand side of Equation (3.13) is the differential of an L^2-form. Because $q[t]^p(\omega) \xrightarrow{t\to\infty} 0$ in L^2, this proves the lemma. □

From here one, we cannot continue exactly as in the classical case, because forms representing zero are not exactly boundaries, and homology sequences are only weakly exact. Instead, we use von Neumann dimensions and suitable subspaces with codimensions tending to zero.

First we address surjectivity of the restriction map

$$r^p\colon \mathcal{H}^p_{(2)}(\overline{M}_\infty) \to \overline{\operatorname{im}(i^p)} = \ker q^p \subseteq H^p_{(2)}(\overline{M}).$$

Consider the differential $d\colon \Omega^{p-1}_{(2)}(\overline{\partial M}) \to \Omega^p_{(2)}(\overline{\partial M})$. This map is unbounded and left Fredholm by elliptic regularity (compare e.g. [19, Lemma 3.3]) and hence the image of the spectral projection $\chi_{(0,\gamma)}(\delta d)$ has von Neumann dimension which tends to zero for $\gamma \to 0$. For given $\epsilon > 0$ choose $\gamma > 0$ such that the image of $\chi_{(0,\gamma)}(\delta d)$ has dimension not bigger than ϵ. Put

$$E^p_\epsilon \ := \ \operatorname{im}(d \circ \chi_{(\gamma,\infty)}(\delta d)) \subseteq \Omega^p_{(2)}(\overline{\partial M}).$$

Since $d \circ \chi_{(-\infty,0]}(\delta d)$ is zero, E^p_ϵ has codimension $\leq \epsilon$ in $\overline{\operatorname{im}(d)}$. Moreover, E^p_ϵ is closed since the restriction of δd to the relevant subspace fulfills $\delta d \geq \gamma$ and hence is invertible.

If, using the well established Hodge decomposition (compare e.g. [37, Theorem 5.10])

$$\begin{aligned} &L^2\Omega^{2n-1}(\overline{M}) = \\ &\quad \overline{(\operatorname{im}(\overline{d}^{2n-2}))} \oplus \overline{(\operatorname{im}(\overline{\delta}^{2n}|_{\{\omega;\omega|_{\overline{\partial M}=0}\}}))} \oplus \ker(\overline{\Delta}_{2n-1}|_{\{\omega;(*\omega)|_{\overline{\partial M}}=0=(\delta\omega)|_{\overline{\partial M}}\}}), \end{aligned} \tag{3.15}$$

we identify $H^p_{(2)}(\overline{M})$ with the space of harmonic forms which fulfill absolute boundary conditions, pulling back to the boundary gives a well defined bounded map $H^p_{(2)}(\overline{M}) \to \Omega^p_{(2)}(\overline{\partial M})$. Let

$$K^p_\epsilon \subseteq H^p_{(2)}(\overline{M})$$

be the inverse image of E^p_ϵ under this map. It is a closed subspace of $H^p_{(2)}(\overline{M})$ which actually is contained in $\ker(q^p)$, the inverse image of $\overline{\operatorname{im}(d(\overline{\partial M}))}$, and has codimension $\leq \epsilon$ in $\ker(q^p)$.

3.16 Lemma. *K^p_ϵ is contained in the image of $r^p\colon \mathcal{H}^p_{(2)}(\overline{M}_\infty) \to \overline{\operatorname{im}(i^p)}$.*

Proof. Let ω be a harmonic form representing an element in K^p_ϵ. Then we have to find a harmonic form $h \in \mathcal{H}^p_{(2)}(\overline{M}_\infty)$ whose restriction to $\overline{M}$ represents the cohomology class of ω. By assumption, $q^p\omega = d\alpha$ for suitable $\alpha \in \Omega^{p-1}_{(2)}(\overline{\partial M})$ in the domain of d. Note that $d\alpha$ itself is smooth by elliptic regularity since ω is harmonic. Choose a smooth function $\psi\colon [0,\infty) \to \mathbb{R}$ with $\psi(t) = 1$ in a neighborhood of 0 and with $\psi(t) = 0$ for $t > 1/2$. Define $\tilde{\alpha} = \alpha \cdot \psi(t) \in \Omega^{p-1}_{(2)}(\overline{\partial M} \times [0,\infty))$. Note that $\tilde{\alpha}$ is an L^2-form in the domain of d, which is smooth in a neighborhood of the boundary. For such forms, all usual integraion by parts formulas hold, a fact we are using frequently in the sequel and which follows e.g. from the methods of [11], or is explained in more detail in [37].

Let $Q\colon \overline{\partial M} \times \{0\} \hookrightarrow \overline{\partial M} \times [0,\infty)$ be the inclusion. Then $Q^{p-1}\tilde{\alpha} = \alpha$ and $Q^p d\tilde{\alpha} = d\alpha$. Define the L^2-form $\tilde{\omega}$ on $\overline{M}_\infty$ to coincide with ω on $\overline{M}$, and with $d\tilde{\alpha}$ on $\overline{\partial M} \times [0,\infty)$. We claim that $\tilde{\omega} \in \ker(d)$, i.e. that $\tilde{\omega}$ is orthogonal to $\delta\phi$ for all smooth ϕ with compact support. This is checked by integration by parts (on $\overline{M}$ and $\overline{\partial M} \times [0,\infty)$ separately): since ω is closed

$$\langle \tilde{\omega}|_{\overline{M}}, \delta\phi|_{\overline{M}}\rangle_{L^2(\overline{M})} = -\int_{\overline{\partial M}} d\alpha \wedge q[0]^{4n-1-p}(*\overline{\phi}),$$

on the other hand

$$\langle \tilde{\omega}|_{\overline{\partial M}\times[0,\infty)}, \delta\phi|_{\overline{\partial M}\times[0,\infty)}\rangle_{L^2(\overline{\partial M}\times[0,\infty))} = -\int_{\overline{\partial M}^-} d\alpha \wedge q[0]^{4n-1-p}(*\overline{\phi}).$$

Because of opposite inward directions the orientation of $\overline{\partial M}$ in the first and second integral are different. Changing the orientation changes the sign of the integral of a differential form. This implies the vanishing of $\langle \tilde{\omega}, \delta\phi\rangle_{L^2(\overline{M}_\infty)}$, which is just the sum of the two terms above.

