GENERAL TOPOLOGY

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Chapter 1

Topological Spaces

1.1 Basic Notions

A ***topology*** on a set X is a set $\mathcal{O}$ of subsets of X, called open sets, with the properties:

(1) The union of an arbitrary family of open sets is open.
(2) The intersection of a finite family of open sets is open.
(3) The empty set $\emptyset$ and X are open.

A ***topological space*** $(X, \mathcal{O})$ consists of a set X and a topology $\mathcal{O}$ on X. The sets in $\mathcal{O}$ are the ***open sets*** of the topological space $(X, \mathcal{O})$. We usually denote a topological space just by the underlying set X. A set $A \subset X$ is ***closed*** in $(X, \mathcal{O})$ if the complement $X \smallsetminus A$ is open in $(X, \mathcal{O})$. Closed sets have properties dual to (1)-(3):

(4) The intersection of an arbitrary family of closed sets is closed.
(5) The union of a finite family of closed sets is closed.
(6) The empty set $\emptyset$ and X are closed.

The properties (1) and (2) of a topology show that we need not specify all of its sets since some of them are generated by taking unions and intersections. We often make use of this fact in the construction of topologies. For this purpose we collect a few general set theoretical remarks.

A subset $\mathcal{B}$ of a topology $\mathcal{O}$ is a ***basis*** of $\mathcal{O}$ if each $U \in \mathcal{O}$ is a union of elements of $\mathcal{B}$. (The empty set is the union of the empty family.) The intersection $A \cap B$ of elements of $\mathcal{B}$ is then in particular a union of elements of $\mathcal{B}$. Conversely assume that a set $\mathcal{B}$ of subsets of X has the property that the intersection of two of its members is the union of members of $\mathcal{B}$, then there exists a unique topology $\mathcal{O}$ which has $\mathcal{B}$ as a basis. It consists of the unions of an arbitrary family of members of $\mathcal{B}$.

A subset $\mathcal{S}$ of $\mathcal{O}$ is a ***subbasis*** of $\mathcal{O}$ if the set $\mathcal{B}(\mathcal{S})$ of intersections of a finite number of elements in $\mathcal{S}$ is a basis of $\mathcal{O}$. (The space X is the intersection of the empty family.) Let $\mathcal{S}$ be any set of subsets of X. Then there exists a unique topology $\mathcal{O}(S)$ which has $\mathcal{S}$ as a subbasis. The set $\mathcal{B}(S)$ is a basis of the topology $\mathcal{O}(S)$. If $\mathcal{O}$ is a topology containing $\mathcal{S}$, then $\mathcal{O}(S) \subset \mathcal{O}$. If $\mathcal{S} \subset \mathcal{O}$, then the topology $\mathcal{O}$ contains $\emptyset$, X, and the elements of $\mathcal{S}$; let $\mathcal{B}(\mathcal{S})$ be the family of all finite intersections of these sets; and let $\mathcal{O}(S)$ be the set of all unions of elements in $\mathcal{B}(S)$. Then $\mathcal{O}(S) \subset \mathcal{O}$, and $\mathcal{O}(S)$ is already a topology, by elementary rules about unions and intersections. Formally, $\mathcal{O}(S)$ may be defined as the intersection of all topologies which contain $\mathcal{S}$. But it is useful to have some information about the sets contained in it.

(1.1.1) Example (Real numbers). The set of open intervals of $\mathbb{R}$ is a basis of a topology on $\mathbb{R}$, the standard topology on $\mathbb{R}$. Thus the open sets are unions of open intervals. A closed interval $[c, d]$ is then closed in this topology. A half-open interval $]a, b], a < b$ is neither open nor closed for this topology. We shall verify later that $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ which are both open and closed (see (1.9.1)). The sets of the form $\{x \mid x < a\}$ and $\{x \mid x > a\}$, $a \in \mathbb{R}$ are a subbasis of this topology. The extended real line $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ has a similar subbasis for its standard topology; here, of course, $-\infty < x < \infty$ for $x \in \mathbb{R}$.

Note that the definition of the standard topology only uses the order relation, and not the algebraic structures of the field $\mathbb{R}$. Despite of the simple language: The real numbers are a very rich and complicated topological space. Many spaces of geometric interest are based on real numbers (manifolds, cell complexes). The real numbers are also important in the axiomatic development of the theory. ◇

(1.1.2) Example (Euclidean spaces). The set of products $\prod_{i=1}^{n}]a_i, b_i[$ of open intervals is a basis for a topology on $\mathbb{R}^n$, the standard topology on the Euclidean space $\mathbb{R}^n$.

Another basis for the same topology are the ε-neighbourhoods $U_\varepsilon(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < \varepsilon\}$ of points a with respect to the Euclidean norm $\| - \|$. This should be known from calculus. We recall it in the next section when we introduce metric spaces. ◇

We fix a topological space X and a subset A. The intersection of the closed sets which contain A is denoted $\overline{A}$ and called ***closure*** of A in X. A set A is ***dense*** in X if $\overline{A} = X$. The ***interior*** of A is the union of the open sets contained in A. We denote the interior by A°. A point in A° is an ***interior point*** of A. A subset is ***nowhere dense*** if the interior of its closure is empty. The ***boundary*** of A in X is $\mathrm{Bd}(A) = \overline{A} \cap (\overline{X \smallsetminus A})$. An open subset U of X which contains A is an ***open neighbourhood*** of A in X. A set B is a ***neighbourhood*** of A if it contains an open neighbourhood of A. If $A = \{a\}$ we talk about neighbourhoods of the point a. A system of neighbourhoods of the point x is a ***neighbourhood basis*** of x if each neighbourhood of x contains one of the system.

One can define a topological space by using neighbourhood systems of points or by using the closure operator (see (1.1.6) and Problem 10).

A map $f\colon X \to Y$ between topological spaces is ***continuous*** if the pre-image $f^{-1}(V)$ of each open set V of Y is open in X. Dually: A map is continuous if the pre-image of each closed set is closed. The identity $\mathrm{id}(X)\colon X \to X$ is always continuous, and the composition of continuous maps is continuous. Hence topological spaces and continuous maps form a category. We denote it by TOP. A map $f\colon X \to Y$ between topological space is said to be ***continuous at*** $x \in X$ if for each neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subset V$; it suffices to consider a neighbourhood basis of x and $f(x)$. The definitions are consistent: See (1.1.5) for various characterizations of continuity.

A ***homeomorphism*** $f\colon X \to Y$ is a continuous map with a continuous inverse $g\colon Y \to X$. A homeomorphism is an isomorphism in the category TOP. Spaces X and Y are ***homeomorphic*** if there exists a homeomorphism between them. One of the aims of geometric and algebraic topology is to develop tools which can be used to decide whether two given spaces are homeomorphic or not. A property $\mathcal{P}$ which spaces can have or not is a ***topological property*** if the following holds: If the space X has property $\mathcal{P}$, then also every homeomorphic space. Often one considers spaces with some additional structure (e.g., a metric, a differential structure, a cell decomposition, a bundle structure, a symmetry) which is not a topological property but may be useful for the investigation of topological problems.

Starting from the definition of a topology and a continuous map one can develop a fairly extensive axiomatic theory — often called ***point set topology***. But in learning about the subject it is advisable to use other material, e.g., what is known from elementary and advanced calculus. On the other hand the general notions of the axiomatic theory can clarify concepts of calculus. For instance, it is often easier to work with the general notion of continuity than with the (ε, δ)-definition of calculus.

A map $f\colon X \to Y$ between topological spaces is ***open*** (***closed***) if the image of each open (closed) set is again open (closed). These properties are not directly related to continuity; but a continuous map can, of course, have these additional and often useful properties.

In the sequel we assume that a map between topological spaces is continuous if nothing else is specified or obvious. A ***set map*** is a map which is not assumed to be continuous at the outset.

(1.1.3) Proposition. *Let A and B be subsets of the space X. Then:* (1) $A \subset B$ *implies* $\overline{A} \subset \overline{B}$. (2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. (3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. (4) $A \subset B$ *implies* $A^\circ \subset B^\circ$. (5) $(A \cap B)^\circ = A^\circ \cap B^\circ$. (6) $(A \cup B)^\circ \supset A^\circ \cup B^\circ$. (7) $X \smallsetminus \overline{A} = (X \smallsetminus A)^\circ$. (8) $\overline{X \smallsetminus A} = X \smallsetminus A^\circ$. (9) $\mathrm{Bd}(A) = \overline{A} \smallsetminus A^\circ$.

Proof. (1) Obviously, $A \subset B \subset \overline{B}$. Since $\overline{B}$ is closed and contains A, we conclude $\overline{A} \subset \overline{B}$. (2) By $A \subset A \cup B$ and (1) $\overline{A} \subset \overline{A \cup B}$ and hence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. From $A \subset \overline{A}$ and $B \subset \overline{B}$ we see $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed, we conclude $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. (3) $A \subset \overline{A}$ and $B \subset \overline{B}$ implies $A \cap B \subset \overline{A} \cap \overline{B}$. Since $\overline{A} \cap \overline{B}$ is closed, we see $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. The proof of (4), (5), and (6) is "dual" to the proof of (1), (2), and (3). (7) $X \smallsetminus \overline{A}$ is open, being the complement of a closed set, and contained in $X \smallsetminus A$. Therefore $X \smallsetminus \overline{A} \subset (X \smallsetminus A)^\circ$. We pass to complements in $X \smallsetminus A \supset (X \smallsetminus A)^\circ$ and see that A is contained in the closed set $X \smallsetminus (X \smallsetminus A)^\circ$, hence $\overline{A}$ is also contained in this set. Passing again to complements, we obtain $X \smallsetminus \overline{A} \subset (X \smallsetminus A)^\circ$. The proof of (8) is "dual" to the proof of (7). (9) follows from the definition of the boundary and (8). □

A point $x \in X$ is called a ***touch point*** of $A \subset X$ if each neighbourhood of x intersects A. It is called a ***limit point*** or ***accumulation point*** of A if each neighbourhood intersects $A \smallsetminus \{x\}$.

(1.1.4) Proposition. *The closure $\overline{A}$ is the set of touch points of A. A set is closed if it contains all its limit points.*

Proof. Let $x \in \overline{A}$ and let U be a neighbourhood of x. Suppose $U \cap A = \emptyset$. Let $V \subset U$ be an open neighbourhood of x. Then $V \cap A = \emptyset$ and $A \subset X \smallsetminus V$, hence $\overline{A} \subset X \smallsetminus V$ and $\overline{A} \cap V = \emptyset$. This contradicts $x \in \overline{A}$, $x \in V$. Therefore $U \cap A \neq \emptyset$.

Suppose each neighbourhood of x intersects A. If $x \notin \overline{A}$, then x is contained in the open set $X \smallsetminus \overline{A}$. Hence $A \cap (X \smallsetminus \overline{A}) \neq \emptyset$, a contradiction. □

The previous proposition says, roughly, that limiting processes from inside A stay in the closure of A.

Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are topologies on X. If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{O}_2$ is ***finer*** than $\mathcal{O}_1$ and $\mathcal{O}_1$ ***coarser*** than $\mathcal{O}_2$. The topology $\mathcal{O}_1$ is finer than $\mathcal{O}_2$ if and only if the identity $(X, \mathcal{O}_1) \to (X, \mathcal{O}_2)$ is continuous. The set of all subsets of X is the finest topology; it is called the ***discrete topology*** and the resulting space a ***discrete space***. All maps $f\colon X \to Y$ from a discrete space X are continuous. The coarsest topology on X consists of $\emptyset$ and X alone; we call it the ***lumpy topology***. All maps into a lumpy space are continuous. If $(\mathcal{O}_j \mid j \in J)$ is a family of topologies on X, then their intersection is a topology.

(1.1.5) Proposition. *Let $f\colon X \to Y$ be a set map between topological spaces. Then the following are equivalent:*

(1) *The map f is continuous.*
(2) *The pre-image of each set in a subbasis of Y is open in X.*
(3) *The map f is continuous at each point of X.*
(4) *For each $B \subset Y$ we have $f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$.*
(5) *For each $B \subset Y$ we have $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.*
(6) *For each $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.*
(7) *The pre-image of each closed set of Y is closed in X.*

Proof. (1) $\Rightarrow$ (2). As a special case.

(2) $\Rightarrow$ (1). We use the construction of the topology from its subbasis. The relations $f^{-1}(\bigcap_j A_j) = \bigcap_j f^{-1}(A_j)$ and $f^{-1}(\bigcup_j A_j) = \bigcup_j f^{-1}(A_j)$ are then used to show that the pre-image of each open set is open.

(1) $\Rightarrow$ (3). Let V be a neighbourhood of $f(x)$. It contains an open neighbourhood W. Its pre-image is open, because f is continuous. Therefore $f^{-1}(V)$ is a neighbourhood of x.

(3) $\Rightarrow$ (1). Let $V \subset Y$ be open. Then V is a neighbourhood of neighbourhood of each of its points $v \in V$. Hence $U = f^{-1}(V)$ is a neighbourhood of each of its points. But a set is open if and only if it is a neighbourhood of each of its points.

(1) $\Rightarrow$ (4). $f^{-1}(B^\circ)$ is open, being the pre-image of an open set, and contained in $f^{-1}(B)$. Now use the definition of the interior.

(4) $\Rightarrow$ (5). We use (1.1.3) and set-theoretical duality

$$\begin{aligned} X \smallsetminus \overline{f^{-1}(B)} &= (X \smallsetminus f^{-1}(B))^\circ = f^{-1}(X \smallsetminus B)^\circ \quad \supset \\ f^{-1}((X \smallsetminus B)^\circ) &= f^{-1}(X \smallsetminus \overline{B}) = X \smallsetminus f^{-1}(\overline{B}). \end{aligned}$$

The inclusion holds by (4). Passage to complements yields the claim.

(5) $\Rightarrow$ (6). We have $f^{-1}(\overline{f(A)}) \supset \overline{f^{-1}f(A)} \supset \overline{A}$, where the first inclusion holds by (5), and the second because of $f^{-1}f(A) \supset A$. The inclusion between the outer terms is equivalent to the claim.

(6) $\Rightarrow$ (7). Suppose $B \subset Y$ is closed. From (6) we obtain

$$f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B} = B.$$

Hence $\overline{f^{-1}(B)} \subset f^{-1}(B)$; the reversed inclusion is clear; therefore equality holds, which means that $f^{-1}(B)$ is closed.

(7) $\Rightarrow$ (1) holds by set-theoretical duality. □

(1.1.6) Proposition. *The neighbourhoods of a point $x \in X$ have the properties:*

(1) *If U is a neighbourhood and $V \supset U$, then V is a neighbourhood.*
(2) *The intersection of a finite number of neighbourhoods is a neighbourhood.*
(3) *Each neighbourhood of x contains x.*
(4) *If U is a neighbourhood of x, then there exists a neighbourhood V of x such that U is a neighbourhood of each $y \in V$.*

Let X be set. Suppose each $x \in X$ has associated to it a set of subsets, called neighbourhoods of x, such that (1)-(4) *hold for this system. Then there exists a unique topology on X which has the given system as system of neighbourhoods of points.*

Proof. Define $\mathcal{O}$ as the set of subsets U of X such that each $x \in U$ has a neighbourhood V with $V \subset U$. Claim: $\mathcal{O}$ is a topology on X. Properties (1) and (3) of a topology are obvious from the definition and property (2) follows from property (2) of the neighbourhood system.

Let $\mathcal{U}_x$ denote the given system of neighbourhoods of x and $\mathcal{O}_x$ the system of neighbourhoods of x defined by $\mathcal{O}$. We have to show $\mathcal{U}_x = \mathcal{O}_x$. By definition of $\mathcal{O}$, the set $U_0 = \{y \in U \mid \exists\, V \in \mathcal{U}_y,\, V \subset U\}$ is open for each $U \subset X$. If $U \in \mathcal{U}_x$, choose V by (4). Then $x \in V \subset U_0 \subset U$ and hence $U \in \mathcal{O}_x$. If $U \in \mathcal{O}_x$, there exist an open $U' \in \mathcal{O}_x$, and therefore $V \in \mathcal{U}_x$ with $V \subset U'$ by definition of $\mathcal{O}$. By (1), $U \in \mathcal{U}_x$.

The uniqueness of the topology follows from the fact that a set is open, if and only if it contains a neighbourhood of each of its points. □

1.1.7 Separation Axioms. We list some properties which a space X may have.

(T_1) Points are closed subsets.

(T_2) Any two points $x \neq y$ have disjoint neighbourhoods.

(T_3) Let $A \subset X$ be closed and $x \notin A$. Then x and A have disjoint neighbourhoods.

