Homomorphisms of quantum groups

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I bought a new car
Outline

1. Multiplicative unitaries
2. Locally compact quantum groups
3. Hopf *-homomorphisms
4. Equivalent pictures of homomorphisms of quantum groups
   - Bicharacters
   - Universal bicharacter
   - Right or left coactions as homomorphisms
   - Morphism as a functor between coaction categories
5. Summary
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Multiplicative unitary

**Definition**

An operator $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be multiplicative unitary on the Hilbert space $\mathcal{H}$ if it satisfies the *pentagon equation*

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

**Examples**

Consider $\mathcal{H}_G = L^2(G, \lambda)$ for a locally compact group $G$ with a right Haar measure $\lambda$. Then, $W_G \in \mathcal{U}(L^2(G \times G, \lambda \times \lambda))$ defined by $W_G T(x, y) = T(xy, y)$ is a multiplicative unitary on $\mathcal{H}_G$. 

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Observations

One can define two non-degenerate, normal, coassociative $\ast$-homomorphisms from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$:

$$\Delta(x) = W(x \otimes I)W^*$$
$$\hat{\Delta}(y) = \text{Ad}(\Sigma) \circ (W^*(I \otimes y)W).$$

for all $x, y \in \mathcal{B}(\mathcal{H})$ and $\Sigma$ is the flip operator acting on $\mathcal{H} \otimes \mathcal{H}$.

Consider the slices/legs of the multiplicative unitaries:

$$\mathcal{C} = \{(\omega \otimes \text{id})W : \omega \in \mathcal{B}(\mathcal{H})_*\}||\cdot||$$
$$\hat{\mathcal{C}} = \{(\text{id} \otimes \omega)W : \omega \in \mathcal{B}(\mathcal{H})_*\}||\cdot||.$$
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$$C = \{(\omega \otimes \text{id})\mathcal{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}$$
$$\hat{C} = \{(\text{id} \otimes \omega)\mathcal{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$
Special class of multiplicative unitaries

Manageability and modularity

- Manageable multiplicative unitary. [Woronowicz, 1997]
- Modular multiplicative unitary. [Sołtan-Woronowicz, 2001]
Nice legs of modular multiplicative unitaries

**Theorem (Sołtan, Woronowicz, 2001)**

Let, $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. Then,

- $C$ and $\hat{C}$ are $C^*$-sub algebras in $\mathcal{B}(\mathcal{H})$ and $W \in \mathcal{UM}(\hat{C} \otimes C)$.
- there exists a unique $\Delta_C \in \text{Mor}(C, C \otimes C)$ such that
  - $(\text{id}_{\hat{C}} \otimes \Delta)W = W_{12}W_{13}$.
  - $\Delta_C$ is coassociative: $(\Delta_C \otimes \text{id}_C) \circ \Delta_C = (\text{id}_C \otimes \Delta_C) \circ \Delta_C$.
  - $\Delta(C)(1 \otimes C)$ and $(C \otimes 1)\Delta(C)$ are linearly dense in $C \otimes C$.
- There exists an involutive normal antiautomorphism $R_C$ of $C$. 
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Locally compact quantum groups

Definition [Sołtan-Woronowicz, 2001]

The pair $\mathbb{G} = (C, \Delta_C)$ is said to be a locally compact quantum group if the C*-algebra $C$ and $\Delta_C \in \text{Mor}(C, C \otimes C)$ comes from a modular multiplicative unitary $\mathbb{W}$. We say $\mathbb{W}$ giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$.

Observation

The unitary operator $\hat{\mathbb{W}} = \text{Ad}(\Sigma)(\mathbb{W}^*)$ gives rise to the quantum group $\hat{\mathbb{G}} = (\hat{C}, \Delta_{\hat{C}})$ which is dual to $\mathbb{G} = (C, \Delta_C)$. 
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From groups to quantum groups

Given a locally compact group $G$:

- $\mathbb{G} = (C_0(G), \Delta)$ is a locally compact quantum group with $\Delta f(x, y) = f(xy)$.
- $\hat{\mathbb{G}} = (C^*_r(G), \hat{\Delta})$ is the dual quantum group of $\mathbb{G}$ with $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in G$.
- $\hat{\mathbb{G}}^u = (C^*(G), \hat{\Delta}^u)$ is a C*-bialgebra which is known as quantum group C*-algebra of $\mathbb{G}$.
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- $\hat{\mathbb{G}}^u = (C^*(G), \hat{\Delta}^u)$ is a C*-bialgebra which is known as quantum group C*-algebra of $\mathbb{G}$. 
Let, $W$ be a modular multiplicative unitary giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$. We write:

- $W$, when we consider it as an unitary operator action on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$
- $W$, when we consider it as in element of $\mathcal{UM}(\hat{C} \otimes C)$.
- $f: A \to B$, when we consider $f \in \text{Mor}(A, B)$ or $f: A \to \mathcal{M}(B)$ where $A$ and $B$ are $C^*$-algebras.
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Let us consider $G = (C, \Delta_C)$ and $H = (A, \Delta)$ be two C*-bialgebras.