By Hodge decomposition, we therefore can write $\tilde{\omega} = h + x$ where $h \in \mathcal{H}^p_{(2)}(\overline{M}_\infty)$ and x lies in the closure of the image of d. If we apply r^p to this equation, we see that the forms ω and $r^p(h)$ represent the same L^2-cohomology class in $H^p_{(2)}(\overline{M})$, which finishes the proof. $\square$

3.17 Corollary. *The map $r^p\colon \mathcal{H}^p_{(2)}(\overline{M}_\infty) \to \overline{\operatorname{im}(i^p)}$ has dense image.*

Proof. The map surjects onto subspaces of arbitrary small codimension. $\square$

Now we have to compare the intersection forms. Again we can not do this directly, but have to restrict our attention to subspaces with small codimension. Observe that $q[0]^p$ defines a map from $\mathcal{H}^p_{(2)}(\overline{M}_\infty)$ to $\overline{\operatorname{im} d(\overline{\partial M})}$. Let

$$\mathcal{H}^p_\epsilon \subseteq \mathcal{H}^p_{(2)}(\overline{M}_\infty)$$

be the inverse image of E_ϵ under this map. On the space of harmonic forms, the pull back map is bounded in the L^2-norm, therefore $\mathcal{H}^p_\epsilon$ is closed. The codimension of $\mathcal{H}^p_\epsilon$ in $\mathcal{H}^p_{(2)}(\overline{M}_\infty)$ is not bigger than ϵ.

3.18 Lemma. *Let $\omega, \eta \in \mathcal{H}^{2n}_\epsilon$, with $q[0]^{2n}\omega = d\alpha$ and $q[0]^{2n}\eta = d\beta$. Define $\tilde{\alpha}$ and $\tilde{\beta}$ as in the proof of Lemma 3.16. Assume, without loss of generality, that $\overline{M}$ has a collar of length 1 which is isometric to a product. Define $\tilde{\alpha}'$ and $\tilde{\beta}'$ as above, but with support on this collar of $\overline{M}$ (i.e. replacing the "outward" $\partial M \times [0,\infty)$ by the "inward" collar). Then $v := r^{2n}(\omega) - d\tilde{\alpha}'$ and $w := r^{2n}(\eta) - d\tilde{\beta}'$ pull back to zero on $\overline{\partial M}$ and represent the same homology classes as $r^{2n}(\omega)$ and $r^{2n}(\eta)$, respectively. Moreover,*

$$\int_{\overline{M}} v \wedge w = \int_{\overline{M}_\infty} \omega \wedge \eta. \tag{3.19}$$

Proof. We only have to prove Equation 3.19. Integration by parts shows that

$$\int_{\overline{M}} v \wedge w = \int_{\overline{M}} \omega \wedge \eta + \int_{\overline{\partial M}} \alpha \wedge d\beta, \tag{3.20}$$

since the additional terms $\int_{\overline{\partial M}} d\alpha \wedge \alpha$ and $\int_{\overline{\partial M}} d\beta \wedge \beta$ vanish as $2d\alpha \wedge \alpha = d\alpha \wedge \alpha + \alpha \wedge d\alpha = d(\alpha \wedge \alpha) = 0$ as α is of odd degree.

We therefore have to show that

$$\int_{\overline{\partial M} \times [0,\infty)} \omega \wedge \eta = \int_{\overline{\partial M}} \alpha \wedge d\beta.$$

Write $\omega - d\tilde{\alpha} = h_1 + x$ and $\eta - d\tilde{\beta} = h_2 + y$, where we restrict to $\overline{\partial M} \times [0,\infty)$ and use the Hodge decomposition for closed forms with vanishing pullback to the boundary. This implies that the harmonic forms h_1 and h_2 also fullfill $q^{2n}(h_1) = 0 = q^{2n}(h_2)$, and $x, y \in \overline{d(\operatorname{im}(d_b))}$, where d_b stands for the differential d, but with domain only the smooth compactly supported forms whose pull back to the boundary is zero. Integration by parts shows that

$$\int_{\overline{\partial M} \times [0,\infty)} (h_1 + x) \wedge (h_2 + y) = \int_{\overline{\partial M} \times [0,\infty)} h_1 \wedge h_2.$$

We can write $h_1 = a(t) + b(t) \wedge dt$ and $h_2 = c(t) + b(t) \wedge dt$, and because of the product structure the fact that h_1 is harmonic implies that the form a is harmonic and the form b (or equivalently $b \wedge dt$) is harmonic. But $0 = q^{2n} h_1 =$

$a(0)$, and a harmonic form which vanishes identically at the boundary is zero, therefore $a = 0$. In the same way, $c = 0$. This implies $h_1 \wedge h_2 = 0$ since $dt \wedge dt = 0$. Consequently

$$0 = \int_{\overline{\partial M} \times [0,\infty)} (\omega - d\tilde{\alpha}) \wedge (\eta - d\tilde{\beta}) = \int_{\overline{\partial M} \times [0,\infty)} \omega \wedge \eta - \int_{\overline{\partial M}} \alpha \wedge d\beta,$$

where the last equation follows from integration by parts (see [11] as in (3.20). This finishes the proof of Lemma 3.18. □