(T_4) Disjoint closed subsets have disjoint neighbourhoods.

We say X satisfies the ***separation axiom*** T_j (or X is a T_j-space) if X has property T_j.

A T_2-space is called ***Hausdorff space*** or ***separated***. A space that satisfies T_1 and T_3 is called ***regular***. A space that satisfies T_1 and T_4 is called ***normal***. A normal space is regular, a regular space is separated. A space X is called ***completely regular*** if it is separated and if for each point $x \in X$ and each closed set A not containing x there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(A) \subset \{0\}$. The separation axioms are of a technical nature, but they serve the purpose of clarifying the concepts. ◇

Felix Hausdorff defined 1912 topological spaces (in fact Hausdorff spaces) via neighbourhood systems [**?**, p.213]. Neighbourhoods belong to the realm of analysis (limits, convergence). Open sets are more geometric, at least psychologically; they are "large" and "vague", like the things we actually see. We will see that open sets are very convenient for the axiomatic development of the theory. But it is not intuitively clear that the axioms for a topology are a good choice. The axioms for neighbourhoods should be convincing, especially if one has already some experience with calculus.

Whenever one has an important category (like TOP) one is obliged to study elementary categorical notions and properties. We will discuss subobjects, quotient objects, products, sums, pullbacks, pushouts, and (in general categorical terms) limits and colimits.

Problems

1. A set $D \subset \mathbb{R}$ is dense in $\mathbb{R}$ if and only if each non-empty interval contains elements of D. Thus $\mathbb{Q}$ is dense or the set of rational numbers with denominator of the form 2^k.

2. Determine the closure, the interior and the boundary of the following subsets of the space $\mathbb{R}$: $]a, b]$, $[2, 3[\cup]3, 4[$, $\mathbb{Q}$.

3. A map $f : X \to \mathbb{R}$ is continuous if the pre-images of $]-\infty, a[$ and $]b, \infty[$ are open where a, b run through a dense subset of $\mathbb{R}$.

4. Give examples for $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ or $(A \cup B)^\circ \neq A^\circ \cup B^\circ$, see (1.1.3).

5. Show by examples that the inclusions (4), (5), (6) in (1.1.5) are not equalities.

6. The union of two nowhere dense subsets is again nowhere dense. The intersection of two dense open sets is dense. Give an example of dense open sets $(A_n \mid n \in \mathbb{N})$ with empty intersections.

7. $x \in \overline{A}$ if and only if each member of a neighbourhood basis of x intersects A.

8. Let A be a subset of a topological space X. How many subsets of X can be obtained from A by iterating the processes closure and complement? There exist subsets of $\mathbb{R}$ where the maximum is attained. Same question for interior and complement; interior and closure; interior, closure and complement.

9. A space is said to satisfy the ***first axiom of countability*** or is ***first countable*** if each point has a countable neighbourhood basis. A space with countable basis is said to satisfy the ***second axiom of countability*** or is ***separable***[1] or ***second countable***. Euclidean spaces are first and second countable.

10. Let X be a space with countable basis. Then each basis contains a countable subsystem which is a basis. A space with countable basis has a countable dense subset. A disjoint family of open sets in a separable space is countable.

11. Consider the topology on the set $\mathbb{R}$ with the halve open intervals $[a, b[$ are a basis. Then the basis sets are open and closed; X has a countable dense subset, but no countable basis.

12. A ***Kuratowski closure operator*** on a set X is a map which assigns to each $A \subset X$ a set $h(A) \subset X$ such that: (1) $h(\emptyset) = \emptyset$. (2) $A \subset h(A)$. (3) $h(A) = h(h(A))$. (4) $h(A \cup B) = h(A) \cup h(B)$. Given a closure operator, there exists a unique topology on X such that $h(A) = \overline{A}$ is the closure of A in this topology.

13. Can one define topological spaces by an operator "interior"?.

14. A space is separated if and only if each point $x \in X$ is the intersection of its closed neighbourhoods. The points of a separated space are closed.

15. Classify topological spaces with 2 or 3 points up to homeomorphism.

16. Let A and Y be closed subsets of a T_4-space X. Suppose U is an open neighbourhood of Y in X. Let $C \subset A$ be a closed neighbourhood of A in $Y \cap A$ which is contained in $U \cap Y$. Then there exists a closed neighbourhood Z of Y which is contained in U and satisfies $Z \cap A = C$.

[1] Do not mix up with "separated".

1.2 Metric Spaces

Many examples of topological spaces arise from metric spaces, and metric spaces are important in their own right. A ***metric*** d on a set X is a map $d\colon X \times X \to [0, \infty[$ with the properties:

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (***triangle inequality***).

We call $d(x, y)$ the ***distance*** between x and y with respect to the metric d. A ***metric space*** (X, d) consists of a set X and a metric d on X.

Let (X, d) be a metric space. The set $U_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is the ε-***neighbourhood*** of x. We call $U \subset X$ ***open with respect to*** d if for each $x \in X$ there exists $\varepsilon > 0$ such that $U_\varepsilon(x) \subset U$. The system $\mathcal{O}_d$ of subsets U

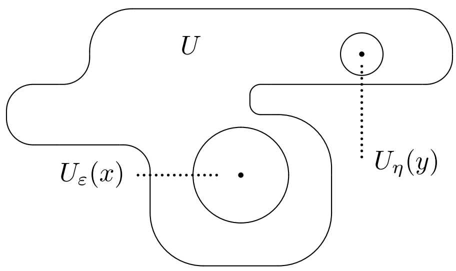

Figure 1.1. Underlying topology

which are open with respect to d is a topology on X, the ***underlying topology*** of the metric space, and the ε-neighbourhoods of all points are a basis for this topology. Subsets of the form $U_\varepsilon(x)$ are open with respect to d. For the proof, let $y \in U_\varepsilon(x)$ and $0 < \eta < \varepsilon - d(x, y)$. Then, by the triangle inequality, $U_\eta(y) \subset U_\varepsilon(x)$. A space $(X, \mathcal{O})$ is ***metrizable*** if there exists a metric d on X such that $\mathcal{O} = \mathcal{O}_d$. Metrizable spaces are first countable: Take the $U_\varepsilon(x)$ with rational ε. A set U is a neighbourhood of x if and only if there exists an $\varepsilon > 0$ such that $U_\varepsilon(x) \subset U$. For metric spaces our definition of continuity is equivalent to the familiar definition of calculus: A map $f\colon X \to Y$ between metric spaces is continuous at $a \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(a, x) < \delta$ implies $d(f(a), f(x)) < \varepsilon$. Continuity only depends on the underlying topology. But a metric is a finer and more rigid structure; one can compare the size of neighbourhoods of different points and one can define uniform continuity. A map $f\colon (X, d_1) \to (Y, d_2)$ between metric spaces is ***uniformly continuous*** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. A sequence $f_n\colon X \to Y$ of maps into a metric space (Y, d) ***converges uniformly*** to $f\colon X \to Y$ if for each $\varepsilon > 0$ there exists N such that for $n > N$ and $x \in X$ the inequality $d(f(x), f_n(x)) < \varepsilon$

holds. If the f_n are continuous functions from a topological space X which converge uniformly to f, then one shows as in calculus that f is continuous.

The Euclidean space $\mathbb{R}^n$ carries the metrics d_1, d_2, and d_∞:

$$\begin{aligned} d_2\big((x_i),(y_i)\big) &= \big(\Sigma_{i=1}^n (x_i - y_i)^2\big)^{1/2} \\ d_1\big((x_i),(y_i)\big) &= \Sigma_{i=1}^n |x_i - y_i| \\ d_\infty\big((x_i),(y_i)\big) &= \max\{|x_i - y_i| \mid i = 1,\dots,n\} \end{aligned}$$

For $n \geq 2$ these metrics are different, but the corresponding topologies are

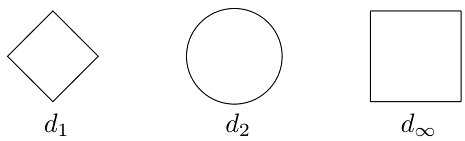

Figure 1.2. ε-neighbourhoods

identical. This holds because an ε-neighbourhood of one metric contains an η-neighbourhood of another metric. The metric d_2 is the ***Euclidean metric***. The topological space $\mathbb{R}^n$ is understood to be the space induced from $(\mathbb{R}^n, d_2)$.

A set A in a metric space (X, d) is ***bounded*** if $\{d(x,y) \mid x, y \in A\}$ is bounded in $\mathbb{R}$. The supremum of the latter set is then the ***diameter*** of A. We define

$$d(x, A) = \inf\{d(x,a) \mid a \in A\}$$

as the ***distance*** of x from $A \neq \emptyset$.

(1.2.1) Proposition. *The map $X \to \mathbb{R}$, $x \mapsto d(x,A)$ is uniformly continuous. The relation $d(x,A) = 0$ is equivalent to $x \in \overline{A}$.*

Proof. Let $a \in A$. The triangle inequality $d(x,y) + d(y,a) \geq d(x,a)$ implies $d(x,y) + d(y,a) \geq d(x,A)$. This holds for each $a \in A$, hence $d(x,y) + d(y,A) \geq d(x,A)$. Similarly if we interchange x and y, hence $|d(x,A) - d(y,A)| \leq d(x,A)$. This implies uniform continuity.

Suppose $d(x,A) = 0$. Given $\varepsilon > 0$, there exists $a \in A$ with $d(x,a) < \varepsilon$, hence $U_\varepsilon(x) \cap A \neq \emptyset$, and we can apply (1.1.4). And conversely. □

(1.2.2) Proposition. *A metrizable space is normal.*

Proof. If A and B are disjoint, non-empty, closed sets in X, then

$$f\colon X \to [0,1], \quad x \mapsto d(x,A)(d(x,A) + d(x,B))^{-1}$$

is a continuous function with $f(A) = \{0\}$ and $f(B) = \{1\}$. Let $0 < a < b < 1$. Then $[0, a[$ and $]b, 1]$ are open in $[0,1]$, and their pre-images under f are disjoint open neighbourhoods of A and B. The points of a metrizable space are closed: $d_y\colon x \mapsto d(x,y)$ is continuous and $d_y^{-1}(0) = \{y\}$, by axiom (1) of a metric. □

In a metric space one can work with sequences. We begin with some general definitions. Let $(x_n \mid n \in \mathbb{N})$ be a sequence in a topological space X. We call $z \in X$ an ***accumulation value*** of the sequence if each neighbourhood of z contains an infinite number of the x_n. The sequence ***converges*** to z if each neighbourhood of z contains all but a finite number of the x_j. We then call z the ***limit*** of the sequence and write as usual $z = \lim x_n$. In a Hausdorff space a sequence can have at most one limit. In a lumpy space each sequence converges to each point of X. Let $\mu\colon \mathbb{N} \to \mathbb{N}$ be injective and increasing. Then $(x_{\mu(n)} \mid n \in \mathbb{N})$ is called a ***subsequence*** of (x_n). We associate three sets to a sequence (x_n): $A(x_n)$ is the set of its accumulation values; $S(x_n)$ is the set of limits of subsequences; $H(x_n) = \bigcap_{n=1}^{\infty} \overline{H(n)}$ with $H(n) = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$.

(1.2.3) Proposition. *For each sequence $A = H \supset S$. If X is first countable, then also $S = H$.*

Proof. $A \subset H$. Let $z \in A$. Since each neighbourhood U of z contains an infinite number of x_n, we see that $U \cap H(n) \neq \emptyset$. Hence z is touch point of $H(n)$. This implies $z \in H$.

$H \subset A$. Let $z \in H$ and U a neighbourhood of z. Then $U \cap H(n) \neq \emptyset$ for each n. Therefore U contains an infinite number of the x_n, i.e., $z \in A$.

$S \subset A$. Let $z \in S$ and $(x_{\mu(n)})$ be s subsequence with limit z. If U is a neighbourhood of z, then, by convergence, there exists $N \in \mathbb{N}$ such that $x_{\mu(n)} \in U$ for $n > N$. Hence U contains an infinite number of x_n, i.e., $z \in A$.

$H \subset S$ if X is first countable. Let $z \in H$. We construct inductively a subsequence which converges to z. Let $U_1 \supset U_2 \subset U_3 \supset \ldots$ be a neighbourhood basis of z. Then $U_j \cap H(n) \neq \emptyset$ for all j and n. Let $x_{\mu(j)}$, $1 \leq j \leq n-1$ be given such that $x_{\mu(j)} \in U_j$. Since $U_n \cap H(\mu(n-1)+1) \neq \emptyset$, there exists $\mu(n) > \mu(n-1)$ with $x_{\mu(n)} \in U_n$. The resulting subsequence converges to z. □

In metric spaces a sequence (x_n) converges to x if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ the inequality $d(x_n, x) < \varepsilon$ holds. A sequence is a ***Cauchy-sequence*** if for each $\varepsilon > 0$ there exists N such that for $m, n > N$ the inequality $d(x_m, x_n) < \varepsilon$ holds. A metric space is ***complete*** if each Cauchy sequence converges. A Cauchy seuqence has at most one accumulation value; if a subsequence converges, then the sequence converges.

Recall from calculus that the spaces $(\mathbb{R}^n, d_i)$ are complete. (See (**??**) for a general theorem to this effect.) Completeness is not a topological property of the underlying space: The interval $]0,1[$ with the metric d_1 is not complete.

Let V be a vector space over the field F of real or complex numbers. A ***norm*** on V is a map $N\colon V \to \mathbb{R}$ with the properties:

(1) $N(v) \geq 0$, $N(v) = 0 \Leftrightarrow v = 0$.
(2) $N(\lambda v) = |\lambda| N(v)$ for $v \in V$ and $\lambda \in F$.
(3) $N(u+v) \leq N(u) + N(v)$.

A ***normed vector space*** (V, N) consists of a vector space V and a norm N on V. A norm N induces a metric $d(x, y) = N(x - y)$. Property (3) of a norm yields the triangle inequality. With respect to the topology $\mathcal{O}_d$, the map $N\colon V \to \mathbb{R}$ is continuous. A normed space which is complete in the induced metric is a ***Banach space***.

If $\langle -, - \rangle\colon V \times V \to \mathbb{R}$ is an inner product on the real vector space V, then $N(v) = \langle v, v \rangle^{1/2}$ yields a norm on V. Similarly for an hermitian inner product on a complex vector space. The ***Euclidean norm*** on $\mathbb{R}^n$ is $\|x\| = (\Sigma_{i=1}^n x_i^2)^{1/2}$. An inner product space which is complete in the associated metric is called a ***Hilbert space***.

Problems

1. A metric δ is ***bounded*** by M if $\delta(x, y) \le M$ for all x, y. Let (X, d) be a metric space. Then $\delta(x, y) = d(x, y)(1 + d(x, y))^{-1}$ is a metric on X bounded by 1 which has the same underlying topology as d.
2. Weaken the axioms of a metric and require only $d(x, y) \ge 0$ and $d(x, x) = 0$ instead of axiom (1); call this a ***quasi-metric***. If d is a quasi-metric on E, one can still define the associated topology $\mathcal{O}_d$. The relation $x \sim y \Leftrightarrow d(x, y) = 0$ is an equivalence relation on E. The space F of equivalence classes carries a metric d' such that $d'(x', y') = d(x, y)$ if x' denotes the class of x. The map $E \to F, x \mapsto x'$ is continuous. Discuss also similar problems if one starts with a map $d\colon E \times E \to [0, \infty]$; the axioms then still make sense.
3. As a joke, replace the triangle inequality by $d(x, y) \ge d(x, z) + d(z, y)$. Discuss the consequences.
4. Let (E, d) be a metric space. Then $d_a\colon E \to \mathbb{R},\ x \mapsto d(x, a)$ is uniformly continuous. The sets $D_\varepsilon(a) = \{x \in E \mid d(a, x) \le \varepsilon\}$ and $S_\varepsilon(a) = \{x \in E \mid d(a, x) = \varepsilon\}$ are closed in E. In Euclidean space, $D_\varepsilon(a)$ is the closure of $U_\varepsilon(x)$; this need not hold for an arbitrary metric space. It can happen that $D_\varepsilon(a)$ is open, and even equal to $U_\varepsilon(x)$. Construct examples.
5. The space $C = C([0, 1])$ of continuous functions $f\colon [0, 1] \to \mathbb{R}$ with the ***sup-norm*** $\|f\| = \sup\{f(x) \mid x \in [0, 1]\}$ is a Banach-space. The integral $f \mapsto \int f$ is a continuous map $C \to \mathbb{R}$. The L^1-norm on C is defined by $\|f\|_1 = \int_0^1 |f(x)| dx$.
6. A metrizable space with a countable dense subset has a countable basis.
7. Let $C(\mathbb{R})$ be the set of continuous functions $\mathbb{R} \to \mathbb{R}$. Let $P \subset C(\mathbb{R})$ be the set of positive functions. For $d \in P$ and $f \in C(\mathbb{R})$ let $U_d(f) = \{g \in C(\mathbb{R}) \mid |f(x) - g(x)| < d(x)\}$. Let $\mathcal{O}$ be the topology with subbasis consisting of the sets $U_d(f)$. Then there is no sequence in P which converges to the zero function n, but $n \in \overline{P}$. (Sequences are too "short"; see the notion of a net.) The topology cannot be generated by a metric. The function n does not have a countable neighbourhood basis.
8. Weaken the axioms of a metric by only requiring $d(x, x) = 0$ in (1). One still can define the topology $\mathcal{O}_d$. The relation $x \sim y \leftrightarrow d(x, y) = 0$ is an equivalence relation

on X. The set of equivalence classes carries a metric d' such that $d'(x', y') = d(x, y)$; here x' denotes the class of x.
9. It is know from calculus that the multiplication $m \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto x \cdot y$ is continuous. Study this from the view point of the general theory by looking at the pre-images of $]a, \infty[$. (Draw a figure.)
10. Let $A \subset \mathbb{R}^n$ be convex. Is then also the closure $\overline{A}$ convex?