**Definition**

A *Hopf *-homomorphism* between them is a morphism $f: C \to A$ that intertwines the comultiplications, that is, the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow{\Delta_C} & & \downarrow{\Delta_A} \\
C \otimes C & \xrightarrow{f \otimes f} & A \otimes A.
\end{array}
\]
Let $G$ and $H$ are two locally compact groups.

- Consider a Hopf $\ast$ homomorphism from $f : C_0(H) \rightarrow C_0(G)$.
- $f$ induces a continuous group homomorphism $\phi : G \rightarrow H$.
- $\phi$ induces a Hopf $\ast$-homomorphism $\hat{f} : C^*_r(G) \rightarrow C^*_r(H)$ if and only if kernel of $\phi$ is amenable.

**Conclusion**

Hopf $\ast$-homomorphisms are not compatible with the duality. But, $\phi$ induces a Hopf $\ast$ morphism $\hat{f}^u : C^*(G) \rightarrow C^*(H)$. 
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- Consider a Hopf $^*$ homomorphism from $f: C_0(H) \to C_0(G)$.
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Let, $\mathbb{G} = (C, \Delta_C)$ and $\mathbb{H} = (A, \Delta_A)$ are two quantum groups.

**Definition**

A unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ is called a *bicharacter from $C$ to $A$* if

\[
(\Delta_{\hat{C}} \otimes \text{id}_A)V = V_{23}V_{13} \quad \text{in } \mathcal{UM}(\hat{C} \otimes \hat{C} \otimes A),
\]

\[
(\text{id}_{\hat{C}} \otimes \Delta_A)V = V_{12}V_{13} \quad \text{in } \mathcal{UM}(\hat{C} \otimes A \otimes A).
\]
Lemma

A unitary \( V \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A) \) comes from a bicharacter \( V \in \mathcal{UM}(\hat{\mathcal{C}} \otimes A) \) (which is necessarily unique) if and only if

\[
\begin{align*}
V_{23} W_{12}^C &= W_{12}^C V_{13} V_{23} & \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_A), \\
W_{23}^A V_{12} &= V_{12} V_{13} W_{23}^A & \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_A).
\end{align*}
\]
An important theorem

Theorem [Woronowicz, 2010]

Let $\mathcal{H}$ be a Hilbert space and let $W \in B(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. If $a, b \in B(\mathcal{H})$ satisfy $W(a \otimes 1) = (1 \otimes b)W$, then $a = b = \lambda 1$ for some $\lambda \in \mathbb{C}$. More generally, if $a, b \in M(K(\mathcal{H}) \otimes D)$ for some C*-algebra $D$ satisfy $W_{12}a_{13} = b_{23}W_{12}$, then $a = b \in \mathbb{C} \cdot 1_{\mathcal{H}} \otimes M(D)$. 
Corollary

Let \((C, \Delta_C)\) be a quantum group. If \(c \in \mathcal{M}(C)\), then
\[
\Delta_C(c) \in \mathcal{M}(C \otimes 1) \quad \text{or} \quad \Delta_C(c) \in \mathcal{M}(1 \otimes C)
\]
if and only if \(c \in \mathbb{C} \cdot 1\).

More generally, if \(D\) is a \(C^*\)-algebra and \(c \in \mathcal{M}(C \otimes D)\), then
\[
(\Delta_C \otimes \text{id}_D)(c) \in \mathcal{M}(C \otimes 1 \otimes D) \quad \text{or}
\]
\[
(\Delta_C \otimes \text{id}_D)(c) \in \mathcal{M}(1 \otimes C \otimes D)
\]
if and only if \(c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)\).
Properties of bicharacters I

Consider \( G = (C, \Delta_C) \), \( H = (A, \Delta_A) \) and \( I = (B, \Delta_B) \) are quantum groups.

- Given a bicharacter \( V \in \mathcal{UM}(\hat{C} \otimes A) \) we have:
  - \((R_{\hat{C}} \otimes R_A)V = V\).
  - \(\hat{V} = \sigma(V^*) \in \mathcal{UM}(A \otimes \hat{C})\) is the dual bicharacter.
Properties of bicharacters I

Consider \( \mathbb{G} = (C, \Delta_C), \; \mathbb{H} = (A, \Delta_A) \) and \( \mathbb{I} = (B, \Delta_B) \) are quantum groups.

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  - \( \hat{V} = \sigma(V^*) \in \mathcal{UM}(A \otimes \hat{C}) \) is the dual bicharacter.
Given two bicharacters $V^{C\to A} \in UM(\hat{C} \otimes A)$ and $V^{A\to B} \in UM(\hat{A} \otimes B)$, there exists unique bicharacter $V^{C\to B} \in UM(\hat{C} \otimes B)$ satisfying

$$V^{C\to B}_{13} = (V^{C\to A}_{12} \ast V^{A\to B}_{23}) \ast V^{C\to A}_{12} (V^{A\to B}_{23})^*.$$

We denote $V^{C\to B} = V^{A\to B} \ast V^{C\to A}$ as composition of $V^{C\to A}$ and $V^{A\to B}$.