Let

$$L_\epsilon^{2n} \subseteq \overline{\operatorname{im}(i^p)}$$

be the closure of the image of $\mathcal{H}_\epsilon^{2n}$ under $r^p \colon \mathcal{H}_{(2)}^p(\overline{M}_\infty) \to \overline{\operatorname{im}(i^p)}$. The codimension of $L_\epsilon^{2n} \subseteq \overline{\operatorname{im}(i^p)}$ is $\leq \epsilon$ because of Corollary 3.17, since the codimension of $\mathcal{H}_\epsilon^p$ in $\mathcal{H}_{(2)}^p(\overline{M}_\infty)$ is $\leq \epsilon$. The intersection form

$$s_{\text{chain}} \colon H_{(2)}^{2n}(\overline{M}, \overline{\partial M}) \times H_{(2)}^{2n}\overline{M}, \overline{\partial M}) \to \mathbb{C}$$

descends to a pairing on $\overline{\operatorname{im}(i^p)}$ which can be restricted to a paring

$$s \colon L_\epsilon^{2n} \times L_\epsilon^{2n} \to \mathbb{C}.$$

Since the codimension of $L_\epsilon^{2n} \subseteq \overline{\operatorname{im}(i^p)}$ is $\leq \epsilon$, we get

$$|\operatorname{sign}^{(2)}(s_{\text{chain}}) - \operatorname{sign}^{(2)}(s)| \ \leq \ \epsilon.$$

Lemma 3.18 implies that the intersection form

$$s_\infty \colon \mathcal{H}_{(2)}(\overline{M}_\infty) \times \mathcal{H}_{(2)}(\overline{M}_\infty) \to \mathbb{C}$$

restricts to a pairing on $\mathcal{H}_\epsilon^{2n}$ which descents to the pairing $s \colon L_\epsilon^{2n} \times L_\epsilon^{2n} \to \mathbb{C}$ above. Since the codimension of $\mathcal{H}_\epsilon^p$ in $\mathcal{H}_{(2)}^p(\overline{M}_\infty)$ is $\leq \epsilon$ we get

$$|\operatorname{sign}^{(2)}(s_\infty) - \operatorname{sign}^{(2)}(s)| \ \leq \ \epsilon.$$

We conclude

$$|\operatorname{sign}^{(2)}(s_\infty) - \operatorname{sign}^{(2)}(s_{\text{chain}})| \leq 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\operatorname{sign}^{(2)}(s_\infty) = \operatorname{sign}^{(2)}(s_{\text{chain}}).$$

This finishes the proof of Theorem 3.10 in the case, where the Riemannian metric is a product metric near ∂M.

The argument also shows that $r^p \colon \mathcal{H}_{(2)}^p(\overline{M}_\infty) \to H_{(2)}^p(\overline{M})$ is injective. This is the case because the intersection pairing is non-degenerate on $\mathcal{H}_{(2)}^p(\overline{M}_\infty)$ (if $0 \neq h \in \mathcal{H}_{(2)}^p(\overline{M}_\infty)$ then h is not perpendicular to $*h$ where $*$ is the Hodge operator), and because on subspaces of arbitrarily small codimension this passes to the image of r^p.

The general version of Theorem 3.10 (without product metric near the boundary) now follows by observing that $H^*_{(2)}(\overline{M},\overline{\partial M})$ is unchanged if we deform the metric on M to a product metric, and that the intersection form also does only depend on the homology. We can deform the metric in such a way that the restriction to the boundary is unchanged (but of course the second fundamental form changes). If one does this, in $\mathrm{sign}^{(2)}_{\mathrm{an}}(\overline{M})$ only the local terms $\int_M L(M)$ and $\int_{\partial M} \Pi_L(\partial M)$ are changed. Exactly the same changes appear in the Atiyah-Patodi-Singer index formula for the ordinary signature on the compact manifold M. We know that the classical index formula also for manifolds without product metric near the boundary computes the signature, which does not depend on the metric. Therefore the overall changes are zero, and the same is true for $\mathrm{sign}^{(2)}_{\mathrm{an}}(\overline{M})$. Since we just argued that $\mathrm{sign}^{(2)}_{\mathrm{forms}}(\overline{M})$ does not depend on the metric on M, Theorem 3.10 follows. □

3.4 The combinatorial L^2-signature

Now we want to give a combinatorial construction of the pairing in (3.8). Assume therefore that instead of a compact connected oriented Riemannian manifold M we have a $4n$-dimensional Poincaré pair (X,Y) over $\mathbb{Q}$. Recall that the Poincaré structure is given by a fundamental class $[X,Y] \in H_{4n}(X,Y;\mathbb{Q})$ with the following property. Let the fundamental chain $[X,Y] \in C_{4n}(X,Y;\mathbb{Q})$, denoted in the same way as the fundamental class, be a closed chain representing the fundamental class. Let $(\overline{X},\overline{Y}) \to (X,Y)$ be a regular covering. Lift this closed chain $[X,Y]$ to the covering $\overline{X}$. The lift will be a closed bounded chain (without compact support) $[\overline{X},\overline{Y}] \in L^\infty C_{4n}(\overline{X},\overline{Y}) = l^\infty(\Gamma)\otimes_{\mathbb{C}\Gamma} C_{4n}(\overline{X},\overline{Y};\mathbb{C})$. There is a duality pairing $L^1C^m(\overline{X},\overline{Y};\mathbb{C}) \times L^\infty C_m(\overline{X},\overline{Y};\mathbb{C}) \to \mathbb{C}$. We call the pairing against $[\overline{X},\overline{Y}]$ "integration over $\overline{X}$". Now the cup product of one L^2-cochains with the complex conjugate of a second one on $\overline{X}$ gives an L^1-cochain which, if the dimensions are right, can be integrated over $\overline{X}$. This passes to reduced L^2-(co)homology