1.3 Subspaces

It is a classical idea and method to define geometric objects (spaces) as subsets of Euclidean spaces, e.g., as solution sets of a system of equations. But it is important to observe that such objects have "absolute" properties which are independent of their position in the ambient space. In the topological context this absolute property is the subspace topology.

It is also an interesting problem to realize spaces with an abstract definition as subsets of a Euclidean space. A typical problem is to find Euclidean models for projective spaces.

Let $(X, \mathcal{O})$ be a topological space and $A \subset X$ a subset. Then

$$\mathcal{O}|A = \{U \subset A \mid \text{there exists } V \in \mathcal{O} \text{ with } \ U = A \cap V\}$$

is a topology on A. It is called the ***induced topology***, the ***subspace topology***, or the ***relative topology***. The space $(A, \mathcal{O}|A)$ is called a ***subspace*** of $(X, \mathcal{O})$; we usually say: A is a subspace of X. A continuous map $f \colon (Y, \mathcal{S}) \to (X, \mathcal{O})$ is an ***embedding*** if it is injective and $(Y, \mathcal{S}) \to (f(Y), \mathcal{O}|f(Y))$, $y \mapsto f(y)$ a homeomorphism.

(1.3.1) Proposition. *Let A be a subspace of X. Then the inclusion $i \colon A \to X$, $a \mapsto a$ is continuous. Let Y be a space and $f \colon Y \to X$ a set-map with $f(Y) \subset A$. Then f is continuous if and only if $\varphi \colon Y \to A$, $y \mapsto f(y)$ is continuous.*

Proof. If $U \subset X$ is open, then $i^{-1}(U) = A \cap U$ is open, by definition of the subspace topology. If φ is continuous, then also $f = i \circ \varphi$. If f is continuous and V open in A, choose U open in X with $U \cap A = V$. Then $\varphi^{-1}(V) = f^{-1}(U)$ is open. □

Property (2) of the next proposition characterizes embeddings i in categorical terms. We call it the ***universal property of an embedding***.

(1.3.2) Proposition. *Let $i \colon Y \to X$ be an injective set map. The following are equivalent:*
(1) *i is an embedding.*

(2) *A set map $g\colon Z \to Y$ from any topological space Z is continuous if and only if ig is continuous.*

$$\begin{tikzcd} X & \\ Y \arrow[u, "i"] & Z \arrow[ul, "ig"'] \arrow[l, dotted, "g"] \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). Let $A = i(Y)$ with the subspace topology of X. If g is continuous, then also the composition ig. Let ig be continuous. Since i is an embedding, $j\colon Y \to A, y \mapsto i(y)$ is a homeomorphism. From (1.3.1) we see that jg is continuous, hence g is continuous.

(2 $\Rightarrow$ (1). We apply (2) to $g = \mathrm{id}(Y)$ and see that i and hence j is continuous. Let $h\colon A \to Y$ be the inverse of j. The composition ih is the inclusion $A \subset X$. Thus, by condition (2), h is continuous. Hence j is a homeomorphism with inverse h. □

Suppose $A \subset B \subset X$ are subspaces. If A is closed in B and B closed in X, then A is closed in X. Similarly for open subspaces. But in general, an open (closed) subset of B must not be open (closed) in X. Note that B is always open and closed in the subspace B. The next proposition will be used many times without further reference. A family $(X_j \mid j \in J)$ of subsets of X is called ***locally finite*** if each point $x \in X$ has a neighbourhood which intersects only finitely many of the X_j.

(1.3.3) Proposition. *Let $f\colon X \to Y$ be a set-map between topological spaces and let X be the union of the subsets $(X_j \mid j \in J)$. If the X_j are open and the maps $f_j = f \mid X_j$ continuous, then f is continuous. A similar assertion holds if the X_j are closed and locally finite.*

Proof. Suppose the X_j are closed and suppose J is finite. Let $C \subset Y$ be closed. Then $f^{-1}(C) = \bigcup_j f^{-1}(C) \cap X_j = \bigcup_j f_j^{-1}(C)$. Since $f_j^{-1}(C)$ is closed in X_j it is also closed in X. Hence we have a finite union of closed sets, i.e., pre-images of closed sets are closed. Similarly for open X_j. If the X_j form a locally finite family, we first conclude that each point has an open neoghbourhood U such that $f|U$ is continuous. □

(1.3.4) Proposition. *Let $f\colon X \to Y$ be an open set map. Then the restriction $f_B\colon f^{-1}(B) \to B$ is open for each $B \subset Y$. Similarly if "open" is replaced by "closed".*

Proof. For $B \subset Y$ and $U \subset X$ the relation

$$f(f^{-1}(B) \cap U) = B \cap f(U)$$

holds for any set map. By definition of the subspace topology, an open set V in $f^{-1}(B)$ has the form $V = f^{-1}(B) \cap U$ for an open $U \subset X$. If f is open, $B \cap f(U)$ is open in B. The relation above shows that f_B is open. □

A subset A of a space X is a ***retract*** of X if there exists a ***retraction*** $r\colon X \to A$, i.e., a continuous map $r\colon X \to A$ such that $r|A = \mathrm{id}(A)$. A continuous map $s\colon B \to E$ is a ***section*** of the continuous map $p\colon E \to B$ if $ps = \mathrm{id}(B)$. In that case s is an embedding onto its image.

Subsets of Euclidean spaces are usually considered as subspaces. We often use the subspaces:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}, \quad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}, \quad E^n = D^n \smallsetminus S^{n-1}.$$

We call S^n the n-dimensional ***unit sphere*** and D^n the n-dimensional ***unit disk*** or ***unit ball***.

(1.3.5) Example (Spheres). Let $e_n = (0, \ldots, 0, 1)$. We define the ***stereographic projection*** $\varphi_N\colon U_N = S^n \smallsetminus \{e_n\} \to \mathbb{R}^n$: the point $\varphi_N(x)$ is the intersection of the line through e_n and x with the hyperplane $\mathbb{R}^n = \mathbb{R}^n \times 0$. One computes $\varphi_N(x_0, \ldots, x_n) = (1 - x_n)^{-1}(x_0, \ldots, x_{n-1})$. An inverse is $\pi_N\colon x \mapsto (1 + \|x\|)^2)^{-1}(2x, \|x\|^2 - 1)$.

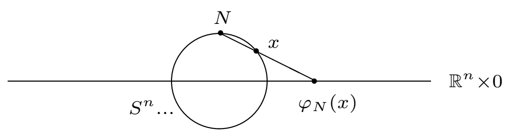

Figure 1.3. Stereographic projection

We also have the stereographic projection $\varphi_S\colon U_S = S^n \smallsetminus \{-e_n\} \to \mathbb{R}^n$ and the relation $\varphi_S \circ \varphi_N^{-1}(y) = \|y\|^{-2}y$ holds. $\diamond$

Many interesting and important spaces are defined as subspaces of vector spaces of matrices. We discuss later the matrix groups (general linear, orthogonal, unitary) as spaces together with a continuous group multiplication (topological groups).

Problems

1. $]0,1]$ is covered by the locally finite family of the $[(n+1)^{-1}, n^{-1}] \mid n \in \mathbb{N}$. If we add $\{0\}$ we obtain a covering of $[0,1]$ which is not locally finite. The conclusion of (1.3.3) does not hold for the latter family.
2. The subspace topology of $\mathbb{R} \subset \overline{\mathbb{R}}$ is the standard topology.
3. A homeomorphism $]0,1[\, \to \mathbb{R}$ extends to a homeomorpism $[0,1] \to \overline{\mathbb{R}}$.
4. Two closed intervals of $\overline{\mathbb{R}}$ with more than two points are homeomorphic.
5. Discuss the extension of the multiplication $m\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \subset \overline{\mathbb{R}}$ as a continuous map to a larger subset of $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Compare this with conventions, known from calculus,

about working with the symbols $\pm\infty$.
6. The boundary of D^n in $\mathbb{R}^n$ is S^{n-1} and $E^n = D^n \smallsetminus S^{n-1}$ its interior.
7. Let $\mathrm{SU}(2)$ be the set of unitary $(2,2)$-matrices with determinant 1, considered as subspace of the vector space of complex $(2,2)$-matrices $\mathrm{M}_2(\mathbb{C}) \cong \mathbb{C}^4$:

$$A \in \mathrm{SU}(2) \quad \Leftrightarrow \quad A = \begin{pmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{pmatrix}, \quad z_0\bar{z}_0 + z_1\bar{z}_1 = 1.$$

Then $\mathrm{SU}(2) \to S^3$, $A \mapsto (z_0, z_1)$ is a homeomorphism of $\mathrm{SU}(2)$ with the unit sphere $S^3 \subset \mathbb{C}^2$.
8. Let $S_0^m = S^{m+n+1} \cap (\mathbb{R}^{m+1} \times 0)$ and $S_1^n = S^{m+n+1} \cap (0 \times \mathbb{R}^{n+1})$. Then $X = S^{m+n+1} \smallsetminus S_1^n$ is homeomorphic to $S^m \times E^n$. A homeomorphism $S^m \times E^n \to X$ is $(x,y) \mapsto (\sqrt{1 - \|y\|^2}x, y)$. The space $Y = S^{m+n+1} \smallsetminus (S_0^m \cup S_1^n)$ is homeomorphic to $S^m \times S^n \times]0,1[$ via $(x,y,t) \mapsto (\sqrt{1-t}x, \sqrt{t}y)$. (Note that the statement uses the product topology, see Section 1.5.)
9. Let (X,d) be a metric space and $A \subset X$. The restriction of d to $A \times A$ is a metric on A. The underlying topology is the subspace topology of $(X, \mathcal{O}_d)$.
10. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps. If f and g are embeddings, then gf is an embedding. If gf and g are embeddings, then g is an embedding. If $gf = \mathrm{id}$, then f is an embedding. An embedding is open (closed) if and only if its image is open (closed). If $f\colon X \to Y$ is a homeomorphism and $A \subset X$, then the map $A \to f(A)$, induced by f, is a homeomorphism.
11. Let $B \subset A \subset X$. Then the subspace topologies on B, considered as subspace of A and of X, coincide. For the closures of B in A and X the relation $\overline{B}^A = \overline{B}^X \cap A$ holds. If A is closed in X, then both closures coincide.
12. The spaces $X = \mathbb{R}^2 \smallsetminus (]-\infty, 0] \times 0)$ and $Y = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ are homeomorphic. The following spaces are pairwise homeomorphic: $X_1 = \mathbb{R}^2 \smallsetminus \{0\}$, $X_2 = \mathbb{R}^2 \smallsetminus D_1(0)$, $X_3 = \mathbb{R}^2 \smallsetminus ([-1,1] \times 0)$, $X_4 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $X_5 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$. (Examples of this type show that homeomorphic spaces can have quite different shape in the sense of elementary geometry. The spaces are also homeomorphic to the product space $S^1 \times \mathbb{R}$. Draw figures.)
13. Let X and Y be topological spaces. Suppose X is the union of A_1 and A_2. Let $f\colon X \to Y$ be a map with continuous restrictions $f_j = f|A_j$. Then f is continuous if $(\overline{X \smallsetminus A_1}) \cap (X \smallsetminus A_2) = \emptyset$, $(\overline{X \smallsetminus A_2}) \cap (X \smallsetminus A_1) = \emptyset$. Show that C is closed in X if the $C \cap A_j$ are closed in A_j.

1.4 Quotient Spaces

In geometric and algebraic topology many of the important spaces are constructed as quotient spaces. They are obtained from a given space by an equivalence relation. Although the quotient topology is easily defined, formally, it takes some time to work with it. In several branches of mathematics quotient object are more difficult to handle than subobjects. Even if one starts with a nice and well-known space its quotient spaces may have strange properties;

usually one has to add a number of hypotheses in order to exclude unwanted phenomena. In other words: Quotient spaces do not, in general, inherit desirable properties from the original space.

The notion of a quotient spaces also makes precise the intuitive idea of gluing and pasting spaces. We will discuss examples after we have introduced compact spaces.

Let X be a topological space and $f\colon X \to Y$ a surjective map onto a set Y. Then $\mathcal{S} = \{U \subset Y \mid f^{-1}(U) \text{ open in } X\}$ is a topology on Y. This is the finest topology on Y such that f is continuous. We call $\mathcal{S}$ the ***quotient topology*** on Y with respect to f. A surjective map $f\colon X \to Y$ between topological spaces is called ***identification*** or ***quotient map*** if it has the property: $U \subset Y$ open $\Leftrightarrow$ $f^{-1}(U) \subset X$ open. If $f\colon X \to Y$ is a quotient map, then Y is called ***quotient space*** of X. This simple definition of a quotient space is a good example for the power of the general concept of a topological space. In general it is impossible to construct quotient spaces in the category of metric spaces.

We recall that a surjective map $f\colon X \to Y$ is essentially the same thing as an equivalence relation on X. If R is an equivalence relation on X, then X/R denotes the set of equivalence classes. The ***canonical map*** $p\colon X \to X/R$ assigns to $x \in X$ its equivalence class. If $f\colon X \to Y$ is surjective, then $x \sim y \Leftrightarrow f(x) = f(y)$ is an equivalence relation R_f on X. There is a canonical bijection $\varphi\colon X/R_f \to Y$ such that $\varphi p = f$. The ***quotient space*** X/R is defined to be the set X/R together with the quotient topology of the canonical map $p\colon X \to X/R$.

Property (2) of the next proposition characterizes quotient maps f in categorical terms. We call it the ***universal property of a quotient map***.

(1.4.1) Proposition. *Let $f\colon X \to Y$ be a surjective set map between topological spaces. The following assertions are equivalent:*

(1) *f is a quotient map.*

(2) *A set map $g\colon Y \to Z$ into any topological space is continuous if and only if gf is continuous.*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "gf"] & \\ Y \arrow[r, dotted, "g"'] & Z \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). If g is continuous, then gf is continuous. Suppose gf is continuous and $U \subset Z$ open. Then $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X, hence, by definition of the quotient topology, $g^{-1}(U)$ is open in Y. Thus g is continuous.