Identity bicharacter:

$$V^{C\to A} = V^{C\to A} \ast W^{C}$$
and

$$V^{C\to A} = W^{A} \ast V^{C\to A}.$$
Properties of bicharacters II

- Given two bicharacters \( V^{C \rightarrow A} \in \mathcal{UM} (\hat{C} \otimes A) \) and \( V^{A \rightarrow B} \in \mathcal{UM} (\hat{A} \otimes B) \), there exists unique bicharacter \( V^{C \rightarrow B} \in \mathcal{UM} (\hat{C} \otimes B) \) satisfying

\[
V^{C \rightarrow B}_{13} = (V^{C \rightarrow A})^* V^{A \rightarrow B} V^{C \rightarrow A} (V^{A \rightarrow B})^* .
\]

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V^{C \rightarrow A} = V^{C \rightarrow A} \ast W^C \quad \text{and} \quad V^{C \rightarrow A} = W^A \ast V^{C \rightarrow A} .
\]
Properties of bicharacters III

- Composition of bicharacters is associative:

\[(V^{B \rightarrow D} \ast V^{A \rightarrow B}) \ast V^{C \rightarrow A} = V^{B \rightarrow D} \ast (V^{A \rightarrow B} \ast V^{C \rightarrow A}).\]

where \(V^{B \rightarrow D} \in \mathcal{UM}(\hat{B} \otimes D)\) where \(\mathbb{J} = (D, \Delta_D)\) is a quantum group.

- Compatibility with duality:

\[
\sqrt{V_{13}^{C \rightarrow B}} = \sqrt{V_{12}^{A \rightarrow B}} \ast \sqrt{V_{23}^{C \rightarrow A}} \ast \sqrt{V_{12}^{A \rightarrow B}} \ast \sqrt{V_{23}^{C \rightarrow A}}.
\]
Properties of bicharacters III

- Composition of bicharacters is associative:
  \[(\mathcal{V}^{B \rightarrow D} \circledast \mathcal{V}^{A \rightarrow B}) \circledast \mathcal{V}^{C \rightarrow A} = \mathcal{V}^{B \rightarrow D} \circledast (\mathcal{V}^{A \rightarrow B} \circledast \mathcal{V}^{C \rightarrow A}).\]

  where \(\mathcal{V}^{B \rightarrow D} \in \mathcal{U}\mathcal{M}(\hat{B} \otimes D)\) where \(\mathbb{J} = (D, \Delta_D)\) is a quantum group.

- Compatibility with duality:
  \[\mathcal{V}_{13}^{C \rightarrow B} = \mathcal{V}_{12}^{A \rightarrow B} \circledast \mathcal{V}_{23}^{C \rightarrow A} \circledast \mathcal{V}_{12}^{A \rightarrow B} \circledast \mathcal{V}_{23}^{C \rightarrow A}.\]
Category of locally compact quantum groups

Proposition [Ng, 1997; Meyer, R., Woronowicz, 2011]

The composition of bicharacters is associative, and the multiplicative unitary $W_C$ is an identity on $C$. Thus bicharacters with the above composition and locally compact quantum groups are the arrows and objects of a category. Duality is a contravariant functor acting on this category.
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Corepresentation and universal bialgebra of a quantum group

**Definition**

A corepresentation of $(\hat{C}, \Delta_{\hat{C}})$ on a C*-algebra $D$ is a unitary multiplier $V \in UM(\hat{C} \otimes D)$ that satisfies

$$(\Delta_{\hat{C}} \otimes \text{id}_D)(V) = V_{23} V_{13}.$$

**Remark**

Similarly corepresentation of $(C, \Delta_C)$ on a C*-algebra $D$ is a unitary multiplier $V \in UM(D \otimes C)$ that satisfies

$$(\text{id}_D \otimes \Delta_C)(V) = V_{12} V_{13}.$$
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Similarly corepresentation of \((C, \Delta_C)\) on a C*-algebra \(D\) is a unitary multiplier \(V \in \mathcal{UM}(D \otimes C)\) that satisfies

\[(\text{id}_D \otimes \Delta_C)(V) = V_{12} V_{13}.\]
Proposition [Sołtan, Woronowicz, 2007]

- There exists a maximal corepresentation $\tilde{V} \in U\mathcal{M}(\hat{C}^u \otimes C)$ of $(C, \Delta_C)$ on a C*-algebra $\hat{C}^u$ such that for any corepresentation $U \in U\mathcal{M}(D \otimes C)$ there exists a unique $\hat{\phi} \in \text{Mor}(\hat{C}^u, D)$ such that $(\hat{\phi} \otimes \text{id}_C)\tilde{V} = U$.