3.21 Definition. *Denote the induced sesquilinear Γ-invariant bounded pairing of Hilbert $\mathcal{N}\Gamma$-modules (in the middle dimension $2n$) by*

$$s_{\mathrm{chain}}\colon H^{2n}_{(2)}(\overline{X},\overline{Y}) \times H^{2n}_{(2)}(\overline{X},\overline{Y}) \to \mathbb{C}. \tag{3.22}$$

Define the combinatorial L^2-signature $\mathrm{sign}^{(2)}_{\mathrm{chain}}(\overline{X},\overline{Y})$ *to be the associated L^2-signature* $\mathrm{sign}_{\mathcal{N}\Gamma}(s_{\mathrm{chain}})$ *of* s_{chain} *as in* (3.7).

To show that this definition makes sense, recall that the definition of the cup-product involves a cellular approximation to the diagonal embedding $X \to X\times X$, which we can lift to an equivariant cellular map $\overline{X} \to \overline{X}\times\overline{X}$. This way, there is a global bound K such that the image of each cell in $\overline{X}$ under the diagonal approximation meets only K cells of $\overline{X}\times\overline{X}$. Remember that the cochain representing the cup-product of a and b maps a cell σ to a certain linear combination of $a(\sigma_1)\cdot b(\sigma_2)$ n (given locally by the diagonal approximation), where $\sigma_1\times\sigma_2$ runs through all cells in the image of σ under the cellular approximation to the diagonal. This implies in a standard way that this cup-product map is continuous from the product of the L^2-cochain spaces to the L^1-cochains.

The result of the pairing between a cochain $\sum_{\sigma\ p\text{-cell}}\lambda_\sigma\sigma$ and a chain of the form $\sum_{\sigma\ p\text{-cell}}\mu_\sigma\sigma$ is the number $\sum_{\sigma\ p\text{-cell}}\lambda_\sigma\overline{\mu_\sigma}$. This is a continuous pairing between L^1-cochains and L^∞-chains.

Taken together, we get a pairing on L^2-cochains with values in the complex numbers. If we restrict in one factor to cochains with compact support, this is the classical pairing. In particular, $\int_{\overline{X}} a\cup b=0$ if $a=\delta(a')$ and a' has compact support and $\delta(b)=0$, since this is true (in the classical situation) if a has compact support and b is completely arbitrary. We want to check the corresponding statement if a is in the closure of the image of δ in the space of L^2-cochains, and b is an L^2-cochain with $\delta(b)=0$. Now $a=\lim_{n\to\infty}\delta(a_n)$, where we can assume that all a_n have compact support, because δ is continuous and the cochains with compact support are dense in the space of L^2-cochain. But then continuity implies the claim that our pairing vanishes on (the closure of the space of) coboundaries and therefore passes to reduced L^2-cohomology. The usual proofs apply to show that the cup product (and the pairing) does not depend on the particular way we constructed it (e.g. the particular cellular approximation to the diagonal embedding).

Note that the construction is homological in nature and therefore depends only on the oriented homotopy type of the pair (X,Y). In particular it is independent of the CW-structure and the choice of the closed cycle representing the fundamental class.

An alternative description of Definition 3.21 can be given using the sequence

$$C_{(2)}^{4n-*}(\overline{X},\overline{Y})\xrightarrow{-\cap[\overline{X},\overline{Y}]}C_*^{(2)}(\overline{X})\to C_*^{(2)}(\overline{X},\overline{Y}). \tag{3.23}$$

Note that this is obtained by tensoring the corresponding $\mathbb{C}\Gamma$-chain map over $\mathbb{C}\Gamma$ with $l^2(\Gamma)$. It induces a selfadjoint bounded Γ-equivariant operator

$$A\colon H^{2n}_{(2)}(\overline{X},\overline{Y})\to H_{2n}^{(2)}(\overline{X},\overline{Y})\xrightarrow[\cong]{g}H^{2n}_{(2)}(\overline{X},\overline{Y}) \tag{3.24}$$

using the canonical identification $H_{2n}^{(2)}(\overline{X},\overline{Y})=H^{2n}_{(2)}(\overline{X},\overline{Y})$ which comes from the cellular Hodge decomposition. Actually, putting any positive inner product on $H_{2n}^{(2)}(\overline{X},\overline{Y})$ will give rise to an identification with its dual space $H^{2n}_{(2)}(\overline{X},\overline{Y})$, and the fact that the Poincaré duality homomorphism is self dual implies that after the identification the homomorphism is self adjoint (with respect to the used inner product), as can be seen by going through the definitions.

3.25 Lemma. *The homological Poincaré duality homomorphism*

$$B\colon H_{2n}^{(2)}\overline{X})\xrightarrow[\cong]{PD^{-1}}H^{2n}_{(2)}(\overline{X},\overline{Y})\xrightarrow{i^*}H^{2n}_{(2)}(\overline{X})\xrightarrow[\cong]{g^{-1}}H_{2n}^{(2)}(\overline{X})$$

has the same L^2-signature as A of (3.24)*, where PD^{-1} is the defined to be the inverse of the isomorphism induced by cup product with the fundamental class (which we abbreviate with PD in this lemma). The corresponding remark holds for the ordinary signature of (X, Y).*

For the calculation of L^2-signatures and ordinary signatures, the Poincaré duality chain map can be replaced by any chain homotopic map, and moreover, it can be "conjugated" with a chain homotopy equivalence and its adjoint.