(2) $\Rightarrow$ (1). Since $\mathrm{id}(Y)$ is continuous, the composition $\mathrm{id}(Y) \circ f = f$ is continuous. Let $\mathcal{T}$ denote the topology on Y and let $\mathcal{S}$ be the quotient topology with respect to f. By (1) $\Rightarrow$ (2), the map $\mathrm{id}\colon (Y, \mathcal{S}) \to (Y, \mathcal{T})$ is continuous. By (2), $\mathrm{id}\colon (Y, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous. Hence $\mathcal{T} = \mathcal{S}$. $\square$

(1.4.2) Proposition. *Let $f\colon X \to Y$ be a quotient map. Let B be open or closed in Y and set $A = f^{-1}(B)$. Then the restriction $g\colon A \to B$ of f is a quotient map.*

Proof. Since $f(X) = Y$, we have $g(A) = B$. Let $B \subset Y$ be open and thus $A \subset X$ open. Suppose $U \subset B$ is such that $g^{-1}(U)$ is open in A. Then $g^{-1}(U)$ is open X. Since f is a quotient map, U is open in Y and hence in B. If $U \subset B$ is open, then $g^{-1}(U)$ is open in A, since g is continuous. This shows that g is a quotient map. A similar reasoning works if B is closed. □

(1.4.3) Proposition. *Let $f\colon X \to Y$ be surjective, continuous and open (or closed). Then f is a quotient map. The restriction $f_B\colon f^{-1}(B) \to B$ is open (or closed) for each $B \subset Y$, hence a quotient map.*

Proof. Suppose $f^{-1}(C)$ is open in X. Since f is surjective, $f(f^{-1}(C)) = C$. Therefore if f is open (closed), then C is open (closed). Now use (1.3.4). □

(1.4.4) Example. Let $f\colon U \to V$ be a continuously differentiable map between open subsets of Euclidean spaces $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$. Suppose that the Jacobian has maximal rank n at each point of U. Then, by the rank theorem of calculus, f is open. ◇

(1.4.5) Example. The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \smallsetminus 0$ is open (see (1.4.4)). Similarly $p\colon \mathbb{R} \to S^1, t \mapsto \exp(2\pi i t)$ is open (see (1.4.3)). The kernel of p is $\mathbb{Z}$. Let $q\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map onto the factor group. There is a bijective map $\alpha\colon \mathbb{R}/\mathbb{Z} \to S^1$ which satisfies $\alpha \circ q = p$. Since p and q are quotient maps, α is a homeomorphism (use part (2) of (1.4.1)). The continuous periodic functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x+1) = f(x)$ therefore correspond to continuous maps $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and to continuous maps $S^1 \to \mathbb{R}$ via composition with q or p. In a similar manner the exponential function exp induces a homeomorphism from the factor group with quotient topology $bbbc/bbbz$ with $\mathbb{C}^*$. ◇

We add some general remarks about working with equivalence relations. An equivalence relation $\sim$ on a set X can be specified by the set

$$R = \{(x,y) \in X \times X \mid x \sim y\}$$

(sometimes called the graph of the relation). A subset $R \subset X \times X$ is an equivalence relation if:

(1) $(x,x) \in R$ for all $x \in X$;
(2) $(x,y) \in R \Rightarrow (y,x) \in R$;
(3) $(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$.

Any subset $S \subset X \times X$ generates an equivalence relation; it is the intersection of all equivalence relations containing S. A quotient spaces Y is often defined by specifying a typical set S; one says, Y is obtained from X by identifying the points a and b whenever $(a,b) \in S$; it is then understood that one works with the equivalence relation generated by S. A subset of X is said to be ***saturated*** with respect to the equivalence relation if it is a union of equivalence classes.

Let A be a subspace of X. We denote by X/A be the quotient space of X where A is identified to a point[2]. In the case that $A = \emptyset$ we set $X/A = X + \{*\}$, the space X with an additional point $\{*\}$ (topological sum). A map $f\colon X \to Y$ factors over the quotient map $X \to X/A$, $A \neq \emptyset$, if and only if f sends A to a point; the induced map $X/A \to Y$ is continuous if and only if f is continuous.

(1.4.6) Example (Interval and circle). Let $I = [0, 1]$ be the unit interval and $\partial I = \{0, 1\}$ its boundary. Then $I \to S^1, t \mapsto \exp(2\pi i t)$ induces a bijective continuous map $q\colon I/\partial I \to S^1$. It is a homeomorphism, see (1.10.6). $\diamond$

(1.4.7) Example (Cylinder). We identify in the square $[0, 1] \times [0, 1]$ the point $(0, t)$ with the point $(1, t)$. The result is homeomorphic to the cylinder $S^1 \times [0, 1]$. $\diamond$

(1.4.8) Example (Möbius band). We identify in the square $[0, 1] \times [0, 1]$ the point $(0, t)$ with the point $(1, 1 - t)$. The result is the ***Möbius band*** M. For more details see (1.11.2). $\diamond$

(1.4.9) Example (Projective plane). Important objects of geometry are the projective spaces. They arise as quotient spaces and not as subspaces of Euclidean spaces. They will be discussed later in detail. We mention here the real projective plane $\mathbb{RP}^2$; one of its definitions is as the quotient space of S^2, by the relation $x \sim -x$ (antipodal points identified), see **??**. $\diamond$

(1.4.10) Example. There exist continuous surjective maps $p\colon I \to I \times I$ (Peano curves, see Section 2.1). A map of this type is a quotient map (1.10.6). Thus, although set-theoretically a quotient of a set is smaller, topologically the quotient can become "larger" (here a 2-dimensional space as a quotient of a 1-dimensional space). $\diamond$

Problems

1. Let $f\colon A \to B$ and $g\colon B \to C$ be continuous. If f and g are quotient maps, then gf is a quotient map. If gf is a quotient map, then g is a quotient map. If $gf = \mathrm{id}$, then g is a quotient map.
2. Let $f\colon X \to Y$ be a closed quotient map. If X is normal, then Y is normal.
3. Identify in $D^n + D^n$ a point $x \in S^{n-1}$ in the first summand with the same point in the second summand. The result is homeomorphic to S^n.
4. Identify in $S^{n-1} \times [0, 1]$ the set $S^{n-1} \times 0$ to a point and the set $S^{n-1} \times 1$ to another point. The result is homeomorphic to S^n.

[2] A notation of this type is also used for factor groups and orbit spaces.

5. Identify in $\mathbb{R}^n + \mathbb{R}^n$ a point $x \in \mathbb{R}^n \smallsetminus 0$ in the first summand with the point $x/\|x\|^2$ in the second summand. The result is homeomorphic to S^n.
6. Identify in $S^k \times D^{l+1} + S^{k+1} \times D^l$ a point $z \in S^k \times S^l$ in the first summand with the same point in the second summand. The result is homeomorphic to S^{k+l+1}.
7. Identify in $S^1 \times [-1,1]$ each point (z,t) with the point $(-z,-t)$. Show that the quotient is homeomorphic to the Möbius band.
8. Find a subspace $M \subset \mathbb{R}^3$ that is homeomorphic to the Möbius band.
9. Show that a space with four points is homeomorphic to a quotient of a Hausdorff space.

1.5 Products and Sums

Let $(X_j, \mathcal{O}_j \mid j \in J)$ be a family of topological spaces. The product set $X = \prod_{j\in J} X_j$ is the set of all families $(x_j \mid j \in J)$ with $x_j \in X_j$. We have the projection $\mathrm{pr}_i \colon X \to X_i$, $(x_j) \mapsto x_i$ into the i-th factor. Let X_j, Y_j be topological spaces and $f_j \colon X_j \to Y_j$ maps. We have the product map $\prod f_j \colon \prod X_j \to \prod Y_j$, $(x_j \mid j \in J) \mapsto (f_j(x_j) \mid j \in J)$.

The family of all pre-images $f^{-1}(U_j)$, $U_j \subset X_j$ open in X_j, is the subbasis for the ***product topology*** $\mathcal{O}$ on X. We call $(X, \mathcal{O})$ the ***topological product*** of the spaces $(X_j, \mathcal{O}_j)$.

(1.5.1) Proposition. *The product topology is the coarsest topology for which all projections* pr_j *are continuous. A set-map* $f \colon Y \to X$ *from a space* Y *into* X *is continuous if and only if all maps* $\mathrm{pr}_j \circ f$ *are continuous. The product* $\prod_j f_j$ *of continuous maps* $f_j \colon X_j \to Y_j$ *is continuous.*

Proof. If pr_j is continuous, then $\mathrm{pr}_j(U)$ is open for open $U \subset X_j$. The product topology has by definition the sets $\mathrm{pr}_j(U)$ as subbasis. If f is continuous, then also $\mathrm{pr}_j \circ f$. For the converse use (1.1.5). □

From (1.5.1) we see that $X = \prod X_j$ together with the projections pr_j is a categorical product of the family (X_j) in the category TOP.

The product of two spaces X_1, X_2 is denoted $X_1 \times X_2$. Similarly $f_1 \times f_2$ for the product of maps. The "identity" $\mathrm{id} \colon X_1 \times (X_2 \times X_3) \to (X_1 \times X_2) \times X_3$ is a homeomorphism. In general, the topological product is associative, i.e., compatible with arbitrary bracketing (this is a general fact for products in categories). The canonical identification $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ is a homeomorphism.

The product of two quotient maps is not, in general, a quotient map. Here is an example. Let $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map which identifies the subset $\mathbb{Z}$ to a point. (Hence this symbol is not the factor group!) Then the product of p with the identity of $\mathbb{Q}$ is not a quotient map. For a proof see (2.10.25). But the product with the identity of a locally compact space is again a quotient map, see (2.2.4) and (2.9.6). The category of compactly generated spaces (see Section 2.10) is designed to remedy this defect.

(1.5.2) Proposition. *A space X is separated if and only if the diagonal $D = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. Let $x \neq y$. Choose disjoint open neighbourhoods U of x and V of y. Then $U \times V$ is open in $X \times Y$ and $D \cap (U \times V) = \emptyset$. Hence $X \times X \smallsetminus D$ is a union of sets of type $U \times V$ and therefore open.

Conversely, let $X \times X \smallsetminus D$ be open. If $x \neq y$ then $(x,y) \in X \times X \smallsetminus D$. By definition of the product topology, there exists a basic open set $U \times V$ such that $(x,y) \in U \times V \subset X \times X \smallsetminus D$. But this means: U and V are disjoint open neighbourhoods of x and y. □

(1.5.3) Proposition. *Let $f, g \colon X \to Y$ be continuous maps into a Hausdorff space. Then the coincidence set $A = \{x \mid f(x) = g(x)\}$ is closed in X.*

Proof. The diagonal map $d \colon X \to X \times X,\ x \mapsto (x,x)$ is continuous. We have $A = ((f \times g)d)^{-1}(D)$. □

(1.5.4) Proposition. *Let $f \colon X \to Y$ be surjective, continuous, and open. Then Y is separated if and only if $R = \{(x_1,x_2) \mid f(x_1) = f(x_2)\}$ is closed in $X \times X$.*

Proof. If Y is separated, then the diagonal D is closed in $Y \times Y$ and therefore $R = (f \times f)^{-1}(D)$ closed.

Suppose R is closed and hence $X \times X \smallsetminus R$ open. Since f is surjective, we have $Y \times Y \smallsetminus D = (f \times f)(X \times X \smallsetminus R)$. If f is open, then also $f \times f$. Hence $Y \times Y \smallsetminus D$ is open and Y separated, by (1.5.2). □

Let $(X_j \mid j \in J)$ be a family of topological spaces. Suppose the X_j are non-empty and pairwise disjoint. The set

$$\mathcal{O} = \{U \subset \textstyle\coprod X_j \mid U \cap X_j \subset X_j \text{ open for all } j\}$$

is a topology on the disjoint union $\coprod X_j$. We call $(\coprod X_j, \mathcal{O})$ the ***topological sum*** of the X_j. A sum of two space is denoted $X_1 + X_2$. The assertions in the next proposition are easily verified from the definitions.

(1.5.5) Proposition. *The topological sum has the properties:*

(1) *The subspace topology of X_j in $\coprod X_j$ is the original topology.*
(2) *Let X be the union of the family $(X_j \mid j \in J)$ of pairwise disjoint subset. Then X is the topological sum of the subspace X_j if and only if the X_j are open.*
(3) *A map $f \colon \coprod X_j \to Y$ is continuous if and only if for each $j \in J$ the restriction $f|X_j \colon X_j \to Y$ is continuous.* □

The topological sum together with the canonical inclusions $X_j \to \coprod X_j$ is a categorical sum in TOP.

(1.5.6) Proposition. *Let $(U_j \mid j \in J)$ be a covering of a space X. We have a canonical map $p\colon \coprod_{j \in J} U_j \to X$ which is the inclusion on each summand. The following are equivalent:*

(1) *The map p is a quotient map.*

(2) *A set map $f\colon X \to Y$ is continuous if and only if each restriction $f|U_j$ is continuous.*

(3) *$U \subset X$ is open if and only if $U \cap U_j$ is open in U_j for each $j \in J$. Similarly if "open" is replaced by "closed".*

These properties hold if the U_j are open, or if the U_j are closed and J is finite. This is essentially a reformulation of (1.3.3). *We say X carries the* ***colimit topology*** *with respect to the family of subspaces $(U_j \mid j \in J)$ if one of the equivalent statements holds.*

Proof. The map is a quotient map means: $U \subset X$ is open if and only if $p^{-1}(U) \subset \coprod X_j$ is open. And the latter is the case if and only if for each $j \in J$ the set $p^{-1}(U_j) = U \cap X_j$ is open in X_j. This shows the equivalence of (1) and (3).

By the universal property of quotient maps, p is a quotient map if and only if f is continuous if fp is continuous. And fp is continuous if and only if each restriction $f|U_j$ is continuous, by definition of the sum topology. This shows the equivalence of (1) and (2). □

(1.5.7) Proposition. *Let X be a set which is covered by a family $(X_j \mid j \in J)$ of subsets. Suppose each X_j carries a topology such that the subspace topologies of $X_i \cap X_j$ in X_i and X_j coincide and these subspaces are closed in X_i and X_j. Give X the quotient topology with respect to the canonical map $p\colon \coprod X_j \to X$. Then the subspace topology of $X_j \subset X$ coincides with the given topology and X_j is closed in X. The space has the colimit topology with respect to the X_j. Similarly if "closed" is replaced by "open".*

Proof. Let $A \subset X_i$ be closed. Then $A \cap X_j$ ist closed in the subspace $X_i \cap X_j$ of X_i. By assumption, it is also closed in this subspace of X_j. Since $X_i \cap X_j$ is closed in X_j, we see that $A \cap X_j$ is closed in X_j. From $p^{-1}p(A) = \coprod_j A \cap X_j$ we see, def definition of the topological sum, that $p^{-1}p(A)$ is closed in $\coprod_j X_j$. Hence $p(A)$ is closed in X, by definiton of the quotient topology. This shows that $p(x_i)$ is closed in X and $p\colon X_i \to p(X_i)$ is bijective, continuous and closed, hence a homeomorphism. □

Problems

1. Let $(X_j \mid j \in J)$ be spaces and $A_j \subset X_j$ non-empty subspaces. Then $\prod_{j \in J} \overline{A}_j =$

$\overline{\prod_{j\in J} A_j}$. The product $\prod_{j\in J} A_j$ is closed if and only if the A_j are closed.
2. The projections $\mathrm{pr}_k\colon \prod_j X_j \to X_k$ are open maps, and in particular quotient maps. (The X_j are non-empty.)
3. Show that $\mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x$ is not a closed map.
4. A discrete space is the topological sum of its points. There is always a canonical homeomorphism $X \times \coprod_j Y_j \cong \coprod_j (X \times Y_j)$. For each $y_0 \in Y$ the map $X \to X \times Y, x \mapsto (x, y_0)$ is an embedding; the image is closed if and only if the point y_0 is closed in Y. If $f\colon X \to Y$ is continuous, then $X \to X \times Y, x \mapsto (x, f(x))$ is an embedding; the image is closed if Y is a Hausdorff space.
5. The topological product of separated spaces is separated. (There exist normal spaces X such that $X \times [0,1]$ is not normal, see [**?**].)
6. The construction of the product topology can be generalized as follows. Let $(Y_j \mid j \in J)$ be a family of topological spaces. Let $f_j\colon X \to Y_j$ be maps from a set X to Y_j. There exists a unique coarsest topology on X such that all f_j are continuous. A map of a topological space Z into X with this topology is continuous if and only if the composition with each f_j is continuous. This topology on X is called the ***initial topology*** with respect to the f_j. An example of an initial topology is also the subspace topology.
7. $\mathrm{Bd}(A) \times \overline{B} \cup \overline{A} \times \mathrm{Bd}(B) = \mathrm{Bd}(A \times B)$.
8. For metric spaces (X_i, d_i), $1 \le i \le n$, the metric $d_\infty((x_i)(y_i)) = \max(d_k((x_i),(y_i))$ induces on $\prod X_i$ the product topology. Similarly for the metric $d_1((x_j),(y_j)) = \sum_{i=1}^n d_i(x_i, y_i)$.
9. Let $((X_j, d_j) \mid j \in \mathbb{N})$ be a countable family of metric spaces with metric bounded by 1. Then $d((x_j),(y_j)) \mid j \in \mathbb{N}) = \sum_{n=1}^\infty 2^{-n} d_n(x_n, y_n)$ defines a metric on the product $\prod_{j=1}^\infty X_j$ which induces the product topology.
10. Let $(X_j \mid j \in J)$ be a family if spaces. There is a topology on $X = \prod_{j\in J} X_j$ which has all products $\prod_{j\in J} U_j$, $U_j \subset X_j$ open, as a subbasis. Call this the ***box topology***. Let Y be the product of a countable number of the discrete space T with two points. Show that the box topology on Y is the discrete topology and is thus different from the product topology.