- There exists a unique $\Delta_{\hat{C}^u} \in \text{Mor}(\hat{C}^u, \hat{C}^u \otimes \hat{C}^u)$ such that:
  - $(\Delta_{\hat{C}^u} \otimes \text{id}_C)\tilde{V} = \tilde{V}_{23} \tilde{V}_{13}$
  - $\Delta_{\hat{C}^u}(\hat{C}^u)(1 \otimes \hat{C}^u)$ and $(\hat{C}^u \otimes 1)\Delta_{\hat{C}^u}$ are linearly dense in $(\hat{C}^u \otimes \hat{C}^u)$.
Proposition [Sołtan, Woronowicz, 2007]

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There exists a unique $\Delta_{\hat{C}^u} \in Mor(\hat{C}^u, \hat{C}^u \otimes \hat{C}^u)$ such that:

- $(\Delta_{\hat{C}^u} \otimes id_C)\tilde{V} = \tilde{V}_{23} \tilde{V}_{13}$
- $\Delta_{\hat{C}^u}(\hat{C}^u)(1 \otimes \hat{C}^u)$ and $(\hat{C}^u \otimes 1)\Delta_{\hat{C}^u}$ are linearly dense in $(\hat{C}^u \otimes \hat{C}^u)$. 
Universal $C^*$-bialgebras associated to a quantum group

**Universal quantum groups $C^*$-algebra**

$(\hat{C}^u, \Delta_{\hat{C}^u})$ is known as *quantum group $C^*$-algebra* or the *universal dual* of $(C, \Delta)$.

**Corollary**

There exists a maximal corepresentation $\mathcal{V} \in \mathcal{U}(\hat{C} \otimes C^u)$ of $(\hat{C}, \Delta_{\hat{C}})$ and $C^*$-bialgebra $(C^u, \Delta_{C^u})$. 
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There exists a maximal corepresentation $\mathcal{V} \in \mathcal{U}(\hat{C} \otimes C^u)$ of $(\hat{C}, \Delta_{\hat{C}})$ and $C^*$-bialgebra $(C^u, \Delta_{C^u})$. 
Reducing morphisms

There exists two Hopf $*$-homomorphisms $\Lambda \in \text{Mor}(C^u, C)$ and $\hat{\Lambda} \in \text{Mor}(\hat{C}^u, \hat{C})$ such that

\[
\begin{align*}
\begin{array}{ccc}
C^u & \xrightarrow{\Lambda} & C \\
\Delta C^u & \\ & \downarrow \\
C^u \otimes C^u & \xrightarrow{\Lambda \otimes \Lambda} & C \otimes C.
\end{array}
\end{align*}
\begin{align*}
\begin{array}{ccc}
\hat{C}^u & \xrightarrow{\hat{\Lambda}} & \hat{C} \\
\Delta \hat{C}^u & \\ & \downarrow \\
\hat{C}^u \otimes \hat{C}^u & \xrightarrow{\hat{\Lambda} \otimes \hat{\Lambda}} & \hat{C} \otimes \hat{C}.
\end{array}
\end{align*}
\]
Results

- Let \((A, \Delta_A)\) be a C*-bialgebra. Bicharacters in \(\mathcal{UM}(\hat{C} \otimes A)\) correspond bijectively to Hopf \(^*\)-homomorphisms from \((C^u, \Delta_{C^u})\) to \((A, \Delta_A)\).

- There is a unique bicharacter \(\chi \in \mathcal{UM}(\hat{C}^u \otimes C^u)\) such that

\[
\mathcal{V}_{23} \tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12} \chi_{13} \mathcal{V}_{23} \quad \text{in} \quad \mathcal{UM}(\hat{C}^u \otimes \mathcal{K}(\mathcal{H}_C) \otimes C^u).
\]

Moreover, \(\chi\) is universal in the following sense:
\[(\text{id}_{\hat{C}^u} \otimes \Lambda) \chi = \tilde{\mathcal{V}}, \quad (\hat{\Lambda} \otimes \text{id}_{C^u}) \chi = \mathcal{V} \quad \text{and} \quad (\hat{\Lambda} \otimes \Lambda) \chi = W.\]

- A bicharacter in \(\mathcal{UM}(\hat{C} \otimes A)\) lifts uniquely to a bicharacter in \(\mathcal{UM}(\hat{C}^u \otimes A^u)\) and hence to bicharacters in \(\mathcal{UM}(\hat{C} \otimes A^u)\) and \(\mathcal{UM}(\hat{C}^u \otimes A)\).
Results

Let \((A, \Delta_A)\) be a C*-bialgebra. Bicharacters in \(\mathcal{UM}(\hat{C} \otimes A)\) correspond bijectively to Hopf *-homomorphisms from \((C^u, \Delta_{C^u})\) to \((A, \Delta_A)\).

