Description of "'Algebraic Topology III"'

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Organisational matters

This course is the continuation of Algebraic Topology I-II of 2016-17.

There are the lectures (twice per week, at the moment scheduled Tue, Fr 12:15–13:55 in Sitzungszimmer, MI.

In addition, there is a tutorial. We will have the usual weekly homework assigment, and the exercises will be discussed there. Assistent for the class (teaching the tutorial, grading) is **Simone Cecchini**, simone.cecchini@uni-goettingen.de. For both of us, office hours are by appointment, or by just knocking at the door.

Communication will be mainly via the stud.ip entry for the class, please register there! I'll try to create (as in previous semesters) course notes.

Literature

A like more or less (and will use)

- Davis and Kirk: Lecture Notes in Algebraic Topology, AMS
- Hatcher: Algebraic topology; on Hatcher's homepage
- Hatcher: Vector bundles and K-theory (unfinished); on Hatcher's homepage
- Kochman: Bordism, stable homotopy and Adams spectral sequence, AMS
- Rudyak: On Thom spectra, orientability, and cobordism; Springer
- Switzer: Algebraic topology—homotopy and homology; Springer
- Whitehead: Elements of homotopy theory; Springer

Here, I recommend in particular Davis and Kirk as well as Hatcher's books.

Prerequisites

- a solid knowledge of homology and cohomology (chain complexes, long exact sequences, cohomology as ring,...),
- CW-complexes, cellular (co)homology
- basics of fundamental group, higher homotopy groups
- basics of fibrations and fiber bundles

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Course content

In this course, we will develop systematically generalized (co)homology theory from a geometric point of view.

We will focus on a number of important examples, in particular on

- bordism theories
- K-theory
- ordinary cohomology

We will also develop more of the homotopy theoretic background. This includes

- more about fibrations and fiber bundles
- more on higher homotopy groups
- classifying spaces
- spectra (of homotopy theory); a generalization of spaces tailor-made to discuss generalized homology theories and transformations between them

We will start (and spend quite a bit of time) with a particularly geometric concept, the concept of bordism.

A quick definition:

0.1 Definition. The n-th (unoriented) bordism group Ω_n^O is defined as the set of diffeomorphism classes of compact manifolds M without boundary modulo the equivalence relation of *bordism*:

Two manifolds M_1, M_2 are *bordant* if there is a compact manifold with boundary W and a diffeomorphism between ∂W and the disjoint union $M_1 \amalg M_2$.

The group operation is defined to be disjoint union.

More generally, if X is a topological space, we define $\Omega_n^O(X)$ as the set of bordism classes of pairs (M, f) where M is a compact n-dimensional manifold without boundary and $f\colon M \to X$ is a continuous map. Again, (M_1, f_1) and (M_2, f_2) are *bordant* if there is a compact manifold W with boundary and with a map $F\colon W \to X$ such that there is a diffeomorphism $\phi\colon \partial W \to M_1 \amalg M_2$ which in addition has to be compatible with the reference maps to X, i.e. $f_1 \amalg f_2 \circ \phi = F|_{\partial W}$: "the restriction of F to the boundary of W coincides with the given maps f_1, f_2.

Again, disjoint union defines a group structure on $\Omega_n^O(X)$.

Given a map $u\colon X \to Y$ we define the induced map

$$\Omega_n^O(u)\colon \Omega_n^O(X) \to \Omega_n^O(Y); [M \xrightarrow{f} X] \mapsto [M \xrightarrow{u\circ f} Y].$$

This way, we get a functor $\Omega_n^O\colon TOP \to ABEL$.

0.2 Remark. The superscript O stands for the *orthogonal groups* $O(n)$ which contain arbitrary, also orientation reversing linear automorphisms, in parallel with the fact that we don't require orientations for M and W.

Obviously, given Definition 0.1 quite a bit of work lies ahead: show the statements about group and functor. We promised to get a homology theory, so we have to extend to pairs, and then establish all the axioms of a homology theory. And then there should be tools to do computations. It turns out that the geometric definition, as appealing as it is, allows for direct computations only in extremely low degrees (essentially $0, 1, 2$, and to some extend 3), where we know a priori quite a lot about manifolds.

The emphasis on the *unoriented* and the decoration O in Ω^O shows already that we should expect other variants of the bordism concept, in particular an oriented version (but there are others with similar importance), and we will develop this in quite some detail.

We will follow mainly Davis-Kirk.

(1) Theory of classifying spaces e.g. of (unoriented or oriented) tangent bundles

(2) theory of stable (structure) group G refining $O(n)$, i.e. with compatible maps $G_n \to O(n)$

(3) G-structure as lift of classifying map for stable normal bundle

(4) construction of Thom spectrum MG, as sequence of Thom spaces $T(V_n)$, where $V_n \to BG_n$ is the universal G_n-vector bundle (pulled back fro mthe universal $O(n)$-vector bundle over $BO(n)$)

(5) definition of G-bordism $\Omega_*^G(X)$

(6) Pontryagin-Thom construction, giving the two inverse to each other maps in the isomorphism
$$\pi_n(X_+ \wedge MG) \cong \Omega_n^G(X).$$

(7) first hints on the calculation of $\pi_n(MG)$ for $G = SO$:

   (a) Thom spaces $MSO(n)$ are simply connected for $n > 3$

   (b) Hurewicz homomorphism $\pi_*(MSO(n)) \to H_*(MSO(n))$ is a rational isomorphism upto degree $2k$ if $H_j(MSO(n)) = 0$ for $j \leq k$

   (c) Thom isomorphism $H_j(MSO(n)) \cong H_{j-n}(BSO(n))$

The main tool to describe, understand and analyze generalized cohomology theories (and to construct new ones out of old ones) is the (homotopy) category of spectra, already alluded to above. "Spectra" are generalizations of spaces. The main idea is that suspension of spectra is defined and is invertible: every spectrum is (at least upto homotopy) the suspension of another spectrum.

When dealing with spaces, there are quite a number of stability phenomena, like the Freudenthal suspension isomorphism, which implies that $\pi_k(\Sigma^n X) \cong \pi_{k+1}(\Sigma^{n+1} X)$ if $k < 2n$ (this means that this isomorphism works for sufficiently high suspensions, depending on k). In the world of spectra, the corresponding phenomena work on the nose and in general; this is the reason why one often speaks of the "stable (homotopy) category".