Proof. Given any self-adjoint Hilbert $\mathcal{N}\Gamma$-module morphism $a\colon V \to W$ and a (not necessarily unitary) Hilbert $\mathcal{N}\Gamma$-isomorphism $f\colon V \to W$, we have

$$\operatorname{sign}^{(2)}(a) = \operatorname{sign}^{(2)}(faf^*).$$

This follows from the fact that the isomorphism f^* intertwines a and faf^*, i.e.

$$\langle (faf^*)x, x\rangle = \langle a(f^*x), (f^*x)\rangle \qquad \forall x \in W,$$

i.e. f^* maps the positive or negative spectral part, respectively, of faf^* to the corresponding part of a, and being an $\mathcal{N}\Gamma$-isomorophism, it preserves the $\mathcal{N}\Gamma$-dimension.

In our case,

$$\begin{aligned}\operatorname{sign}^{(2)}(B) = \operatorname{sign}^{(2)}(PD^* \circ B \circ PD) &= \operatorname{sign}^{(2)}(PD^* g^{-1} i^*)\\ &= \operatorname{sign}^{(2)}((g^{-1} \circ i_* \circ PD)^*) = \operatorname{sign}^{(2)}(A),\end{aligned}$$

since first $i^*\colon H^{2n}_{(2)}(\overline{X}, \overline{Y}) \to H^{2n}_{(2)}(\overline{X})$ is dual to $i_*\colon H^{(2)}_{2n}(\overline{X}) \to H^{(2)}_{2n}(\overline{X}, \overline{Y})$, and therefore $i^* \circ g^{-1}$ is adjoint to $g^{-1} \circ i_*$ by the usual relations between dual and adjoint on Hilbert spaces, and secondly $A^* = A$.

The statement about the chain homotopy invariance follow trivially from the fact that L^2-signature and signature depend on the homological Poincaré duality map only, which is not affected by passing to a chain homotopic map, and "conjugation" with a chain homotopy equivalence and its adjoint corresponds to "conjugation" by an isomorphism and its adjoint. We have just checked that this does not change the L^2-signature.

The identical argument applies to the ordinary signature (which can be considered as the L^2-signature for the trivial one-sheeted covering). □

The standard relations between cup- and cap-product and "integration" of homology against cohomology classes imply

3.26 Proposition. *The operator in* (3.24) *is the operator associated in* (3.6) *to the pairing appearing in Definition* (3.21)*. In particular we get*

$$\operatorname{sign}^{(2)}_{\text{chain}}(\overline{X}, \overline{Y}) = \dim_{\mathcal{N}\Gamma}(\chi_{(0,\infty)}(A)) - \dim_{\mathcal{N}\Gamma}(\chi_{(-\infty,0)}(A)). \tag{3.27}$$

Suppose (X, Y) happens to be an oriented cocompact smooth manifold with boundary, and the CW-structure is given by a smooth triangulation. Then by the L^2-de Rham isomorphism of Dodziuk [6, Theorem 1] and its version for manifolds with boundary ([37, Corollary 1.7] or [12]), L^2-simplicial and L^2-de Rham cohomology are isomorphic.

For reasons of completeness, we will prove that the pairings which give rise to $\operatorname{sign}^{(2)}_{\text{forms}}$ and $\operatorname{sign}^{(2)}_{\text{chain}}$ are compatible with respect to this isomorphism. It would perhaps be more satisfactory to prove that the isomorphism is compatible with the products. However, we don't want to discuss the L^1-version of the Hodge-de Rham theorem (and note that the product of two L^2-forms is an L^1-form), so we use this shortcut. The advantage is that we can give a "local" proof of the weaker result, which holds on the chain level. Note that, in contrast, there is no good way to describe a good cup product on the level of cochains of a simplicial complex which is at the same time graded commutative and associative, as is the case for the wedge product of differential forms. Similar and related work has e.g. been done in [21, Section 7], and his methods could be used as well. Another version would use an intermediate simplicial L^2-de Rham complex as in the treatment in [8] of multiplicativity of the ordinary de Rham isomorphism. Actually, this method is used in [12] to prove the de Rham theorem for L^2-cohomology (as well as L^p-cohomology), but without taking care of the multiplicative structure. We believe that the combination of [8] and [12] proves that the L^2-de Rham isomorphism preserves the multiplicative structure.

We choose to give a direct argument, using some calculations of [31].

To start with, we recall a possible definition of the cup product on the cochain level of a simplicial complex (using the Alexander-Whitney approximation).

So, assume X is a simplicial complex. Choose an orientation of X, i.e. an orientation of each simplex of X. Next, we choose a local ordering of the chain complex, i.e. a total ordering of the vertices of every simplex with the compatibility condition that, if a simplex σ is the face of a simplex τ, then the restriction of the ordering on the simplices of τ should give the ordering on σ. Customarily, such a local ordering is obtained by globally ordering all the vertices of the simplicial complex, but that is by no means necessary for the following cup product construction, and for us it will later be much more convenient to use local orderings.

Observe that we do not require that the ordering is compatible with the orientation (later on, we will use different local orderings, but the same orientation).

If $e_0,\dots,e_n$ are the ordered vertices of a simplex σ, then $\langle e_0,\dots,e_n\rangle := \epsilon(e_0,\dots,e_n)\sigma$ is a chain, where $\epsilon(e_0,\dots,e_n)=1$ if $(e_0,\dots,e_n)$ represents the orientation of σ, and $\epsilon(e_0,\dots,e_n)=-1$, otherwise.

Following the conventions in [32], the cup product of a p-cochain a and a q-cochain b is defined by

$$a\cup b(\langle e_0,\dots,e_n\rangle) = a(\langle e_0,\dots,e_p\rangle))\cdot b(\langle e_p,\cdots,e_n\rangle). \tag{3.28}$$

Note in particular that, if a is the elementary cochain corresponding to $\langle e_0,\dots,e_p\rangle$ (i.e. maps this simplex to one, and all other simplices to zero), and b is the elementary cochain of $\langle e_p,\dots,e_n\rangle$, then $a\cup b$ is the elementary cochain of $\langle e_0,\dots,e_n\rangle$.