There is a unique bicharacter \(\mathcal{X} \in \mathcal{UM}(\hat{C}^u \otimes C^u)\) such that

\[ V_{23} \tilde{V}_{12} = \tilde{V}_{12} \mathcal{X}_{13} V_{23} \quad \text{in} \quad \mathcal{UM}(\hat{C}^u \otimes \mathbb{K}(\mathcal{H}_C) \otimes C^u). \]

Moreover, \(\mathcal{X}\) is universal in the following sense:

\((\text{id}_{\hat{C}^u} \otimes \Lambda)\mathcal{X} = \tilde{V}, (\hat{\Lambda} \otimes \text{id}_{C^u})\mathcal{X} = V \) and \((\hat{\Lambda} \otimes \Lambda)\mathcal{X} = W.\)

A bicharacter in \(\mathcal{UM}(\hat{C} \otimes A)\) lifts uniquely to a bicharacter in \(\mathcal{UM}(\hat{C}^u \otimes A^u)\) and hence to bicharacters in \(\mathcal{UM}(\hat{C} \otimes A^u)\) and \(\mathcal{UM}(\hat{C}^u \otimes A).\)
Let \((A, \Delta_A)\) be a C*-bialgebra. Bicharacters in \(\mathcal{UM}(\hat{C} \otimes A)\) correspond bijectively to Hopf *-homomorphisms from \((C^u, \Delta_{C^u})\) to \((A, \Delta_A)\).

There is a unique bicharacter \(\mathcal{X} \in \mathcal{UM}(\hat{C}^u \otimes C^u)\) such that

\[
\tilde{V}_{23} \tilde{V}_{12} = \tilde{V}_{12} \mathcal{X}_{13} \tilde{V}_{23} \quad \text{in} \quad \mathcal{UM}(\hat{C}^u \otimes K(\mathcal{H}_C) \otimes C^u). \]

Moreover, \(\mathcal{X}\) is universal in the following sense:
\((\text{id}_{\hat{C}^u} \otimes \Lambda) \mathcal{X} = \tilde{V}, (\hat{\Lambda} \otimes \text{id}_{C^u}) \mathcal{X} = V\) and \((\hat{\Lambda} \otimes \Lambda) \mathcal{X} = W\).

A bicharacter in \(\mathcal{UM}(\hat{C} \otimes A)\) lifts uniquely to a bicharacter in \(\mathcal{UM}(\hat{C}^u \otimes A^u)\) and hence to bicharacters in \(\mathcal{UM}(\hat{C} \otimes A^u)\) and \(\mathcal{UM}(\hat{C}^u \otimes A)\).
Let \((A, \Delta_A)\) be a C*-bialgebra. Bicharacters in \(\mathcal{UM}(\hat{C} \otimes A)\) correspond bijectively to Hopf *-homomorphisms from \((C^u, \Delta_{C^u})\) to \((A, \Delta_A)\).

There is a unique bicharacter \(\mathcal{X} \in \mathcal{UM}(\hat{C}^u \otimes C^u)\) such that
\[
\mathcal{V}_{23}\tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12}\mathcal{X}_{13}\mathcal{V}_{23} \quad \text{in} \quad \mathcal{UM}(\hat{C}^u \otimes \mathbb{K}(\mathcal{H}_C) \otimes C^u).
\]

Moreover, \(\mathcal{X}\) is universal in the following sense:
\((\text{id}_{\hat{C}^u} \otimes \Lambda)\mathcal{X} = \tilde{\mathcal{V}}, (\hat{\Lambda} \otimes \text{id}_{C^u})\mathcal{X} = \mathcal{V}\) and \((\hat{\Lambda} \otimes \Lambda)\mathcal{X} = \mathcal{W}\).

A bicharacter in \(\mathcal{UM}(\hat{C} \otimes A)\) lifts uniquely to a bicharacter in \(\mathcal{UM}(\hat{C}^u \otimes A^u)\) and hence to bicharacters in \(\mathcal{UM}(\hat{C} \otimes A^u)\) and \(\mathcal{UM}(\hat{C}^u \otimes A)\).
Theorem [Ng, 1997; Meyer, R., Woronowicz, 2011]

There is an isomorphism between the categories of locally compact quantum groups with bicharacters from $C$ to $A$ and with Hopf $^*$-homomorphisms $C^u \to A^u$ as morphisms $C \to A$, respectively. The bicharacter associated to a Hopf $^*$-homomorphism $\varphi: C^u \to A^u$ is $(\Lambda_{\hat{C}} \otimes \Lambda_A \varphi)(\mathcal{X}^C) \in UM(\hat{C} \otimes A)$.

Furthermore, the duality on the level of bicharacters corresponds to the duality $\varphi \mapsto \hat{\varphi}$ on Hopf $^*$-homomorphisms, where $\hat{\varphi}: \hat{A}^u \to \hat{C}^u$ is the unique Hopf $^*$-homomorphism with $(\hat{\varphi} \otimes \text{id}_{A^u})(\mathcal{X}^A) = (\text{id}_{\hat{C}^u} \otimes \varphi)(\mathcal{X}^C)$. 
There is an isomorphism between the categories of locally compact quantum groups with bicharacters from $C$ to $A$ and with Hopf $^*$-homomorphisms $C^u \to A^u$ as morphisms $C \to A$, respectively. The bicharacter associated to a Hopf $^*$-homomorphism $\varphi: C^u \to A^u$ is $(\Lambda_{\hat{C}} \otimes \Lambda_A \varphi)(\mathcal{X}^C) \in UM(\hat{C} \otimes A)$. Furthermore, the duality on the level of bicharacters corresponds to the duality $\varphi \mapsto \hat{\varphi}$ on Hopf $^*$-homomorphisms, where $\hat{\varphi}: \hat{A}^u \to \hat{C}^u$ is the unique Hopf $^*$-homomorphism with $(\hat{\varphi} \otimes id_{A^u})(\mathcal{X}^A) = (id_{\hat{C}^u} \otimes \varphi)(\mathcal{X}^C)$. 