1.6 Pullback and Pushout

1.6.1 Pullback. Let $f\colon X \to B$ and $g\colon Y \to B$ be continuous maps. Let $Z = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$ with the subspace topology of $X \times Y$. We have the projections onto the factors $F\colon Z \to Y$ and $G\colon Z \to X$. The commutative diagram

$$\begin{tikzcd} Z \arrow[r, "F"] \arrow[d, "G"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback in TOP, by (1.2.1) and (1.5.1). The space Z is sometimes written

$Z = X \times_B Y$ and called the product of X and Y over B (the product in the category TOP_B of spaces over B). $\diamond$

Pullbacks allow to convert liftings into sections. A ***lifting*** of f ***along*** g is a map $\varphi\colon X \to Y$ such that $g\varphi = f$.

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[ur, dotted, "\varphi"] \arrow[r, "f"] & B \end{tikzcd}$$

A section of G is a map $\Sigma\colon X \to Z$ such that $G\Sigma = \mathrm{id}(X)$. If Σ is a section, then $F\Sigma$ is a lifting. If φ is a lifting, then there exists a unique section Σ such that $F\Sigma = \varphi$. Let $A \subset X$ and $a\colon A \to Y$ be given. Then $\varphi|A = a$ if and only if $F\sigma|A = a$.

Let $\sigma\colon B \to Y$ be a section of g. There exists a unique section $\Sigma\colon X \to Z$ of G such that $\sigma f = F\Sigma$. We call Σ the ***induced section***.

Let $g\colon Y \to B$ and $A \subset B$ be given. Let $G\colon X = g^{-1}(A) \to A$ be the restriction of g. Then

$$\begin{tikzcd} X \arrow[r, "\subset"] \arrow[d, "G"] & Y \arrow[d, "g"] \\ A \arrow[r, "\subset"] & B \end{tikzcd}$$

is a pullback.

1.6.2 Pushout. Let $j\colon A \to X$ and $f\colon A \to B$ be continuous maps and form a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

in the category SET of sets. Then Y is obtainable as a quotient of the set $X + B$. We give Y the quotient topology via $\langle F, J \rangle\colon X + B \to Y$. Then the resulting diagram is a pushout in TOP. The space Y is sometimes written $X +^A B$ and called the sum of X and B under A (the sum in the category TOP^A of spaces under A). $\diamond$

Pushouts allow to convert extensions into retractions. An extensions of f over j is a map $\tau\colon X \to B$ such that $\tau j = f$. A retraction of J is a map $R\colon Y \to B$ such that $RJ = \mathrm{id}(B)$. If R is a retraction, then RF is an extension. If τ is an extension, there exists a unique retraction R such that $RF = \tau$.

Problems

1. Consider the commutative diagrams

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[d, "h"] \\ D \arrow[r, "d"] & E \arrow[r, "e"] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "ba"] \arrow[d, "f"] & C \arrow[d, "h"] \\ D \arrow[r, "ed"] & F. \end{tikzcd}$$

Suppose the first square is a pushout. Then the second square is a pushout if and only if the third square is a pushout. (This holds in any category.)
2. Let A and B be subspaces of X such that $X = A \cup B^\circ = A^\circ \cup B$. Then the diagram of inclusions

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

is a pushout in TOP.

1.7 Clutching Data

Suppose the set X is the union of a family $(U(j) \mid j \in J)$ of subsets $U(j)$. Moreover, suppose that for each $j \in J$ there is given a bijection $h_j \colon U_j \to U(j)$ with some other set U_j. We interprete h_j as a parametrization (or coordinate description) of $U(j)$. Situations of this type occur for manifolds or bundles (charts, bundle charts). Let

$$U_i \supset U_i^j = h_i^{-1}(U(i) \cap U(j)).$$

The bijection

$$g_i^j = h_j^{-1} h_i \colon U_i^j \to U_j^i$$

is then called a coordinate transformation. It often happens that the U_i are topological spaces; we then want to provide X with a topology such that the h_i become embeddings. For this purpose it is necessary that the h_i are compatible.

(1.7.1) Proposition. *Suppose the U_i are topological spaces, the $U_i^j \subset U_i$ open subspaces and the g_i^j homeomorphisms. Then there exists a unique topology on X such that the $U(i)$ are open subsets and the $h_i \colon U_i \to U(i)$ homeomorphisms.*

Proof. Let $h \colon \coprod_{j \in J} U_j \to X$ be the canonical map which coincides on U_i with $h_i \colon U_i \to U(i) \subset X$. Give X the quotient topology with respect to h. We claim that h is an open map. Let $U \subset U_i$ be open. Then

$$h^{-1}h(U) = \coprod_{j \in J} g_i^j(U \cap U_i^j).$$

From our hypotheses we conclude that this set is open. By definition of the quotient topology, $h(U)$ is open. In particular $h(U_i) = U(i)$ is open and $h_i \colon U_i \to U(i)$ bijective and open, hence a homeomorphism. An open set in $\coprod U_j$ is a union of open sets in the summands, hence the image under h is a union of open sets.

Conversely, assume X carries a topology such that the $h_i \colon U_i \to U(i)$ are homeomorphisms onto open subsets. Then the $U_i^j \subset U_i$ are open and the g_i^j are homeomorphisms. Suppose $h^{-1}(U)$ is open. Then the set

$$h^{-1}(U) \cap U_i = h_i^{-1}(U \cap U(i))$$

is open, hence $U \cap U(i) \subset U(i)$ open, and therefore $U = \bigcup(U \cap U(i))$ open. This shows that h is a quotient map. □

An important method for the construction of spaces is to paste spaces. Let $(U_j \mid j \in J)$ be a family of sets. Assume that for each pair $(i,j) \in J \times J$ a subset $U_i^j \subset U_i$ is given as well as a map $g_i^j \colon U_i^j \to U_j^i$. We require the axioms:

(1) $U_j = U_j^j$ and $g_j^j = \mathrm{id}$.

(2) For each triple $(i,j,k) \in J \times J \times J$ the inclusion $g_i^j(U_i^j \cap U_i^k) \subset U_j^k$ holds; thus we have an induced map

$$g_i^j \colon U_i^j \cap U_i^k \to U_j^i \cap U_j^k.$$

We require $g_j^k \circ g_i^j = g_i^k$, considered as maps from $U_i^j \cap U_i^k$ to $U_k^j \cap U_k^i$.

Then we call the families (U_j, U_j^k, g_j^k) a ***clutching datum***. We apply (1) and (2) for the triples (i,j,i) and (j,i,j) and conclude that g_i^j and g_j^i are inverse bijections between the sets U_i^j and U_j^i.

Given a clutching datum, we have the equivalence relation on the disjoint sum $\coprod_{j \in J} U_j$:

$$x \in U_i \sim y \in U_j \quad \Leftrightarrow \quad x \in U_i^j \text{ and } g_i^j(x) = y.$$

Let X denote the set of equivalence classes and let $h_i \colon U_i \to X$ be the map which sends $x \in U_i$ to its class. Then we have:

(1.7.2) Lemma. *The map h_i is injective. Set $U(i) = \mathrm{image}\ h_i$. Then we have $U(i) \cap U(j) = h_i(U_i^j)$.* □

Conversely, assume that X is a quotient of $\coprod_{j \in J} U_j$ such that each $h_i \colon U_i \to X$ is injective with image $U(i)$. Let $U_i^j = h_i^{-1}(U(i) \cap U(j))$ and $g_i^j = h_j^{-1} \circ h_i \colon U_i^j \to U_j^i$. Then the (U_i, U_i^j, g_i^j) are a clutching datum. If we apply the construction above to this datum, we get back X and the h_i.

We now turn our attention to a topological situation.

(1.7.3) Proposition. *Let (U_i, U_i^j, g_i^j) be a clutching datum. Assume that the U_i are topological spaces, the $U_i^j \subset U_i$ open (closed) subsets, and the $g_i^j\colon U_i^j \to U_j^i$ homeomorphisms. Let X carry the quotient topology with respect to the quotient map $p\colon \coprod_{j\in J} U_j \to X$. Then the following holds:*

(1) *The map h_i is a homeomorphism onto an open (closed) subset of X. If the $U_i^j \subset U_i$ are open, then p is open.*

(2) *Suppose the U_i are Hausdorff spaces and the U_i^j open. Then X is a Hausdorff space if and only if for each pair (i,j) the map $\gamma_i^j\colon U_i^j \to U_i \times U_j$, $x \mapsto (x, g_i^j(x))$ is a closed embedding.*

Proof. (1) is a consequence of (1.7.1).

(2) Suppose the conditions in (2) hold. We have to show that the diagonal $D \subset X \times X$ is closed. Since p is open, the map $p \times p$ is a quotient map. We have to verify that $(p \times p)^{-1}(D)$ is closed. This is the case if and only if the intersection with $U_k \times U_l \subset (\coprod U_i) \times (\coprod U_j)$ is closed. This intersection is the image of γ_k^l. □

(1.7.4) Example. Let $U_1 = U_2 = \mathbb{R}^n$ and $V_1 = V_2 = \mathbb{R}^n \smallsetminus 0$. Let $\varphi = \mathrm{id}$. Then the graph of φ in $\mathbb{R}^n \times \mathbb{R}^n$ is not closed. The resulting locally Euclidean space is not Hausdorff. If we use $\varphi(x) = x \cdot \|x\|^{-2}$, then the result is homeomorphic to S^n, see (1.3.5). ◇

1.8 Adjunction Spaces

Special cases of pushout diagrams are used to define adjunction spaces. Let $j\colon A \subset X$ be an inclusion and $f\colon A \to Y$ a continuous map. We identify in the topological sum $X + Y$ for each $a \in A$ the point $a \in X$ with the point $f(a) \in Y$, i.e., we consider the equivalence relation on $X + Y$ with equivalence classes $\{z\}$ for $z \notin A + f(A)$ and $f^{-1}(z) + \{z\}$ for $z \in f(A)$. The quotient space Z is sometimes denoted by $Y \cup_f X$ and called the ***adjunction space*** obtained by ***attaching*** X via f to Y. The canonical inclusions $X \to X + Y$ and $Y \to X + Y$ induce maps $F\colon X \to Y \cup_f X$ and $J\colon Y \to Y \cup_f X$. The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[d, "J"] \\ X \arrow[r, "F"] & Z \end{tikzcd} \quad = Y \cup_f X$$

is a pushout in TOP.

(1.8.1) Proposition. *The data of the pushout have the properties:*

(1) *J is an embedding.*

(2) *If A is closed in X, then J is a closed embedding.*
(3) *If A is closed in X, then F restricted to $X \smallsetminus A$ is an open embedding.*
(4) *If X, Y are T_1-spaces (T_4-spaces), then $Y \cup_f X$ is a T_1-space (T_4-space).*
(5) *If f is a quotient map, then F is a quotient map.*
(6) *If $A \neq \emptyset$ and X, Y are connected, then $Y \cup_f X$ is connected.*
(7) *If $A \neq \emptyset$ and X, Y are path connected, then $Y \cup_f X$ is path connected.*

Proof. (1) Let $U \subset Y$ be open. The pre-image of $J(U)$ in $X + Y$ is $f^{-1}(U) + U$. Since j is an embedding, there exists an open subset $V \subset X$ such that $f^{-1}(U) = A \cap V$. The set $V + U$ is saturated, its image W in Z is open, and $W \cap J(Y) = J(U)$. Hence $J(U)$ is open in Z.

(2) Let $C \subset Y$ be closed. Then $p^{-1}(J(C)) = f^{-1}(C) + C$ is closed in $X + Y$, because $f^{-1}(C)$ is closed in A and A is closed in X. Hence $J(C)$ is closed.

(3) If $U \subset X \smallsetminus A$ is open, then $p^{-1}F(U) = U$ is open in $X + Y$. Hence $F(U)$ is open.

(4) (T_1) The points of $Y \cup_f X$ have pre-images in $X + Y$ of the form $\{z\}$ for $z \notin A + j(A)$ or $f^{-1}(z) + \{z\}$. Since these sets are closed, points are closed in $Y \cup_f X$.

(T_4) Let C and D be disjoint closed subsets of $Y \cup_f X$. Choose a function $a\colon Y \to [0,1]$ such that $a(C \cap Y) \subset \{0\}$ and $a(D \cap Y) \subset \{1\}$. Define a function $b\colon A \cup p^{-1}(C) \cap X \cup p^{-1}(D) \cap X \to [0,1]$ which equals f on A, is zero on $p^{-1}(C) \cap X$ and one on $p^{-1}(D) \cap X$. By the Tietze extension theorem (2.3.1), b can be extended to $c\colon X \to [0,1]$. The function a and c together yield a function $d\colon Y \cup_f X \to [0,1]$ with $d(C) \subset \{0\}$, $d(D) \subset \{1\}$. One can also use (**??**) instead of (2.3.1).

(5) Let $g\colon Y \cup_f X \to Z$ be given. Assume gF is continuous. Since $gFj = gJf$ and f is a quotient map, gJ is continuous. The functions gF and gJ together yield, by the pushout property, a continuous function g.

(6) and (7) are left as exercises. □

Because of (2) and (3) we identify $X \smallsetminus A$ with the open subspace $F(X \smallsetminus A)$ and Y with the closed subspace $J(Y)$. In this sense, $A \cup_f X$ is the union of the disjoint subsets $X \smallsetminus A$ and Y.

(1.8.2) Proposition. *Suppose we have closed subspaces $A \subset X' \subset X$ and a subspace $Y' \subset Y$. Let $f\colon A \to Y$ have an image in Y' and denote by $g\colon A \to Y'$ the restriction of f. Then $Y' \cup_g X'$ is a subspace of $Y \cup_f X$. This subspace is open (closed) if X' and Y' are open (closed).*

Proof. Let $C' + D' \subset X' + Y'$ be a saturated closed set. There exist closed sets $C \subset X$ with $C' = C \cap X'$ and $D \subset Y$ with $D \cap Y' = D'$. The set $C + D$ is saturated, since $A \subset X'$. Hence $p(C + D)$ is closed in $Y \cup_f X$ and has intersection $p(C' + D')$ with $Y' \cup_g X'$.

If X', Y' are open (closed), then $X' + Y'$ is a saturated open (closed) subset of $X + Y$. Hence $Y' \cup_g X' = p(X' + Y')$ is open (closed) in $Y \cup_f X$. □

Let $x\colon X \to A$ be a retraction of j. Since $fr\colon X \to Y$, $\mathrm{id}\colon Y \to Y$ satisfy $\mathrm{id} \circ f = fr \circ j$ we have, by the pushout property, an induced retraction $R\colon Y \cup_f X \to Y$ of J.

(1.8.3) Proposition. *The space $Y \cup_f X$ is a Hausdorff space, provided the following holds: Y is a Hausdorff space, X is regular, and A is a retract of an open neighbourhood in X.*

Proof. Let $z_1, z_2 \in Y \cup_f X = Z$ be different points. We distinguish three cases. If $z_1, z_2 \in X \smallsetminus A$, they can be separated by neighbourhoods in $X \smallsetminus A$, since this space is Hausdorff. Since $X \smallsetminus A$ is open in Z, the same neighbourhoods separate in Z.

Let $z_1 \in X \smallsetminus A$ and $z_2 \in Y$. Since X is regular, there exist open disjoint sets U, V of X with $z_1 \in U \subset X \smallsetminus A$ and $A \subset V$. Then U and $V \cup_f Y$ are disjoint neighbourhoods of z_1 and z_2.

