Seminar on Elliptic Boundary Problems for Dirac Operator

WS 20/21

Schedule of Talks

0. **Crash-Course Sobolev-Spaces and Operator theory (Thorsten Hertl)**
   Basic concepts of unbounded operators: domain, closed operator/closure, adjoints, symmetric versus selfadjoint, composition of operators, spectrum of an operator. Example: Differential Operator on Unit interval.

   Introduction to Sobolev Spaces for manifolds (with boundary): Introduce them for half-spaces (for all real parameter) and then to all manifolds with boundary via patching. Introduce the needed concepts from §11 (in particular the restriction operator). State, without proof or just oral proof sketch, the Rellich-Lemma (inclusions $H^{s+1}(X) \hookrightarrow H^s(X)$ is compact and the (weak) Sobolev-embedding theorem (intersection of all Sobolev spaces are yield the smooth functions).

   Crash-Course Pseudodifferential operators: Basic definition of PDO's on $\mathbb{R}^n$ and their principal symbols, Examples DO, multiplication operator, infinite smoothing. PDO on vector bundles that look locally like PDO on euclidean spaces. Elementary properties without proof:

   - for an algebra
   - symbol exact sequence
   - elliptic pseudos: existence of pseudo-inverse, inverse (if exits) is also a pseudo
   - lifting jack & Fredholm property
   - if pseudo is selfadjoint and ellitpic, then it is diagonalisable and the spectrum consists of eigenvalues only.

1. **Spinorbundles and Connections (§1 + §2)**
   Introduce Clifford algebras and their representation. Instead of proving Lemma 1.3 present one or two example(s). Extend these definitions to bundles (Definition 2.1). Lemma 2.2 is quite fundamental but prove Prop 2.5 instead.

2. **Dirac Operators and Connection Laplacian (§3 + §4)**
   Introduce Dirac operators and discuss the different types considered in this book (compatible Dirac, generalised Dirac, and Dirac type / present examples that show these classes not to be equal). Prove Proposition 3.4 (Green's formula). Introduce the Connection Laplacian (Def 4.1) and the Dirac Laplacian. Prove Prop 4.3 if time is left and

state the difference between the Dirac and the Connection Laplacian (Bochner's identity). As an important example, show how the Dirac operator decompose on the product manifold $X = Y \times [0,1]$.

3. **Chirality and UCP (§7 + §8)**
   Prove theorem 7.3. It is quite illuminating to compare the Signature operator and the ?Euler operator? but only do this if the time permit this. The proof of the UCP relies on several estimation. It is fine to represent not all of them.

4. **Invertible Doubles (§9)**
   Focus on the construction of the operator $\tilde{A}$ on the (extended) double. If you have not enough time to prove both, Lemma 9.2 and Proposition 9.3, prove the latter. The results of Prop 9.4 and 9.5 can be black-boxed, just draw the conclusion. Discuss Remark 9.6.

5. **Glueing Constructions (§10)**
   Discuss index density method of the heat-kernel (Gilkey 1996, Section 1.6 and 1.7) prove the four propositions (the strategy is always the same). Another helpful source might Bleecker-Booß, p.304ff and Gromov-Lawson p.119ff and the references therein.

6. **Sobolev Spaces and Spectral Projections (§11 + §14)**
   Recall shortly the definition of Sobolev spaces for manifolds with boundary. Prove Theorem 11.4 under the assumption of Lemma 11.5. Also the generalisation to bundles can be left to the audience. Define the operators on page 71,72.

   Prove Proposition 14.2. You also need to explain the Cauchy integral formula for projections (see Kato Section III.6.4, but not in full detail), and sketch how the full symbol is derived. If you have time left, you can explain why a selfadjoint elliptic operator has only eigenvalues in its spectrum.

7. **Calderon Projectors (§12)**
   Explain everything what appears in Theorem 12.4. You do not need to prove the theorem for the employed methods require more knowledge than one can expect from the audience. Instead prove Lemma 12.3 and Lemma 12.7. Also Lemma 12.8 may be stated without proof. Also prove Corollary 14.3.

8. **Existence of Traces (§13)**
   The aim is the proof of Theorem 13.1. Lemma 13.2 and Prop 13.3 may be stated without

proof. If time permits prove either Prop 13.5 or Theorem 13.8 but at least present both and draw the conclusion (Remark 13.10).

9. **Elliptic Boundary Problems (§18)**
   Discuss the different kinds of elliptic boundary conditions The main task is to prove Theorem 8.5 and and Corollary 18.15. You may black box the pure functional analysis statements 18.6 and 18.7. Also Lemma 18.13 and Proposition 18.14 can be stated without proof (but be careful, the proof of Corollary 18.15 requires concepts that are introduced in the proof of 18.13 or 18.14). Caution! Inequality (18.8) cannot be assumed. Instead you need to show that spec(σ_T) must be compact.

10. **Regularity of Solutions (§19)**
    The main objectives are the lifting jack (Theorem 19.6) and that the realisation A_R has closed image (Proposition 19.11). Discuss everything. If you think that you have not enough time you may omit the proof of Theorem 19.5 and Lemma 19.9.

11. **Fredholmness of (global) Realisations (p.188-200)**
    The main focus lies on the proof of Theorem 20.4, Corollary 20.5, and Theorem 20.8. You may state Lemma 20.1 but do not prove it directly. Instead, derive it from Corollary 20.5. You may state Proposition 20.7 if you have time left, but do not waste time to prove it for we do not need this statement later.

12. **Exchange of the boundary (p.201-210)**
    Prove Theorem 20.12/20.13, and Corollary 20.14. Also mention that the index is a homotopy invariant (Remark 20.15).Discuss Example 21.1 to show that indices of global boundary are in general not homotopy invariant and prove the Theorem 21.5 and the Cobordism theorem (Corollary 21.6).

13. **APS I (§22 A + B)**
    There's too much material to cover everything in detail. Important is to prove Lemma 22.2 and to make the proof strategy clear. The proof Proposition 22.4 may be skipped because the employed techniques are similar to the proof of Proposition 22.8. You do not need to prove all of Corollary 22.9 but make it transparent, why η enters the game.

14. **APS II (§22 C + D)**
    The main objective in part C is the explanation of *Duhamel's principle* and the con-

struction of the heat kernel of the Dirac operator out of the heat kernels of the auxiliary operators. Well definedness of $\mathcal{E}$ is more important than the proof of the estimation in Theorem 22.14.

Give the proof of the Atiyah-Patodi-Singer theorem presented on p.239-240 and fill in what's missing (if something is missing).

15. Applications

Literatur

[BBW93] Bernheim Booss-Bavnbek and Krzysztof Wojciechowski, *Elliptic Boundary Value Problems for Dirac Operators*, Mathematics: Theory & Applications, Birkhäuser, 1993.

[BBB13] Bernheim Booss-Bavnbek and David Bleecker, *Index Theory in Mathematics and Physics*, International Press of Boston, 2013.

[Kat76] Tosio Kato, *Perturbation Theory for Linear Operator*, Springer, 1976.

[Gil96] Peter B Gilkey, *Invariance Theory, Heat Equation, and the Atiyah-Singer Index Theorem*, Mathematics lecture series, 11, Publish or Perish Inc, 1996.

[Gro83] Mikhael and Lawson Gromov H Blaine, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publications Mathématiques de l'IHÉS **58** (1983), 83–196.