Theorem [Ng, 1997; Meyer, R., Woronowicz, 2011]
1. Multiplicative unitaries

2. Locally compact quantum groups

3. Hopf *-homomorphisms

4. Equivalent pictures of homomorphisms of quantum groups
   - Bicharacters
   - Universal bicharacter
   - Right or left coactions as homomorphisms
   - Morphism as a functor between coaction categories

5. Summary
Definition

A right or left coaction of \((A, \Delta_A)\) on a C*-algebra \(C\) is a morphism \(\alpha_R : C \rightarrow C \otimes A\) or \(\alpha_L : C \rightarrow A \otimes C\) for which following diagram in the left or the right hand side commutes:

\[
\begin{align*}
C & \xrightarrow{\alpha_R} C \otimes A \\
\downarrow{\alpha_R} & \downarrow{id_C \otimes \Delta_A} \\
C \otimes A & \xrightarrow{\alpha_R \otimes \text{id}_A} C \otimes A \otimes A,
\end{align*}
\]

\[
\begin{align*}
C & \xrightarrow{\alpha_L} A \otimes C \\
\downarrow{\alpha_L} & \downarrow{\Delta_A \otimes \text{id}_C} \\
A \otimes C & \xrightarrow{id_A \otimes \alpha_L} A \otimes A \otimes C.
\end{align*}
\]
Right quantum group homomorphisms

**Definition**

A right quantum group homomorphism from \((C, \Delta_C)\) to \((A, \Delta_A)\) is a morphism \(\Delta_R : C \to C \otimes A\) for which following two diagrams commute:

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_R} & C \otimes A \\
\downarrow{\Delta_R} & & \downarrow{id_C \otimes \Delta_A} \\
C \otimes A & \xrightarrow{\Delta_R \otimes \text{id}_A} & C \otimes A \otimes A.
\end{array}
\]

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_R} & C \otimes A \\
\downarrow{\Delta_C} & & \downarrow{\Delta_C \otimes \text{id}_A} \\
C \otimes C & \xrightarrow{\text{id}_C \otimes \Delta_R} & C \otimes C \otimes A.
\end{array}
\]
A left quantum group homomorphism from \((C, \Delta_C)\) to \((A, \Delta_A)\) is a morphism \(\Delta_L : C \rightarrow A \otimes C\) for which following two diagram commute:
Theorem [Meyer, R., Woronowicz, 2011]

For any right quantum group homomorphism $\Delta_R : C \to C \otimes A$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

$$(\text{id}_{\hat{C}} \otimes \Delta_R)(W) = W_{12} V_{13}.$$  

This unitary is a bicharacter.

Conversely, let $V$ be a bicharacter from $C$ to $A$, and let $\mathcal{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_R(x) := \mathcal{V}(x \otimes 1)\mathcal{V}^*$$

for all $x \in C$

defines a right quantum group homomorphism from $C$ to $A$.  

These two maps between bicharacters and right quantum group homomorphisms are inverse to each other.
Theorem [Meyer, R., Woronowicz, 2011]

For any right quantum group homomorphism $\Delta_R : C \to C \otimes A$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

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Conversely, let $V$ be a bicharacter from $C$ to $A$, and let $\mathcal{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_R(x) := \mathcal{V}(x \otimes 1)\mathcal{V}^* \quad \text{for all } x \in C$$

defines a right quantum group homomorphism from $C$ to $A$. These two maps between bicharacters and right quantum group homomorphisms are inverse to each other.
Theorem [Meyer, R., Woronowicz, 2011]

For any left quantum group homomorphism \( \Delta_L : C \to A \otimes C \)
there is a unique unitary \( V \in \mathcal{UM}(\hat{C} \otimes A) \) with

\[
(id_{\hat{C}} \otimes \Delta_L)(W) = V_{12}W_{13}.
\]

This unitary is a bicharacter.

Conversely, let \( V \) be a bicharacter from \( C \) to \( A \), let \( \hat{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A) \) be the corresponding concrete bicharacter. Then

\[
\Delta_L(x) := (R_A \otimes R_C)(\hat{V}^*(1 \otimes R_C(x))\hat{V}) \quad \text{for all } x \in C
\]

is a left quantum group homomorphism from \( C \) to \( A \). These two maps between bicharacters and left quantum group homomorphisms are bijective and inverse to each other.
Left quantum group homomorphisms and bicharacters

Theorem [Meyer, R., Woronowicz, 2011]

For any left quantum group homomorphism $\Delta_L: C \to A \otimes C$, there is a unique unitary $V \in UM(\hat{C} \otimes A)$ with

$$(\text{id}_\hat{C} \otimes \Delta_L)(W) = V_{12}W_{13}.$$  

This unitary is a bicharacter.

