Organization of the talks: *Of Sounds and Shapes: An Introduction to Spectral Geometry*

The main reference for this seminar is the book *Spectral Theory in Riemannian Geometry* by **Olivier Lablée**, *EMS* (2015).

Talk 1

The goal of this talk is to introduce the notion of **bounded and unbounded linear operators** on Hilbert spaces. Then we will learn about the following properties about linear operator: **closeness, symmetry and self-adjointness**. Here is a detailed list of the topics which should appear in this talk:

- Definition of bounded and unbounded linear operators on a Hilbert space.
- Lax-Milgram Theorem (without proof), von Neumann Lemma.
- Definition and characterization of a closed operator.
- Definition of symmetric and self-adjoint operators, characterization of such operators.
- Examples.

References: Sections 2.(1+2+3)

Talk 2

The aim of this talk is to introduce the notions of **spectrum and resolvent** of a linear operator on an Hilbert space. Here is a detailed list of the topics which should appear in this talk:

- Definition of the spectrum, the point spectrum and the resolvent.
- Definition of the resolvent operator.
- Properties of the resolvent.
- Characterization of the spectrum of a self-adjoint operator.
- Examples.

References: Section 2.4.

Talk 3

The aim of this talk is to give a detailed account of the spectral theory for **compact operators**. Here is a detailed list of the topics which should appear in this talk:

- Definition of compact operators on a Hilbert space and their characterizations.
- The set of compact operators is an algebra.
- Riesz-Schauder theory.
- Characterization of the spectrum of a self-adjoint compact operator.

References: Section 2.5.

Talk 4

The goal of this talk is to present the spectral theory for an unbounded self-adjoint operator with compact resolvent. Here is a detailed list of the topics which should appear in this talk:

- Multiplication operators.
- Spectral Theorem (multiplication operator form).
- Definition of closable operators.
- Unbounded operators with compact resolvent and their spectrum.
- Spectral Theorem for unbounded self-adjoint operator with compact resolvent.

References: Sections 2.6.1, 2.7.(1+2).

Talk 5

The aim of this talk is to introduce the basic notions and concepts of **Differential Geometry**. Moreover, we will give the definition of Riemannian metric. Here is a detailed list of the topics which should appear in this talk:

- Definition of tangent bundle, vector fields and tangent maps.
- Definition of cotangent bundle and differential forms.
- Definition of connection.
- Definition of Riemannian metric and the Levi-Civita connection.
- Examples.

References: Sections 3.2.(1+2).

Talk 6

The goal of this talk is to give a basic introduction to **Riemannian Gemometry**. Here is a detailed list of the topics which should appear in this talk:

- Riemannian manifolds are metric spaces.
- Geodesics, injectivity radius, the exponential maps: the Hops-Rinow Theorem (without proof).
- The curvature tensor, the Ricci curvature, the scalar curvature.
- The volume form, integration, the gradient, the divergence formula.
- Examples.

References: Sections 3.2.(3+4+5).

Talk 7

In this talk consist of two part. First, the basic tools to do **analysis on manifolds** will be explained and the **Laplace operator** will be defined. In the second part we will see some classical example (motivated by physics) where the Laplacian plays a central role: the wave equation, the heat equation and the Schrödinger equation. Here is a detailed list of the topics which should appear in this talk:

- Distributions on a Riemannian manifold.
- Sobolev spaces on a Riemannian manifold and related theorems (mostly without proof).
- The Laplace operator and the Green formula.
- The wave equation on a string.
- The heat equation.
- The eigenvalue problem on closed manifolds. The Dirichlet eigenvalue problem. Neuman eignevalue problem.

References: Sections 3.2, 4.1.(1+2), 4.2.(2+3+4).

Talk 8

In this talk the **Spectral Theorem for the Laplacian** is presented and we will start to prove it. Here is a detailed list of the topics which should appear in this talk:

- The Laplacian is symmetric positive.

- Statement of the Spectral Theorem for the Laplacian.
- The spectrum of a Riemannian manifold.
- Variational generic abstract eigenvalue problems.

References: Sections 4.3, 4.4.1.

Talk 9

The goal of this talk is to complete the proof of the Spectral Theorem in the closed, Dirichlet and Neumann cases. Here is a detailed list of the topics which should appear in this talk:

- The eigenvalue problem on closed manifolds.
- The Dirichlet eigenvalue problem.
- The Neumann eigenvalue problem.

References: Sections 4.4.(2+3+4).

Talk 10

The topic of this talk is the **Minmax principle** and some applications. Here is a detailed list of the topics which should appear in this talk:

- Definition of the Rayleigh quotient.
- Eigenvalues as Rayleigh quotients and related results.
- The Minmax principle.
- Properties of the first eigenvalue, Cheeger inequality.

References: Sections 4.5.(1+2+3). Perhaps an additional rference on Cheeger inequality.

Talk 11

This talk consists of two different parts. The first one is about the **Monotonicity domain principle**. The second part is about the calculation of the spectrum in some simple examples. Here is a detailed list of the topics which should appear in this talk:

- The monotonicity domain principle.
- A result about perturbed metrics.
- Explicit calculation of the spectrum for: flat tori, rectangular domains with boundary conditions, spheres, (maybe projective spaces and others).

References: Sections 4.5.(4+5), 5.1.(1+2+3).

Talk 12

The aim of this talk is to present some qualitative properties of the spectrum (like asymptotics: what is the approximate number of eigenvalues below E for large E?). In particular we will learn about the Lichnerowicz-Obata Theorem, the Cheeger inequality, the Weyl formula. Here is a detailed list of the topics for this talk:

- The Lichnerowicz-Obata Theorem.
- The Cheeger inequality, Li-Yau, Faber-Krahn, Szegö-Weinberger (all of them without proof).
- The Weyl formula (proof for the flat tori).
- The spectral partition function and its relation with the spectrum.
- Spectrum of surfaces and Courant's nodal Theorem (without proof).

References: Section 5.2.(1+3), 5.3, 5.4.

Talk 13

The main topic of this talk is **Milnor's counterexample**, which provides the example of two manifolds which are isospectral but not isometric. Here is a detailed list of the topics which should appear in this talk:

- Lenght spectrum and trace formulas (just an overview).
- Milnor's counterexample.
- Prescribing the spectrum on a manifold: Colin de Vérdier's Theorem (sketch).

References: Section 7.(1+2+3+4).

Talk 14

The goal of this talk is to introduce the **heat kernel** of a Riemannian manifold, study its asymptotic expansion and its trace. We deduce as corollary the Minakshisundaram-Pleijel formula and the Weyl formula in its general form. Here is a detailed list of the topics which should appear in this talk:

- The heat equation on a Riemannian manifold.
- Definition of the heat kernel of a Riemannian manifold.
- Asymptotic expansion of the heat kernel (sketch or complete proof).
- The trace of the heat kernel is the spectral partition function.
- the Minakshisundaram-Pleijel formula and the Weyl formula.

References: Section 7.(5+6).