The de Rham map $\int$ maps a (sufficiently smooth) p-form ω to a p-cochain of the simplicial cochain complex of a smooth triangulation of the manifold. The value of $\int(\omega)$ on a p-simplex σ simply is the integral of ω over σ. This is a chain map.

An inverse map W from the cochain complex to differential forms (going back to Whitney) is given by mapping an elementary p-cochain σ with vertices $(e_0, \dots, e_p)$ to the "barycentre form"

$$W(\sigma) := p! \sum_{i=0}^{p} (-1)^i x_i \; dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_p,$$

where the hat means, as usual, that the corresponding entry is omitted, and the x_i are defined to be the barycentric coordinates with non-zero values in the stars of the vertices e_i. The form $W(\sigma)$ is non-zero only on the open star of σ.

Dodziuk [6], compare also [12], proves that W indeed induces an isomorphism on reduced L^2-cohomology. The inverse is essentially induced by $\int$. In particular, it is easily established that $\int \circ W = \mathrm{id}$. However, since $\int$ is (below the top degree) not defined for all L^2-forms, one has to be somewhat careful here. This is the main reason why we don't prove that the de Rham isomorphism is multiplicative for L^p-cohomology (where the product of an L^p and an L^q-form is an L^r-form with $1/r = 1/p + 1/q$).

3.29 Lemma. *If c_1 and c_2 are elements of the simplicial L^2-cochain complex such that the degrees add up to $4n$, then on the $4n$-dimensional manifold X*

$$s_{\text{chain}}(c_1, c_2) = \sum_{\sigma \, oriented \; 4n \; simplex \; of \; X} (c_1 \cup \overline{c_2})(\sigma) = \int_X W(c_1 \cup \overline{c_2}),$$

where the sum is over all $4n$-simplices with orientation induced from X.

Proof. The first equality is the definition of the pairing. For the second one observe that

$$\begin{aligned} \int_X W(c_1 \cup \overline{c_2}) &= \sum_{\sigma \text{ } 4n\text{-simplex of } X} \int_\sigma W(c_1 \cup \overline{c_2}) \\ &= \sum_{\sigma \text{ } 4n\text{-simplex of } X} (\int \circ W)(c_1 \cup \overline{c_2})(\sigma) = \sum_{\sigma \text{oriented } 4n\text{-simplex of } X} (c_1 \cup \overline{c_2})(\sigma). \end{aligned}$$

We used the fact that $\int \circ W$ is identically the identity map. □

Since we already know that W induces an isomorphism, from this it suffices to check for the compatibility of the two pairings that

$$\langle W(c_1), W(c_2) \rangle := \int_X W(c_1) \wedge \overline{W(c_2)} = \int_X W(c_1 \cup \overline{c_2}) \qquad (3.30)$$

for c_1 and c_2 cochains as in Lemma 3.29, since the right hand side equals $s_{\text{chain}}(c_1, c_2)$.

We are only interested in the result on cohomology. Therefore, we can define the cup product on the cochain level appropriately. Recall, as already observed above, that many choices are possible. Our description depends e.g. on the chosen local ordering.

First, we assume that our triangulation is the barycentric subdivision of some other triangulation (if it is not yet, pass to the barycentric subdivision). There, a canonical local ordering is defined: a simplex σ of the barycentric subdivision is by definition a chain $s_0 \subset s_1 \subset \cdots \subseteq s_k$ of simplices of the original triangulation with vertices $s_0,\ldots, s_k$; and the ordering on the latter is given by inclusion, or, equivalently, by ordering according to the dimension.

In the latter description, on our $4n$-dimensional simplicial complex X we define a collection of local orderings parameterized by the symmetric group Σ_{4n+1} of permutations of $\{0,\ldots,4n\}$ with vertices s_i, s_j of the simplex σ above satisfying $s_i <_\tau s_j$ under the ordering induced by $\tau \in \Sigma_{4n+1}$ if and only if $\tau(s_i) < \tau(s_j)$.We denote the cup product induced by this local ordering by $\cup_\tau$.

The cup product to be used for Equation (3.30) is then the average of all the $\cup_\tau$:

$$c_1 \cup c_2 := \frac{1}{(4n+1)!} \sum_{\tau\in\Sigma_{4n+1}} c_1 \cup_\tau c_2.$$

We now prove Equation (3.30) with this definition of the cup product.

Proof. Let $v_1(c_1,c_2) := \int_X W(c_1)\wedge \overline{W(c_2)}$ and $v_2(c_1,c_2) := \int_X W(c_1\cup \overline{c_2})$ for simplicial L^2-cochain c_1, c_2.

Then v_1 and v_2 are sesquilinear and jointly continuous. For the latter we use the fact that W is a continuous map from L^2-cochain to L^2-forms as well as from L^1-cochain to L^1-forms (this follows from its "local" character). Moreover, the wedge as well as our cup product are continuous from L^2 to L^1 by an appropriate application of the Hölder inequality (again, the "local" definition of the cup product is used here).

The span of the elementary cochains given by the (oriented) simplices of the triangulation (defined after Equation (3.28)) is dense in the space of all L^2-cochains. Consequently, it suffices to prove that $v_1(c_1,c_2) = v_2(c_1,c_2)$ if c_1 and c_2 are two cochains corresponding to oriented simplices $\sigma_1 = (e_0,\ldots,e_p) = \langle e_0,\ldots,e_p\rangle$ or $\sigma_2 = (f_0,\ldots,f_q) = \langle f_0,\ldots,f_q\rangle$, respectively.