Conversely, let $V$ be a bicharacter from $C$ to $A$, let $V \in U(H_C \otimes H_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_L(x) := (R_A \otimes R_C)(\hat{V}^*(1 \otimes R_C(x))\hat{V})$$  

for all $x \in C$

is a left quantum group homomorphism from $C$ to $A$.

These two maps between bicharacters and left quantum group homomorphisms are bijective and inverse to each other.
Lemma

Let $\Delta_L : C \to A \otimes C$ and $\Delta_R : C \to C \otimes B$ be a left and a right quantum group homomorphism. Then the following diagram commutes:

\[
\begin{array}{c}
C \xrightarrow{\Delta_L} A \otimes C \\
\Delta_R \downarrow \quad \downarrow \text{id}_{A \otimes \Delta_R} \\
C \otimes B \xrightarrow{\Delta_L \otimes \text{id}_B} A \otimes C \otimes B.
\end{array}
\]
Commutation relation between left and right homomorphisms

Lemma

$\Delta_L$ and $\Delta_R$ are associated to the same bicharacter $V \in UM(\hat{C} \otimes A)$ if and only if the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \otimes C \\
\downarrow{\Delta_C} & & \downarrow{\text{id}_C \otimes \Delta_L} \\
C \otimes C & \xrightarrow{\Delta_R \otimes \text{id}_C} & C \otimes A \otimes C.
\end{array}
\]
1 Multiplicative unitaries

2 Locally compact quantum groups

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   • Morphism as a functor between coaction categories

5 Summary
Lemma

Right or left quantum group homomorphisms are injective and satisfies

\[
\Delta_R(C) (1 \otimes A) \text{ is linearly dense in } C \otimes A \\
\Delta_L(C) (A \otimes 1) \text{ is linearly dense in } A \otimes C
\]

Equivalently right and left quantum group homomorphisms are injective and continuous as coactions.

- Let \( C^*\text{alg}(A) \) or \( C^*\text{alg}(A, \Delta_A) \) denote the category of \( C^* \)-algebras with a continuous, injective \( A \)-coaction.
- \( A \)-equivariant morphisms as arrows in \( C^*\text{alg}(A) \).
Coaction category

**Lemma**

Right or left quantum group homomorphisms are injective and satisfies

$$\Delta_R(C)(1 \otimes A) \text{ is linearly dense in } C \otimes A$$

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Equivalently right and left quantum group homomorphisms are injective and continuous as coactions.

- Let $\mathcal{C}^*\text{alg}(A)$ or $\mathcal{C}^*\text{alg}(A, \Delta_A)$ denote the category of $C^*$-algebras with a continuous, injective $A$-coaction.
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Homomorphisms of quantum groups
Coaction category

Lemma

Right or left quantum group homomorphisms are injective and satisfies

\[ \Delta_R(C)(1 \otimes A) \text{ is linearly dense in } C \otimes A \]
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- Let \( \mathcal{C}^*\text{alg}(A) \) or \( \mathcal{C}^*\text{alg}(A, \Delta_A) \) denote the category of \( \mathcal{C}^* \)-algebras with a continuous, injective \( A \)-coaction.
- \( A \)-equivariant morphisms as arrows in \( \mathcal{C}^*\text{alg}(A) \).
Assumptions

- \((A, \Delta_A)\) and \((B, \Delta_B)\) be locally compact quantum groups.
- \(\alpha: C \rightarrow C \otimes A\) be a continuous right coaction of \((A, \Delta_A)\) on a \(C^*\)-algebra \(C\).
- \(\Delta_R: A \rightarrow A \otimes B\) be a right quantum group homomorphism.
- \(\mathcal{F}_R: \mathcal{C}^*\mathcal{A}lg(A) \rightarrow \mathcal{C}^*\mathcal{A}lg\) be the functor that forgets the \(A\)-coaction.
Theorem [Meyer, R., Woronowicz, 2011]

There is a unique continuous coaction $\gamma$ of $(B, \Delta_B)$ on $C$ such that the following diagram commutes:

$$
\begin{align*}
C & \xrightarrow{\alpha} C \otimes A \\
\gamma & \downarrow \quad \downarrow \text{id}_{C \otimes \Delta_R}
\end{align*}
$$

$$
\begin{align*}
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} C \otimes A \otimes B.
\end{align*}
$$

This construction is a functor $F : C^*\text{alg}(A) \rightarrow C^*\text{alg}(B)$ with $\forall \gamma \circ F = \forall \gamma$ as any $A$-equivariant morphisms $D \rightarrow D'$ are also $B$-equivariant for $D, D' \in C^*\text{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism $\Delta_R$. 
**Theorem [Meyer, R., Woronowicz, 2011]**

There is a unique continuous coaction $\gamma$ of $(B, \Delta_B)$ on $C$ such that the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & C \otimes A \\
\downarrow{\gamma} & & \downarrow{\text{id}_C \otimes \Delta_R} \\
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} & C \otimes A \otimes B.
\end{array}
\]

This construction is a functor $F : C^*\text{alg}(A) \to C^*\text{alg}(B)$ with $\Phi \circ F = \Phi$ as any $A$-equivariant morphisms $D \to D'$ are also $B$-equivariant for $D, D' \in C^*\text{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism $\Delta_R$. 