Let $z_1, z_2 \in Y$. Since Y is Hausdorff, we can choose open disjoint neighbourhoods W_j of z_j in Y. Let $r\colon U \to A$ be a retraction of an open set $U \subset X$ onto A. The sets $r^{-1}f^{-1}(W_j)$ are open in U and X; they are disjoint, since the $f^{-1}(W_j)$ are disjoint and r is a retraction. The sets $r^{-1}f^{-1}(W_j) + W_j$ are open and saturated in $X + Y$. Hence their images in $Y \cup_f X$ are separating neighbourhoods of z_1 and z_2. □

(1.8.4) Proposition. *Let a commutative diagram* (1.8.1) *with closed embeddings j and J be given. Suppose F induces a bijection $X \smallsetminus A \to Z \smallsetminus Y$. Then the diagram is a pushout, provided:*

(1) *$F(X) \subset Z$ is closed.*
(2) *$F\colon X \to F(X)$ is a quotient map.*

(2) *holds if X is compact and Z Hausdorff.*

Proof. Let $g\colon X \to U$ and $h\colon Y \to U$ be given such that $gj = hf$. The diagram is a set-theoretical pushout. Therefore there exists a unique set map $\varphi\colon Z \to U$ with $\varphi F = g$, $\varphi J = h$. Since J is a closed embedding, $\varphi | J(Y)$ is continuous. Since F is a quotient map, $\varphi | F(X)$ is continuous. Thus φ is continuous, since $F(X)$ and $J(Y)$ are closed sets which cover Z. □

Problems

1. The transitivity of pushouts in the diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "j'"] \arrow[d, "F"] & X \arrow[d, "F'"] \\
Y \arrow[r, "J"] & Y \cup_f B \arrow[r] & (Y \cup_f B) \cup_F X
\end{tikzcd}$$

yields a canonical homeomorphism $Y \cup_f X \cong (Y \cup_f B) \cup_F X$. The transitivity of pushouts in the next diagram yields a canonical homeomorphism

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "J"] & Z \arrow[d] \\ X \arrow[r] & Y \cup_f X \arrow[r] & Z \cup_g (Y \cup_f X) \end{tikzcd}$$

$Z \cup_{fg} X \cong Z \cup_g (Y \cup_f X)$. (Here we view J as a closed inclusion.)
2. Let $A_i \subset X_i$ be closed subsets and $f_i \colon A_i \to Y$ be continuous maps ($i = 1, 2$). Then $Y \cup_{f_i} X_i$ are closed subsets of $Y \cup_{\langle f_1, f_2 \rangle} (X_1 + X_2) = Z$.
3. Let D be locally compact and C a closed subspace. Suppose the left diagram is a pushout with closed inclusions j and J.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times C \cup A \times D \arrow[r, "f'"] \arrow[d, "j'"] & X \times C \cup B \times D \arrow[d, "J'"] \\ X \times D \arrow[r, "F \times 1"] & Y \times D \end{tikzcd}$$

Then the right diagram is a pushout $j' = 1 \times i \cup j \times 1$, $J' = 1 \times i \cup J \times 1$, and $f' = F \times 1 \cup f \times 1$.

1.9 Connected Spaces

A space is ***connected*** if it is not the topological sum of two non-empty subspaces. Thus X is disconnected if and only if X contains a subset X which is open, closed, and different from $\emptyset$ and X. A ***decomposition*** of X is a pair U, V of open, non-empty, disjoint subsets with union X. A space X is disconnected if and only if there exists a continuous surjective map $f \colon X \to \{0, 1\}$; a decomposition is given by $U = f^{-1}(0)$, $V = f^{-1}(1)$. The continuous image of a connected space is connected. A subset A of a space X is called connected if it is a connected space in the subspace topology; thus this means that there do not exist open subset U, V in X such that $U \cap A$ and $V \cap A$ are non-empty but $U \cap V \cap A$ is empty.

(1.9.1) Theorem. *A subset $A \subset \mathbb{R}$ is connected if and only if it is an interval.*

Proof. An interval is a subset which contains with x, y also $[x, y]$. Suppose A is not an interval. Then there exists $z \in \mathbb{R}$ such that $A \cap\,]-\infty, z[$ and $A \cap\,]z, \infty[$ are non-empty; these sets are then a decomposition of A.

Suppose A is an interval. Let $U, V \subset \mathbb{R}$ be open sets which yield a decomposition $A \cap U$, $A \cap V$. Assume $x \in A \cap U$, $y \in A \cap V$, $x < y$. Let $z = \sup(U \cap [x, y])$. If $z \in U$, then $z < y$ and z cannot be the supremum, since U is open. If $z \in V$, there would exist $u < z$ with $u \in [x, y] \cap U \cap V$, and this contradicts $A \cap U \cap V = \emptyset$. $\square$

The intermediate value theorem of calculus is a direct consequence: If $f\colon A \to \mathbb{R}$ is continuous and $f(x) < z < f(y)$, then $z \in f(A)$, because otherwise the pre-images of $]-\infty, z[$ and $]z, \infty[$ would give a decomposition of the interval A.

(1.9.2) Proposition. *Let $(A_j \mid j \in J)$ be a family of connected subsets of X such that $A_i \cap A_j \neq \emptyset$ for all i, j. Then $\bigcup_j A_j = Y$ is connected. Let A be connected and $A \subset B \subset \overline{A}$. Then B is connected.*

Proof. A continuous map $f\colon Y \to \{0,1\}$ is constant on each A_j. Since $A_i \cap A_j \neq \emptyset$, the value $f(A_i)$ is independent of i.

Let U, V be open subsets of X with $B \subset (U \cup V)$ and $B \cap U \cap V = \emptyset$. Since A is connected, we have, say, $U \cap A = \emptyset$. Hence $A \subset X \smallsetminus U$, $\overline{A} \subset X \smallsetminus U$, $U \cap \overline{A} = \emptyset$. □

The union of the connected sets in X which contain x is thus a closed connected subset. We call it the ***component*** $X(x)$ of x in X. If $y \in X(x)$, then $X(y) = X(x)$. A component of X is a maximal connected subset. Any space is the disjoint union of its components. A space is ***totally disconnected*** if its components consist of single points.

A continuous map $w\colon [a,b] \to X$ is a ***path*** in X from $w(a)$ to $w(b)$, *connecting* a and b. Given w, then $[0,1] \to X, t \mapsto w((1-t)a + tb)$ is also a path from $w(a)$ to $w(b)$. It therefore often suffices to work with the unit interval $I = [0,1]$. Let $u, v\colon I \to U$ be paths with $u(1) = v(0)$. The ***product path*** $w = u * v$ is defined by $w(t) = u(2t)$ for $t \leq 1/2$ and $w(t) = v(2t-1)$ for $t \geq 1/2$. The path $u^-(t) = u(1-t)$ is the ***inverse path*** of u. The ***constant***

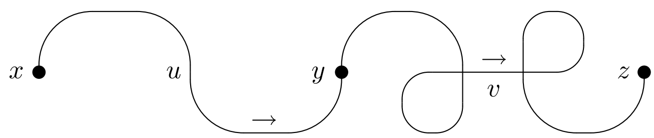

Figure 1.4. The product path $u * v$

path c_x with value x is, of course, $t \mapsto c_x(t) = x$. From these remarks we infer: Being connectible by paths is an equivalence relation on X; the classes are called the ***path components*** of X. If there is just one path component we call X ***path connected*** or ***pathwise connected***. Since intervals are connected, a path connected space is connected.

A space X is ***locally (pathwise) connected*** if for each $x \in X$ and each neighbourhood U of x there exists a (pathwise) connected neighbourhood V of x which is contained in U. Both properties are inherited by open sets.

(1.9.3) Proposition. *The components of a locally connected space X are open. The path components of a locally pathwise connected space Y are open and coincide with the components.*

Proof. Let K be a (path-)component of x. Let V be connected neighbourhood of x. Then $K \cup V$ is connected. Hence $K \cup V \subset K$. This shows that K is open.

Let Y be connected and K a path component. Then $Y \smallsetminus K$ is a union of path components, hence open. In the case that $Y \neq K$ we would obtain a decomposition. □

(1.9.4) Proposition. (1) *Suppose the open set A is a component of the open set B in X. Then* $\mathrm{Bd}(A) \subset \mathrm{Bd}(B)$.

(2) *If A is open in X and connected, then A is a component of* $X \smallsetminus \mathrm{Bd}(A)$.

Proof. (1) A is closed in B, hence $A = B \cap \overline{A}$. We conclude

$$\mathrm{Bd}(A) = \overline{A} \smallsetminus A = \overline{A} \smallsetminus (X \cap \overline{A}) = \overline{A} \cap (X \smallsetminus B) \subset \overline{B} \cap (X \smallsetminus B) = \mathrm{Bd}(B).$$

(2) Let B be the component of $X \smallsetminus \mathrm{Bd}(A)$ which contains A. If $A \neq B$, then B would intersect A as well as $X \smallsetminus A$. Since $B \cap \mathrm{Bd}(A) = \emptyset$, the set B would contain points of $X \smallsetminus \overline{A}$, and thus $A \cap B$ and $(X \smallsetminus \overline{A}) \cap B$ would form a decomposition of B. □

(1.9.5) Proposition. *Let X be connected and let $A \subset X$ be a connected subspace. Let C be a component of $X \smallsetminus A$. Then:*

(1) *If V is open and closed in $X \smallsetminus C$, then $C \cup V$ is connected.*

(2) *$X \smallsetminus C$ is connected.*

Proof. Let U_1, U_2 be a decomposition of $C \cup V$. Then the connected set C is contained in U_1, say, and hence U_2 contained in V. Hence U_2 is open and closed in V, and (since V is open and closed in $X \smallsetminus C$) open and closed in $X \smallsetminus C$, thus altogether open and closed in $(X \smallsetminus C) \cup (C \cup V) = X$. This contradicts the connectedness of X.

(2) Let U_3, U_4 be a decomposition of $X \smallsetminus C$. We show that then $A \cap U_3, A \cap U_4$ is a decomposition of A. Suppose $A \cap U_3 = \emptyset$. Then, by (1), $C \cup U_3$ is connected and contained in $X \smallsetminus A$. Since C is a proper subset of $C \cup U_3$, this contradicts the fact that C is a component $X \smallsetminus A$. Hence $A \cap U_3 \neq \emptyset$, and similarly $A \cap U_4 \neq \emptyset$. □

Problems

1. Suppose there exists a homeomorphism $\mathbb{R} \to X \times Y$. Then X or Y is a point.

2. A product $\prod_j X_j$ is connected if each X_j is connected. The component of $(x_j) \in \prod_j X_j$ is the product of the components of the x_j.

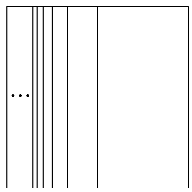

Figure 1.5. The comb space

3. A countable metric space is totally disconnected.
4. Let $f\colon X \to Y$ be surjective. If X is (path) connected, then Y is (path) connected.
5. Let C be a countable subset of $\mathbb{R}^n$, $n \geq 2$. Show that $\mathbb{R}^n \smallsetminus C$ is path connected.
6. The unitary group $\mathrm{U}(n)$ and the general linear group $\mathrm{GL}_n(\mathbb{C})$ are path connected. The orthogonal group $\mathrm{O}(n)$ and the general linear group $\mathrm{GL}_n(\mathbb{R})$ have two path components; one of them consists of the matrices with positive determinant. (These groups are defined as subspaces of the appropriate vector spaces of (n,n)-matrices.)
7. Let $U \subset \mathbb{R}^n$ be open. The path components of U are open and coincide with the components. The set of path components is finite or countably infinite. An open subset of $\mathbb{R}$ is a disjoint union of open intervals.

1.10 Compact Spaces

A collection (family, set) of subsets of X is a ***covering*** or ***cover*** of $A \subset X$ if their union contains A. If X is a topological space, a covering is ***open***(***closed***) if each of its members is open (closed). A covering $\mathcal{B}$ is a ***subcovering*** of $\mathcal{A}$ if it is a subfamily (subset) of $\mathcal{A}$. A covering is ***finite*** if the family (set) is finite.

A space X is ***compact*** if each open covering has a finite subcovering. (In some texts, this property is called ***quasi-compact***. In that case the property compact includes the Hausdorff property.) By passage to complements we see: If X is compact, then any family of closed sets with empty intersection contains a finite family with empty intersection. A set A in a space X is ***relatively compact*** if its closure is compact. We recall from calculus the fundamental ***Heine–Borel Theorem***:

(1.10.1) Theorem. *The unit interval $I = [0,1]$ is compact.* □

Proof. Let $[0,1]$ be covered by a collection $\mathcal{A}$ of open subsets of $\mathbb{R}$. Let $T \subset [0,1]$ be the set of t such that $[0,t]$ can be covered by a finite number of members of $\mathcal{A}$. It contains 0, hence is non-empty. Let $z = \sup T$. Then z is contained in an open interval J which is contained in a menber of $\mathcal{A}$, and J together with a finite covering of $[0, t-\varepsilon]$, $t - \varepsilon \in J$, shows $z \in T$ and that $z < 1$ is impossible. □

(1.10.2) Example. A continuous map $f\colon X \to \mathbb{R}$ from a non-empty compact space assumes a maximum; if not, the covering by the open sets $f^{-1}]-\infty, a[$, $a \in f(X)$ would not contain a finite subcovering. $\diamond$

(1.10.3) Proposition. *Let X be compact, $A \subset X$ closed and $f\colon X \to Y$ continuous. Then A and $f(X)$ are compact.*

Proof. Adjoin the open set $X \smallsetminus A$ to an open covering of A and use the compactness of X.

The pre-images of the sets in an open covering of $f(X)$ form an open covering of X. Now use the compactness of X. $\square$

(1.10.4) Theorem. *Let B, C be compact subsets of spaces X, Y, respectively. Let $\mathcal{A}$ be a family of open subsets of $X \times Y$ which covers $B \times C$. Then there exist open neighbourhoods U of B in X and V of C in Y and a finite subcovering of $\mathcal{A}$ which covers $U \times V$ In particular the product of two compact spaces is compact.*

Proof. We begin with the case $B = \{b\}$. For each $c \in C$ there exist open neighbourhoods M_c of b in X and N_c of c in Y such that $M_c \times N_c$ is contained in some member A_c of $\mathcal{A}$; this follows from the definition of the product topology. Then $(N_c \mid c \in C)$ is an open covering of C. Let $(N_c \mid c \in F)$ be a finite subcovering. Set $U = \bigcap_{c \in F} M_c$ and $V = \bigcup_{c \in F} N_c$. Then

$$\{b\} \times C \subset U \times V \subset \bigcup\nolimits_{c \in F} A_c.$$

Let now B be an arbitrary compact subset. By the first part of the proof, we can find an open neighbourhood U_b of b and an open neighbourhood V_b of C such that $U_b \times V_b$ is contained in a finite union $A(b)$ of members of $\mathcal{A}$. Then $(U_b \mid b \in B)$ is an open covering of B. Let $(U_b \mid b \in G)$ be a finite subcovering and set $U = \bigcup_{b \in G} U_b$, $V = \bigcap_{b \in G} V_b$. Then

$$B \times C \subset U \times V \subset \bigcup\nolimits_{b \in G} A(b)$$

proves the claim. $\square$

(1.10.5) Proposition. *Let B and C be disjoint compact subsets of a Hausdorff space X. Then B and C have disjoint open neighbourhoods. Hence a compact Hausdorff space is normal. A compact subset C of a Hausdorff space X is closed.*

Proof. By assumption, $B \times C \subset X \times X \smallsetminus D = W$, D diagonal. Since X is separated, W is open. By (1.10.4), we can find open neighbourhoods U of B and V of C such that $U \times V \subset W$, i.e., $U \cap V = \emptyset$. Let $x \in X \smallsetminus C$. Then $\{x\}$ and C have disjoint open neighbourhoods. Hence $X \smallsetminus C$ is open. $\square$

The next theorem often allows to show that a given map is a homeomorphism without an explicit construction or discussion of an inverse map. It also provides us with a useful criterion for quotient maps and embeddings.

(1.10.6) Theorem. *Let $f\colon X \to Y$ be a continuous map from a compact space into a Hausdorff space.*
(1) *f is closed.*
(2) *If f is injective, then f is an embedding.*
(3) *If f is surjective, then f is a quotient map.*
(4) *If f is bijective, then f is a homeomorphism.*

Proof. Let $A \subset X$ be closed. By (1.10.3) and (1.10.5), $f(A)$ is compact and closed. An injective closed (or open) continuous map is an embedding. A surjective closed (or open) continuous map is a quotient map. A bijective embedding or quotient map is a homeomorphism. □

(1.10.7) Proposition. *Let X be a compact Hausdorff space and $f\colon X \to Y$ a quotient map. The following assertions are equivalent:*
(1) *Y is a Hausdorff space.*
(2) *f is closed.*
(3) *$R = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is closed in $X \times X$.*

Proof. (1) $\Rightarrow$ (3). The set R is closed as pre-image of the diagonal $D \subset Y \times Y$ under $f \times f$.

(3) $\Rightarrow$ (2). Let $A \subset X$ be closed. We have to show that $f(A)$ is closed. Since f is a quotient map, it suffices to show that $f^{-1}f(A)$ is closed. We have

$$f^{-1}f(A) = \mathrm{pr}_1(R \cap \mathrm{pr}_2^{-1}(A)).$$

Since R is closed, so is $R \cap \mathrm{pr}_2^{-1}(A)$. Hence this intersection is compact as a closed subset of the Hausdorff space $X \times X$. Hence $\mathrm{pr}_1(R \cap \mathrm{pr}_2^{-1}(A))$ is compact, and therefore closed, since X is separated.