Let us first consider the case that σ_1 and σ_2 have no vertex in common. Then the cup product of c_1 and c_2 is zero. At the same time, the supports of $W(c_1)$ and $W(c_2)$ (being the open stars of the simplices σ_1 and σ_2) have empty intersection. In this case therefore $v_1(c_1,c_1) = 0 = v_2(c_1,c_2)$.

Secondly, assume σ_1 and σ_2 have 2 or more vertices in common. Then $W(c_1)\cup \overline{W(c_2)} = 0$ since each summand contains the square of a one-form dx_j for some barycentre function x_j. Similarly, $c_1 \cup \overline{c_2}(\sigma) = 0$ for each *non-degenerate* simplex and in particular for each (non-degenerate) $4n$-simplex, so again $v_1(c_1,c_2) = 0 = v_2(c_1,c_2)$.

Finally, for the interesting case, assume $f_0 = e_p$ is the only vertex which both simplices have in common ($f_0 = e_p$ is no real loss of generality, we could replace an oriented simplex by the negative of a simplex with the wrong orientation and the whole argument would go through). Evidently, only the case $p+q=4n$ is of interest, in which case $(e_0,\dots,e_p=f_0,f_1,\dots f_q)$ spans a $4n$-simplex. Let σ be the oriented simplex with these vertices and with orientation induced from X. Observe that σ is spanned by σ_1 and σ_2, but the orientation it gets that way differs from its orientation by $\epsilon(e_0,\dots,f_q)=:*(\sigma_1,\sigma_2)$. The latter notation is used in [31, p. 23].

The support of $W(\sigma_1)\cup\overline{W(\sigma_2)}$ is the interior of this $4n$-simplex. Therefore, its integral over any other $4n$-simplex is zero.

Moreover, $c_1\cup c_2$ vanishes on all $4n$-simplices apart from σ (as follows immediately from the formula for the cup product), and hence $W(c_1\cup c_2)=(c_1\cup c_2)(\sigma)$. It remains to compute this number. Our definition of the cup product involves one summand for each of the $(4n+1)!$ permutations of the simplices of σ. The contribution of such a permutation can only be nontrivial, when the first $p+1$ simplices $(e_0,\dots,e_p)$ are mapped to themselves and the last $q+1$-simplices $(f_0,\dots,f_q)$ are also mapped to themselves, in particular, $e_p=f_0$ has to be fixed by such a permutation. Observe that we obtain exactly $p!\cdot q!$ permutations with non-trivial contribution.

$$\begin{aligned} c_1\cup\overline{c_2}(\langle e_0,\dots,e_p=f_0,\dots,f_q\rangle) &= \\ \frac{1}{(p+q+1)!}\sum_{\pi\in\Sigma_p,\psi\in\Sigma_q} & c_1(\langle e_{\pi(0)},\dots,e_{\pi(p-1)},e_q\rangle)\overline{c_2(\langle f_0,f_{\psi(1)},\dots f_{\psi(q)}\rangle)}. \end{aligned} \tag{3.31}$$

The definition of the chain

$$\langle e_{\pi(0)},\dots,e_{\pi(p-1)},e_p\rangle$$

differs from the simplex $(e_{\pi(0)},\dots,e_{\pi(p-1)},e_p)$ by a sign which makes up for the (possible) change of orientation compared to the oriented simplex spanned by $e_0,\dots,e_p$. This implies that the value of the expression in (3.31) does not depend on the particular permutation. For our cup product, we therefore get

$$\begin{aligned} c_1\cup\overline{c_2}(\langle e_0,\dots,e_p=f_0,\dots,f_q\rangle) &= \\ \frac{p!\cdot q!}{(p+q+1)!}c_1(\langle e_0,\dots,e_p\rangle)c_2(\langle f_0,\dots,f_q\rangle) &= \frac{p!\cdot q!}{(p+q+1)!} \end{aligned}$$

by the definition of c_1 and c_2. Finally, observe that

$$c_1\cup\overline{c_2}(\sigma)=*(\sigma_1,\sigma_2)c_1\cup\overline{c_2}(\langle e_0,\dots,f_q\rangle)=*(\sigma_1,\sigma_2)\frac{p!\cdot q!}{(1+p+q)!}.$$

It remains to calculate $\int_\sigma W(c_1)\wedge\overline{W(c_2)}$. This is carried out in [31, Appendix] and we obtain indeed

$$\int_\sigma W(c_1)\wedge\overline{W(c_2)}=*(\sigma_1,\sigma_2)\frac{p!\cdot q!}{(p+q+1)!}.$$

This finishes the proof of the claim. $\square$

In particular, it follows that:

3.32 Proposition. *Assume M is a compact oriented smooth manifold with boundary ∂M. Then*

$$\operatorname{sign}^{(2)}_{\text{chain}}(\overline{M}, \overline{\partial M}) = \operatorname{sign}^{(2)}_{\text{forms}}(\overline{M}, \overline{\partial M}).$$

3.5 The L-theoretic L^2-signature

3.33 Definition. *Consider a Poincaré space X of dimension $d = 4n$ over $\mathbb{Q}$. Let $\overline{X} \to X$ be a regular Γ-covering. We have already mentioned its symmetric signature $\sigma(\overline{X}) \in L^0(\mathbb{Z}\Gamma)$ in* (2.4)*. Define its* L-theoretic L^2-signature

$$\operatorname{sign}^{(2)}_L(\overline{X}) \;\in\; \mathbb{R}$$

as the image of $\sigma(X)$ under the map $\operatorname{sign}^{(2)} \colon L^0(\mathbb{Q}\Gamma) \to \mathbb{R}$ introduced in (2.5).