Sutanu Roy (Göttingen)
There is a unique continuous coaction $\gamma$ of $(B, \Delta_B)$ on $C$ such that the following diagram commutes:

$$
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C & \xrightarrow{\alpha} & C \otimes A \\
\gamma \downarrow & & \downarrow \text{id}_C \otimes \Delta_R \\
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} & C \otimes A \otimes B.
\end{array}
$$

This construction is a functor $F: \mathcal{C}^*\text{alg}(A) \to \mathcal{C}^*\text{alg}(B)$ with $\mathcal{F}or \circ F = \mathcal{F}or$ as any $A$-equivariant morphisms $D \to D'$ are also $B$-equivariant for $D, D' \in \mathcal{C}^*\text{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism $\Delta_R$.
Assumptions

- \((A, \Delta_A)\) and \((B, \Delta_B)\) be locally compact quantum groups.
- \(\alpha: C \to C \otimes A\) be a right quantum group homomorphism where \((C, \Delta_C)\) is a quantum group.
- \(\beta: A \to A \otimes B\) be another right quantum group homomorphism.
- \(F_\alpha: \mathcal{C}^\ast\text{alg}(C) \to \mathcal{C}^\ast\text{alg}(A)\) and \(F_\beta: \mathcal{C}^\ast\text{alg}(A) \to \mathcal{C}^\ast\text{alg}(B)\) be the associated functors.
- \(V_C \to B = V_A \to B \ast V_C \to A\).
Proposition

There exists $\gamma: C \to C \otimes B$ which is the unique right quantum group homomorphism that makes the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha} & C \otimes A \\
\gamma \downarrow & & \downarrow \text{id}_{C \otimes B} \\
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} & C \otimes A \otimes B.
\end{array}
$$

which satisfies $F_B \circ F_\alpha = F_\gamma$.

Moreover, $V^{C \to B}$ is the bicharacter associated to $\gamma$. 
Proposition

There exists $\gamma : C \to C \otimes B$ which is the unique right quantum group homomorphism that makes the following diagram commute:

$$
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C & \xrightarrow{\alpha} & C \otimes A \\
\gamma \downarrow & & \downarrow \text{id}_{C \otimes B} \\
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} & C \otimes A \otimes B.
\end{array}
$$

which satisfies $F_\beta \circ F_\alpha = F_\gamma$.

Moreover, $\psi_{C \to B}$ is the bicharacter associated to $\gamma$. 
There exists $\gamma : C \to C \otimes B$ which is the unique \textit{right quantum group homomorphism} that makes the following diagram commute:

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & C \otimes A \\
\downarrow{\gamma} & & \downarrow{\text{id}_{C \otimes B}} \\
C \otimes B & \xrightarrow{\alpha \otimes \text{id}_B} & C \otimes A \otimes B.
\end{array}
\]

which satisfies $F_\beta \circ F_\alpha = F_\gamma$.

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Outline

1. Multiplicative unitaries
2. Locally compact quantum groups
3. Hopf *-homomorphisms
4. Equivalent pictures of homomorphisms of quantum groups
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5. Summary
Summary

- Multiplicative unitaries are the fundamental objects.
- Every modular/manageable multiplicative unitary $W \in \mathcal{UM}(\hat{C} \otimes C)$ admits a unique lift to $\mathcal{X} \in \mathcal{UM}(\hat{C}^u \otimes C^u)$. Hence they are *basic* in sense of Ng and hence the *birepresentations* (bicharacters in our terminology) are indeed the correct notion of homomorphisms between quantum groups.
Summary

- Multiplicative unitaries are the fundamental objects.
- Every modular/manageable multiplicative unitary $W \in \mathcal{UM}(\hat{\mathbb{C}} \otimes \mathbb{C})$ admits a unique lift to $\mathcal{X} \in \mathcal{UM}(\hat{\mathbb{C}}^u \otimes \mathbb{C}^u)$. Hence they are basic in sense of Ng and hence the birepresentations (bicharacters in our terminology) are indeed the correct notion of homomorphisms between quantum groups.
Vaes introduced the notion of homomorphisms between quantum groups (von Neumann algebraic setting) as Hopf*-homomorphisms between universal C*-bialgebras which is equivalent to the bicharacters.

Last but not least, bicharacters induces a functor between coaction categories via left/right quantum group homomorphism which is a new realization of homomorphisms between quantum groups.
Vaes introduced the notion of homomorphisms between quantum groups (von Neumann algebraic setting) as Hopf*-homomorphisms between universal C*-bialgebras which is equivalent to the bicharacters.

Last but not least, bicharacters induces a functor between coaction categories via left/right quantum group homomorphism which is a new realization of homomorphisms between quantum groups.
More details.....

http://arxiv.org/abs/1011.4284/v2
Thank you for your attention!