(2) $\Rightarrow$ (1). Let y_1 and y_2 be different points of Y. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X; this uses the fact that the $\{y_j\}$ are closed as image of points and points are closed in X. Since the compact Hausdorff space X is normal, we have disjoint open neighbourhoods U_j of $f^{-1}(y_j)$. Since f is closed, the sets $f(X \smallsetminus U_j)$ are closed; and $V_j = Y \smallsetminus f(X \smallsetminus U_j)$ is therefore open. By construction, $y_j \in V_j$ and $V_1 \cap V_2 = \emptyset$. □

(1.10.8) Proposition. *Let K be compact. Then $\mathrm{pr}\colon X \times K \to X$ is a closed map.*

Proof. Let $A \subset X \times K$ be closed and suppose $x \notin \mathrm{pr}(A)$. Then $\{x\} \times K$ is contained in the open complement V of A. By (1.10.4), there exists an open neighbourhood U of x such that $U \times K \subset V$. Therefore U is contained in the complement of $\mathrm{pr}(A)$. □

(1.10.9) Proposition. *A discrete closed subset F of a compact space K is finite.*

Proof. For each $x \in F$, the set $F \smallsetminus \{x\}$ is closed in F, since F carries the discrete topology. Since F is closed in K, also $F \smallsetminus \{x\}$ is closed in K. The intersection $\bigcap_{x\in F}(F \smallsetminus \{x\})$ is empty. Hence a finite number of $F \smallsetminus \{x\}$ have empty intersection, and this means that F is finite. □

The set $\{n^{-1} \mid n \in \mathbb{N}\}$ is a discrete infinite subset of the compact space $[0,1]$, but, of course, not closed.

(1.10.10) Proposition. *Let (x_n) be a sequence in a compact space. Then the set A of its accumulation values is non-empty and closed.*

Proof. We use (1.2.3) and the equality $A = H$. The set H is closed and therefore compact. If $H = \emptyset$, then, by compactness, a finite intersection of the $\overline{H(n)}$ would be empty; contradiction. □

Let X be a union of subspaces $X_1 \subset X_2 \subset \cdots$. We say X carries the ***colimit-topology with respect to the filtration*** (X_i) if $A \subset X$ is open (closed) if and only if each intersection $A \cap X_n$ is open (closed) in X_n. This topology has the universal property: A map $f\colon X \to Y$ is continuous if and only if each restriction $f|X_n$ is continuous. If only the X_i are given as topological spaces, we can define a topology on X as the colimit topology. We then call X the ***colimit*** of the ascending sequence (X_i). (This is a colimit in the categorical sense.)

(1.10.11) Proposition. *Suppose X is the colimit of the sequence $X_1 \subset X_2 \subset \cdots$. Suppose points in X_i are closed. Then each compact subset K of X is contained in some X_k.*

Proof. Suppose this is not the case. Then there exists a countably infinite set $F \subset K$ such that each intersection $F \cap X_n$ is finite. For each subset S of F, the intersection $S \cap X_n$ is a finite union of points, hence closed. By definition of the colimit topology, each subset of F is closed in X and therefore also in F. Thus F is discrete and, by the previous proposition, finite. □

Problems

1. D^n/S^{n-1} is homeomorphic to S^n. For the proof verify that

$$D^n \to S^n, \quad x \mapsto (2\sqrt{1-\|x\|^2}x, 2\|x\|^2 - 1)$$

induces a bijection $D^n/S^{n-1} \to S^n$.
2. $\mathbb{R}^n/D^n$ is homeomorphic to $\mathbb{R}^n$.

3. Let A and B be compact subsets of the metric space (E,d). Then there exists $(a,b) \in A \times B$ such that for all (x,y) the inequality $d(x,y) \geq d(a,b)$ holds. In particular, there exists a pair of points in A which realizes the diameter of A.
4. Let $f\colon X \times C \to \mathbb{R}$ be continuous. Assume that C is compact and set $g(x) = \sup\{f(x,c) \mid c \in C\}$. Then $g\colon X \to \mathbb{R}$ is continuous.
5. Let X be a countable compact Hausdorff space. Then each point has a neighbourhood basis consisting of sets which are open and closed.
6. $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.
7. $A \subset \mathbb{R}^n$ is compact if and only if continuous functions $A \to \mathbb{R}$ are bounded.
8. Let A be a compact subset of X and $p\colon X \to X/A$ the quotient map. Then for every Y the product $p \times \mathrm{id}\colon X \times Y \to X/A \times Y$ is a quotient map.
9. Let $A \subset \mathbb{R}$ and $CA = A \times [0,1]/A \times 1$ (the formal cone on A). The map $f\colon CA \to \mathbb{R}^2$, $(a,t) \mapsto ((1-t)a, t)$ is injective and continuous. If A is compact, then f is an embedding. If $A \neq \emptyset$ is an open interval, then f is not an embedding (consider neighbourhoods of the cone point $A \times 1$; it has no countable neighbourhood basis).
10. Let $(f_n\colon X \to \mathbb{R})$ be a sequence of continuous functions on a compact space X. Suppose that for $t \in X$ and $n \in \mathbb{N}$ the inequality $f_n(t) \leq f_{n+1}(t)$ holds. If the sequence converges pointwise to a continuous function f, then the convergence is uniform (***Theorem of Dini***).
11. Let $\mathbb{R}^\infty$ be the vector space of all sequences $(x_1, x_2, \dots)$ of real numbers which are eventually zero. Let $\mathbb{R}^n$ be the subspace of sequences with $x_j = 0$ for $j > n$. Give $\mathbb{R}^\infty$ the colimit topology with respect to the subspaces $\mathbb{R}^n$. Then addition of vectors is a continuous map $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. Scalar multiplication is a continuous map $\mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. (Thus $\mathbb{R}^\infty$ is a topological vector space.) A neighbourhood basis of zero consists of the intersection of $\mathbb{R}^\infty$ with products of the form $\prod_{i\geq 1}]-\varepsilon_i, \varepsilon_i[$. The space has also the colimit topology with respect to the set of finite-dimensional linear subspaces. One can also consider the metric topology with respect to the metric $d((x_i),(y_i)) = (\sum_i (x_i - y_i)^2)^{1/2}$; denote it by $\mathbb{R}^\infty_d$. The identity $\mathbb{R}^\infty \to \mathbb{R}^\infty_d$ is continuous. The space $\mathbb{R}^\infty$ is separated and not homeomorphic to $\mathbb{R}^\infty_d$.

1.11 Examples

We discuss in this section a number of examples of quotient spaces. We study by way of example some standard surfaces. The notion of a quotient space gives precision to the intuitive idea of pasting and gluing. Everybody should understand the surfaces that can be obtained from a square by pairwise identification of its sides. These are: Cylinder (1), Möbius Band (2), Sphere (3), Torus (4), Projective Plane (5), Klein Bottle (6). We use a symbolic notation for the identification process as shown in figure 1.6. We write $I = [0,1]$ for the unit interval.

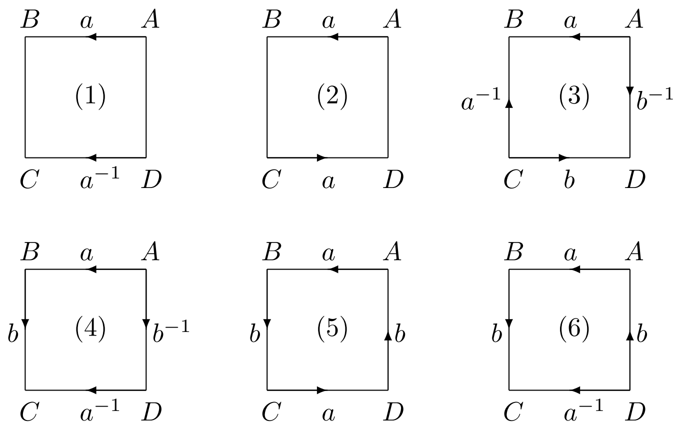

Figure 1.6. Construction of surfaces from $I \times I$

1.11.1 Cylinder. Here we identify $(t, 0)$ with $(t, 1)$. The quotient space is $I \times I/\partial I \cong I \times S^1$. In figure (1) $A = D$ and $B = C$ after the identification. ◇

1.11.2 Möbius band. Here we identify $(t, 0)$ with $(1 - t, 1)$. A realization as a subset of $\mathbb{R}^3$ is shown in the figures **??** and **??**. After the identification $A = C$ and $B = D$.

M can also be defined as the quotient of $S^1 \times [-1, 1]$ with respect to $(z, t) \sim (-z, -t)$. In this way it is obtained as a quotient of the cylinder $S^1 \times [-1, 1]$.

The Möbius band is often called "one-sided". If you paint the paper-model, then you can only use a single colour. But the Möbius band is proud of itself and wants to be treated as an abstract space (not embedded in Euclidean space.) As such the term one-sided is meaningless. The formal mathematical analysis is more subtle. In technical terms, we have a non-orientable surface; it is impossible to orient the tangent spaces in a continuous manner, the notion "clock-wise" cannot be transported in a continuous manner over the Möbius band. This property of the Möbius band is a "global" property. ◇

1.11.3 Sphere. We leave it as an exercise to show that the identifications $(1, t) \sim (1 - t, 0)$ and $(0, s) \sim (1, 1 - s)$ yield a space which is homeomorphic to the 2-sphere S^2. After the identification $A = C$ ◇

1.11.4 Torus. The 2-dimensional torus is the space (group) $S^1 \times S^1$. Figure **??** shows standard realizations as a surface in $\mathbb{R}^3$.

We have the homeomorphism $p\colon I/\partial I \to S^1$ induced by $t \mapsto \exp(2\pi i t)$. Therefore the torus can also be obtained as the space $T = I/\partial I \times I/\partial I$. This space is a quotient space of $I \times I$. It is also obtainable via a pushout

$$\begin{tikzcd} \partial(I \times I) \arrow[r, "c"] \arrow[d] & I/\partial I \vee I/\partial I \arrow[d] \\ I \times I \arrow[r] & T. \end{tikzcd}$$

The standard way to interprete this construction of the torus is by identifying opposite edges in the square as shown in (4) of figure 1.6.

The edge $a = AB$ is identified with the edge $a^{-1} = DC$, the edge $b = BC$ is identified with the edge $b^{-1} = AD$. After the identification we have $A = B = C = D$. If we run along the boundary of the square starting at A counter-clockwise, then the map c has the effect: a runs around the first summand $I/\partial I$ and b runs around the second summand; a^{-1} and b^{-1} are given by the inverse paths. ◇

1.11.5 Projective plane. The real projective plane P^2 is defined as the quotient of S^2 by the relation $x \sim -x$, see **??** for more details. Let $[x_0, x_1, x_2]$ denote the equivalence class of $x = (x_0, x_1, x_2)$. We can also obtain P^2 from S^1 by attaching a 2-cell

$$\begin{tikzcd} S^1 \arrow[r, "\varphi"] \arrow[d, "j"] & P^1 \arrow[d, "J"] \\ D^2 \arrow[r, "\Phi"] & P^2. \end{tikzcd}$$

Here $P^1 = \{[x_0, x_1, 0]\} \subset P^2$ and $\varphi(x_0, x_1) = [x_0, x_1, 0]$. The space P^1 is homeomorphic to S^1 via $[x_0, x_1, 0] \mapsto z^2$ with $z = x_0 + ix_1$; and φ corresponds to the standard map $z \mapsto z^2$ under this homeomorphism The map Φ is $x = (x_0, x_1) \mapsto [x_0, x_1, \sqrt{1 - \|x\|^2}]$.

Another interpretation of the pushout: P^2 is obtained from D^2 by identifying opposite points of the boundary S^1. (Part (5) in figure 1.6 illustrates a similar definition starting from the square. After the identification we have $A = C$ and $B = D$.) The subspace $\{(x_0, x_1) \mid \|x\| \geq 1/2\}$ becomes in P^2 a Möbius band M. Thus P^2 is obtainable from a Möbius band M and a 2-disk D by identification of the boundary circles by a homeomorphism. (The reader should find a Möbius band in figure 1.6 (5) by comparison with (2).) The projective plane cannot be embedded into $\mathbb{R}^3$. There exist models in $\mathbb{R}^3$ with self-intersections (technically, the image of a smooth immersion.) The projective plane is a non-orientable surface. ◇

1.11.6 Klein bottle. In order to obtain the Klein bottle from a square one first forms the cylinder as in the case of the torus. The remaining boundary circles are then identified in the opposite manner as in the case of the circle.

Also this surface cannot be realized as a subspace of $\mathbb{R}^3$. As Felix Klein already observed, one can obtain a model in $\mathbb{R}^3$ with self-intersections by pushing one end of the pipe through its wall and then glue the parallel end from the inside, see figure **??**. The Klein bottle is a non-orientable surface.

After the identification in figure 1.6 we have (6) $A = B = C = D$. The Klein bottle K can also be obtained from two Möbius band M by an identification of their boundary curves with a homeomorphism, $K = M \cup_{\partial M} M$. In figure 1.7 the dotted area will give a the Möbius band. $\diamond$

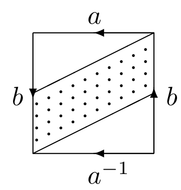

Figure 1.7. Möbius band in the Klein bottle

The space $M/\partial M$ is homeomorphic to the projective plane P^2. If we identify the central ∂M to a point, we obtain a map $q\colon K = M \cup_{\partial M} M \to P^2 \vee P^2$.

There is a free involution on $T = S^1 \times S^1$ given by $(z, w) \mapsto (\overline{z}, -w)$. The orbit space is another model of the Klein bottle. The classification theory of compact connected surfaces presents a surface as a quotient space of a regular $2n$-gon, see e.g. [**?**, p.75] [**?**] [**?**]. The edges are identified in pairs by a homeomorphism. The surface F is obtained as a pushout of the type

$$\begin{array}{ccc} S^1 & \xrightarrow{\varphi} & \bigvee_1^n S^1 \\ \downarrow & & \downarrow \\ D^2 & \xrightarrow{\Phi} & F. \end{array}$$

Problems

1. Show that the two definitions of the Möbius band in 1.11.2 yield homeomorphic spaces.
2. Construct a Möbius band in $\mathbb{R}^3$ such that its boundary curve is a standard circle.
3. Show that the following quotient spaces are pairwise homeomorphic. They yield

different descriptions of the real projective plane $\mathbb{R}\mathrm{P}^2$.

$$\begin{aligned}
&P_1 = \mathbb{R}^3 \smallsetminus 0/\sim, \qquad x \sim \lambda x,\ \lambda \in \mathbb{R} \smallsetminus 0\\
&P_2 = S^2/\sim, \qquad x \sim -x\\
&P_3 = D^2/\sim, \qquad z \in S^1 \sim -z \in S^1\\
&P_4 = [0,1] \times [0,1]/\sim, \qquad (0,t) \sim (1,1-t),\ (s,0) \sim (1,1-s)\\
&P_5 = \text{the space } P^2 \text{ defined by the pushout in 1.11.5}\\
&P_6 = M \cup_{S^1} D^2 \text{ Möbius band and 2-disk identified by}\\
&\qquad \text{a homeomorphism between their boundaries}\\
&P_7 = M/\partial M
\end{aligned}$$

4. Show that the three definitions of the Klein bottle in 1.11.6 as the quotient of $I \times I$, as a quotient of $S^1 \times S^1$ and as $M \cup_{\partial M} M$ yield homeomorphic spaces.
5. Verify 1.11.3.

1.12 Compact Metric Spaces

The next result is of fundamental importance. It is impossible to prove geometric results about continuous maps without subdivision and approximation procedures. In most of these procedures (1.12.1) will be used.