3.34 Lemma. *In the situation of Definition 3.33, we have*

$$\operatorname{sign}^{(2)}_L(\overline{X}) \;=\; \operatorname{sign}^{(2)}_{\text{chain}}(\overline{X}).$$

Proof. Let $\mathcal{U}(\Gamma)$ be the algebra of operators affiliated to $\mathcal{N}\Gamma$. Algebraically $\mathcal{U}(\Gamma)$ is the Ore localization of $\mathcal{N}\Gamma$ and has the property that it is a von Neumann regular ring, i.e. any finitely generated submodule of a finitely generated projective $\mathcal{U}\Gamma$ module is a direct summand [18, Theorem 8.22]. There is a commutative square

$$\begin{array}{ccc} L^0(\mathcal{N}\Gamma) & \longrightarrow & K_0(\mathcal{N}\Gamma) \\ \big\downarrow{\cong} & & \big\downarrow{\cong} \\ L^0(\mathcal{U}\Gamma) & \longrightarrow & K_0(\mathcal{U}\Gamma) \end{array}$$

where the vertical maps are change of rings maps and isomorphisms [18, Theorem 9.31]. Since $\mathcal{U}\Gamma$ is von Neumann regular, the $\mathcal{U}\Gamma$-chain complex

$$\cdots \xrightarrow{0} H_*(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma) \xrightarrow{0} \cdots$$

given by the homology and the trivial differentials consists of finitely generated projective $\mathcal{U}\Gamma$-chain modules and there is a $\mathcal{U}\Gamma$-chain homotopy equivalence

$$i_* \colon H_*(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma) \to C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma$$

which is up to homotopy characterized by the property that it induces the identity on homology. The symmetric Poincaré structure on $C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma$ induces one on $H_*(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma)$ and i_* is an $\mathcal{U}\Gamma$-chain homotopy equivalence of symmetric $\mathcal{U}\Gamma$-Poincaré complexes. This implies for their classes in $L^0(\mathcal{U})$ [28, Proposition 1.2.1].

$$[C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma] \;=\; [H_*(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma)].$$

Elementary algebraic surgery in the sense of [28, Section 1.5] shows that the class $[H_*(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma)]$ in $L^0(\mathcal{U})$ is given by the sesquilinear non-degenerate pairing on the middle homology group $H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma)$. Let

$$\mathbf{P}H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{N}\Gamma)$$

be the projective part of the finitely generated $\mathcal{N}\Gamma$-module $H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{N}\Gamma)$ in the sense of [18, Definition 6.1]. It is a finitely generated projective $\mathcal{N}(\Gamma)$-module [18, Theorem 6.7] and inherits a sesquilinear non-degenerate pairing from the Poincaré structure. There is a canonical isomorphism

$$\big(\mathbf{P}H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{N}\Gamma)\big) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma) \xrightarrow{\cong} H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma)$$

which is compatible with the pairings (see [18, Theorem 6.7 and Lemma 8.33]). We have shown that the image of $\sigma(\overline{X})$ under the change of rings maps $L^0(\mathbb{Q}\Gamma) \to L^0(\mathcal{N}\Gamma)$ agrees with the class represented by the Poincaré pairing on

$$\mathbf{P}H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{N}\Gamma).$$

We conclude from the definitions, Proposition 3.26 and [18, Theorem 6.24] that the map

$$L^0(\mathcal{N}\Gamma) \xrightarrow{\cong} K_0(\mathcal{N}\Gamma) \to \mathbb{R}$$

sends the class represented by the Poincaré pairing on $\mathbf{P}H_{2n}(C_*(\overline{X};\mathbb{Q}) \otimes_{\mathbb{Q}\Gamma} \mathcal{N}\Gamma)$ to $\operatorname{sign}^{(2)}_{\text{chain}}(\overline{X})$. We conclude from the definition of $\operatorname{sign}^{(2)}_L(\overline{X})$ that $\operatorname{sign}^{(2)}_{\text{chain}}(\overline{X}) = \operatorname{sign}^{(2)}_L(\overline{X})$ holds. □

If X is a closed oriented smooth Riemannian manifold then, as we have seen above, the signature operator twisted with the canonical non-trivial flat $\mathcal{N}\Gamma$-bundle on X has an index in $K_0(\mathcal{N}\Gamma)$.

It is now a fundamental result, due to Mishchenko and Kasparov, that this index is equal to the element given by the symmetric signature (they are actually using the group C^*-algebra $C^*\Gamma$, but the argument for the von Neumann algebra is the same). For an extensive treatment of these facts (and a generalization to more general C^*-algebra-module bundles), compare [21]. In particular, we get the following result (see also [16, pages 728-729]).

3.35 Theorem. *Let M be a closed oriented smooth Riemannian manifold of dimension $4n$. Let $\overline{M} \to M$ be a regular Γ-covering. Then*

$$\operatorname{sign}^{(2)}_L(\overline{M}) = \operatorname{sign}^{(2)}_K(\overline{M}).$$

3.6 Künneth formula

3.36 Proposition. *The L^2-signature is multiplicative: if X and Y are two Poincaré spaces with a regular Γ_X-covering $\overline{X} \to X$ and a regular Γ_Y-covering $\overline{Y} \to Y$, then we get a regular $\Gamma_X \times \Gamma_Y$-covering $\overline{X} \times \overline{Y} \to X \times Y$ and we have*

$$\operatorname{sign}^{(2)}(\overline{X} \times \overline{Y}) = \operatorname{sign}^{(2)}(\overline{X}) \cdot \operatorname{sign}^{(2)}(\overline{Y}).$$

Proof. This follows, as in the classical compact case, in a straightforward way from the Künneth formula for L^2-cohomology. □

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