(1.12.1) Theorem (Lebesgue-Lemma). *Let X be a compact metric space. Let $\mathcal{A} = (A_j \mid j \in J)$ be an open covering of X. Then there exists $\varepsilon > 0$ such that for each $x \in X$ the ε-neighbourhood $U_\varepsilon(x)$ is contained in some A_j. (An ε with this property is called a* **Lebesgue number** *of the covering.)*

Proof. For $x \in X$ choose $\varepsilon(x) > 0$ such that $U_{2\varepsilon(x)}(x)$ is contained in some member of $\mathcal{A}$. By compactness there exists a finite subset $E \subset X$ such that the $W_z = U_{\varepsilon(x)}(z)$, $z \in E$ cover X. Let $\varepsilon = \min(\varepsilon(z) \mid z \in E)$. Given $x \in X$ there exists $z \in E$ such that $x \in W_z$. Let $y \in U_\varepsilon(x)$. Then $d(y,z) \leq d(y,x) + d(x,z) < \varepsilon + \varepsilon(z) \leq 2\varepsilon(z)$. Hence $U_\varepsilon(x) \subset U_{2\varepsilon(z)}(z)$, and the latter set is contained in a member of $\mathcal{A}$. □

(1.12.2) Proposition. *Let $\mathcal{U} = (U_j \mid j \in J)$ be an open covering of $B \times [0,1]$. For each $b \in B$ there exists an open neighbourhood $V(b)$ of x in B and $n = n(b) \in \mathbb{N}$ such that for $0 \leq i < n$ the set $V(b) \times [i/n, (i+1)/n]$ is contained in some U_j.*

Proof. Given $(b,t) \in B \times [0,1]$ choose open neighbourhoods $V(b,t)$ of b in B and $W(b,t)$ of t in $[=,1]$ such that $V(b,t) \times W(b,t)$ is contained in some member of $\mathcal{U}$. Suppose $[0,1]$ is covered by $\mathcal{W} = (W(b,t_1), \ldots, W(b,t_k))$. Set $V(b) = \bigcap_{i=1}^k V(b,t_i)$ and take $1/n(b)$ as a Lebesgue number of $\mathcal{W}$. □

(1.12.3) Proposition. *Let X be a compact metric space and W a neighbourhood of the diagonal D_X in $X \times X$. Then there exists $\delta > 0$ such that $U_\delta(D_X) = \{(x,y) \mid d(x,y) < \delta\}$ is contained in W.*

Proof. Given $x \in X$ choose $\delta(x) > 0$ such that $U_{2\delta(x)}(x) \times U_{2\delta(x)}(x) \subset W$. Suppose X is covered by the $W_z = U_{\delta(z)}(z)$, where z runs through a finite subset $E \subset X$, and set $\delta = \min(\delta(z) \mid z \in E)$. We claim that $U_\delta(D_X) \subset W$. Let $\delta(x,y) < \delta$. There exists $z \in E$ such that $x \in W_z$, hence $d(x,z) < \delta(z)$. Also $d(y,z) \leq d(y,x) + d(x,z) < 2\delta(z)$. Hence $x, y \in U_{2\delta(z)}(z)$ and therefore $(x,y) \in W$, by our choice of the $\delta(z)$. □

(1.12.4) Proposition. *A continuous map of a compact metric space into a metric space is uniformly continuous.*

Proof. The assertion can be expressed as follows: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $(f \times f)^{-1}(U_\varepsilon(D_Y)) \supset U_\delta(D_X)$. We apply (1.12.3). □

(1.12.5) Theorem. *Let $K \subset \mathbb{R}^n$ be a compact convex subset with non-empty interior. Then there exists a homeomorphism of pairs $(D^n, S^{n-1}) \to (K, \partial K)$ which sends $0 \in D^n$ to a preassigned $x \in K^\circ$.*

Proof. (1) Let $K \subset \mathbb{R}^n$ be closed and compact and $0 \in K^\circ$. Then the ray from 0 intersect ∂K in exactly one point. Proof:

Let R be a ray, let $p, q \in R \cap K$ be different from 0, and suppose $\|p\| < \|q\|$. Choose $\varepsilon > 0$ such that $D_\varepsilon(0) \subset K$. The set J of all segments from points in $D_\varepsilon(0)$ to q is contained in K, by convexity, and $p \in J^\circ$, by a geometric argument that we leave to the reader; hence $p \notin \partial K$.

Let $x \in S^{n-1}$ and consider the ray $R_x = \{tx \mid t \geq 0\}$. Then $R_x \not\subset K$, by compactness. If $R_x \cap \partial K$ were empty, then $K^\circ \cap R_x \neq \emptyset$ and $(\mathbb{R}^n \smallsetminus K) \cap R_x \neq \emptyset$, and this contradicts the connectedness of R_x.

(2) Let $0 \in K^\circ$. The map $f \colon \partial K \to S^{n-1}$, $x \mapsto x/\|x\|$ is a homeomorphism. Proof: The map is continuous and, by (1), bijective, hence a homeomorphism by (1.10.6).

(3) Without essential restriction we can assume $0 \in K^\circ$. Let f be as in (2). The continuous map $\varphi \colon S^{n-1} \times [0,1] \to K, (x,t) \mapsto tf^{-1}(x)$ factors over $q \colon S^{n-1} \times [0,1] \to D^n, (x,t) \mapsto tx$ and yields a bijective map $k \colon D^n \to K$. By (1.10.6), q is a quotient map; by (1.4.1) k is continuous and, by (1.10.6) again, a homeomorphism. □

(1.12.6) Corollary. *Let K be as in* (1.12.5) *and $x \in K^\circ$. Then there exists a retraction of $\partial K \subset K \smallsetminus \{x\}$, i.e., a map $K \smallsetminus \{x\} \to \partial K$ which is the identity on ∂K.* □

A metric space X is called ***pre-compact*** if for each $\varepsilon > 0$ there exists a finite covering of X by sets of diameter less than ε. This latter property is equivalent to: For each $\varepsilon > 0$ there exists a finite set $F \subset X$ such that for each $x \in X$ we have $d(x, F) < \varepsilon$.

(1.12.7) Proposition. *Let X be a metric space. The following assertions about $A \subset X$ are equivalent:*

(1) *$\overline{A}$ is compact.*
(2) *Each sequence in A has an accumulation point in X.*
(3) *Each sequence in A has a subsequence which converges in X.*
(4) *$\overline{A}$ is pre-compact and complete.*

Proof. For the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) see (1.2.3) and (1.10.10).

(3) $\Rightarrow$ (4). Let $(a_n \mid n \in \mathbb{N})$ be a Cauchy sequence in $\overline{A}$. There exists $b_n \in A$ such that $d(b_n, a_n) < 2^{-n}$. By (3), there exists a convergent subsequence of (b_n) with limit b and the corresponding subsequence of (a_n) has the same limit. Hence (a_n) itself converges to b. Thus $\overline{A}$ is complete.

If, for $\alpha > 0$, the set $\overline{A}$ is not covered by a finite number of sets $B(x, \alpha) = \{y \mid d(x, y) \le \alpha\}$, we could find inductively a sequence (x_n) such that $d(x_i, x_j) > \alpha$ for all $i \ne j$. This sequence does not have a convergent subsequence.

(4) $\Rightarrow$ (1). Let $(U_j \mid j \in J)$ be an open covering of $\overline{A}$. We define inductively a sequence of set $B(x_n, 2^{-n}) = B_n$ as follows: Suppose $\overline{A}$ is not covered by a finite subfamily of the U_j. There exists a finite covering of $\overline{A}$ by sets $B(x, 1)$ and hence a $B(x_1, 1)$ which is not covered by finitely many U_j. Suppose $B(x_{n-1}, 2^{-(n-1)})$ is not covered by a finite number of U_j. We choose a finite covering of $\overline{A}$ by sets $B(y_k, 2^{-n})$. Among the $B(y_k, 2^{-n})$ which intersect $B(x_{n-1}, 2^{-(n-1)})$ we find a $B(x_n, 2^{-n})$ which is not covered by a finite number of U_j. For $n \le p < q$ we have $d(x_p, x_q) \le d(x_p, x_{p+1}) + \cdots + d(x_{q-1}, x_q)$. Since $B_{n-1} \cap B_n \ne \emptyset$, we have $d(x_{n-1}, x_n) \le \frac{1}{2^{n-1}} + \frac{1}{2^n} \le \frac{1}{2^{n-2}}$ and hence

$$d(x_p, x_q) \le \frac{1}{2^{p-1}} + \cdots + \frac{1}{2^{q-2}} \le \frac{1}{2^{n-2}}.$$

Thus (x_p) is a Cauchy sequence. It has, by (3), a convergent subsequence, with limit x say. There exists λ such that $x \in U_\lambda$. For some $\alpha > 0$ we have $B(x, \alpha) \subset U_\lambda$. There exists n such that $d(x, x_n) < \frac{\alpha}{2}$, $\frac{1}{2^n} < \frac{\alpha}{2}$. Hence

$$B(x_n, 2^{-n}) \subset B(x, \alpha) \subset U_\lambda,$$

and this contradicts the fact that $B(x_n, 2^{-n})$ is not covered by a finite number of U_j. □

Problems

1. Let $A \subset \mathbb{R}^2$ be a compact, non-empty set with the property: For each $a \in A$ there exists a unique affine line $G_a \subset \mathbb{R}^2$ with $G_a \cap A = \{a\}$ and such that $A \smallsetminus \{a\}$ lies on

one side of G_a. Then A is homeomorphic to S^1.
2. A countable set $A \subset \mathbb{R}$ is not equal to its set of accumulation points.
3. Let X be a compact metric space and $f\colon X \to X$ a set map preserving the distance $d(f(x), f(y)) = d(x, y)$. Then f is a homeomorphism. (Suppose $y \notin f(x)$. Consider the sequence $y = y_0, y_n = f(y_{n-1})$.)
4. There exists a continuous map $f\colon D^n \times E^n \to D^n \to D^n$ such that for each $y \in E^n$ the map $f_y\colon x \mapsto f(x, y)$ is a homeomorphism which is the identity on S^{n-1} and sends y to 0.
5. Let E_1 and E_2 be finite subsets of $\mathbb{R}^n$ with the same cardinality. There exists a continuous map $h\colon \mathbb{R} \times [0, 1] \to \mathbb{R}$ with the properties: h_0 is the identity. $h_1(E_1) = E_2$. Each h_t is a homeomorphism. There exists a compact set $K \subset \mathbb{R}^n$ such that h_t is the identity on $\mathbb{R}^n \smallsetminus K$. (Here $h_t\colon x \mapsto h(x, t)$.)

Chapter 2

Topological Spaces: Further Results

2.1 The Cantor Space. Peano Curves

We construct inductively sets

$$C(0) = [0,1] \supset C(1) \supset C(2) \supset \dots$$

with the properties: $C(n)$ is the disjoint union of 2^n intervals of length 3^{-n}. The set $C(n)$ is obtained from $C(n-1)$ by deleting from each of its intervals the open middle third. The intersection $C = \cap_{n \geq 0} C(n)$ is compact. The space C is called the ***Cantor space***. The endpoints of the intervals which are deleted in the n-th step from the 2^{n-1} intervals are the numbers

$$\textstyle\sum_{i=1}^{n-1} \frac{a(i)}{3^i} + \frac{2}{3^n}, \qquad a(i) \in \{0,2\}.$$

The numbers remaining after the n-th step have the form

$$\textstyle\sum_{i=1}^{n} \frac{a(i)}{3^i} + \frac{x}{3^n}, \qquad a(i) \in \{0,2\}, \quad 0 \leq x \leq 1;$$

these are the numbers which have in its 3-adic presentation up to the place n only coefficients $a(i) \in \{0,2\}$. This implies that C consists of the numbers with 3-adic development $\sum_{i \geq 1} a(i) 3^{-i}$, $a(i) \in \{0,2\}$. A number $x \in [0,1]$ has at most one development of this form. Therefore we obtain a bijection

$$p \colon C \to P = \textstyle\prod_{i \geq 1} \{0,2\}, \quad \sum_{i \geq 1} a(i) 3^{-i} \mapsto (a(i) \mid n \in \mathbb{N}).$$

The projection $p_k \colon C \to \{0,2\}$, $\sum a(i) 3^{-i} \mapsto a(k)$ is continuous. In order to verify this, note that the conditions $a(k) = 0, 2$ determine closed sets. This is clear, since after $a(1), \dots, a(k-1)$ have been fixed the rest is homeomorphic with C (multiplication by 3^{k-1}.) If we provide P with the product topology, p becomes a bijective continuous map of a compact space into a separated one. Hence P is compact and p a homeomorphism.

Since p_k is continuous, $\varphi = \sum_{k \geq 1} 2^{-k-1} p_k$ is a uniformly convergent series of continuous functions, hence φ is continuous. The image of φ is $[0,1]$, since the 2-adic developments of its elements are contained in the image.

By grouping even and odd factors of P we see that P is homeomorphic to $P \times P$, hence C is homeomorphic to $C \times C$. (Similarly for a finite or countably infinite number of factors.) We see that there exist continuous surjective maps $f \colon C \to [0,1]^n$. A map of this form has a continuous extension to $[0,1]$: use in each open interval of the complement the affine extension of the values of f in the end points. (One can also refer to the Tietze extension theorem.)

The existence of surjective continuous maps $[0,1] \to [0,1] \times [0,1]$ was discovered 1890 by Peano [**?**]. They are called ***Peano curves***. We remark that the arguments show that a countable product of factors $[0,1]$ is compact; the same then holds for a countable product of factors which are homeomorphic to compact subsets of Euclidean spaces. Thus in these cases it is not necessary to use the theorem of Tychonoff. All this tells us that the unit interval $[0,1]$ is a highly non-trivial topological space.

The space C is nowhere dense in $[0,1]$. The total length of the removed intervals is 1; hence C has measure 0. There don't exist continuously differentiable Peano curves. ◇

2.2 Locally Compact Spaces

A space is ***locally compact*** if each neighbourhood of a point x contains a compact neighbourhood. An open subset of a locally compact space is again locally compact.

Let X be a Hausdorff space and assume that each point has a compact neighbourhood. Let U be a neighbourhood of x and K a compact neighbourhood. Since K is normal, $K \cap U$ contains a closed neighbourhood L of x in K. Then L is compact and a neighbourhood of x in X. Therefore X is locally compact. In particular, a compact Hausdorff space is locally compact.

Let X be a topological space. An embedding $f \colon X \to Y$ is a ***compactification*** of X if Y is compact and $f(X)$ dense in Y.

A compactification by a single point is called an ***Alexandroff compactification*** or the ***one-point compactification***. The additional point is the ***point at infinity***. In a general compactification $f \colon X \to Y$, one calls the points in $Y \smallsetminus f(X)$ the points at infinity.

(2.2.1) Theorem. *Let X be a locally compact Hausdorff space. Up to homeomorphism, there exists a unique compactification $f\colon X \to Y$ by a compact Hausdorff space such that $Y \smallsetminus f(X)$ consists of a single point.*

Proof. Let $Y = X \cup \{\infty\}$. Define a topology on Y as follows: The open sets are the open sets of X and the sets $Y \smallsetminus K$ for $K \subset X$ compact. One verifies that this is a topology on Y which induces on X the original topology. Two points of X are still separable by disjoint open neighbourhoods. If $x \in X$ and K a compact neighbourhood, then K and $Y \smallsetminus K$ are disjoint neighbourhoods of x and ∞. Thus Y is separated. The space Y is compact, since each open covering contains a set of the form $Y \smallsetminus K$ with compact K.

Let $Y' = X \cup \{\infty'\}$ be another space with the stated properties. Let $F\colon Y \to Y'$ be the map which is the identity on X and sends ∞ to ∞'. Then F is bijective and continuous in each point of X. We show continuity in ∞. The complement of an open neighbourhood of ∞' is, as a closed subset of the compact Hausdorff space Y', compact. Hence the pre-image of this neighbourhood by F is open. By (1.10.6), F is a homeomorphism. □

(2.2.2) Proposition. *Let the locally compact space be a union of compact subsets $(K_i \mid i \in \mathbb{N})$. Then there exists a sequence $(U_i \mid i \in \mathbb{N})$ of open subsets with the properties:*

(1) *For each i the closure $\overline{U_i}$ is compact.*
(2) *For each i we have $\overline{U_i} \subset U_{i+1}$.*
(3) $X = \bigcup_{i=1}^{\infty} U_i$.

Proof. Let K be a compact subset of X. Then there exists a compact set L which contains K in its interior. For the proof, choose a compact neighbourhood $K(x)$ of $x \in X$. A finite number $K(x_1)^\circ, \dots, K(x_n)^\circ$ cover K, and the union L of the $K(x_j)$ has the desired property. We call L a thickening of K. Let U_1 be the interior of a thickening of K_1. Inductively, we let U_{n+1} be the interior of a thickening of $\overline{U}_n \cup K_{n+1}$. □

(2.2.3) Theorem. *Let the locally compact Hausdorff space $M \neq \emptyset$ be a union of closed subsets M_n, $n \in \mathbb{N}$. Then at least one of the M_n contains an interior point.*

Proof. Suppose this is not the case. Since M is locally compact, there exists a compact set K with $K^\circ = V \neq \emptyset$. There exists $v_1 \in V \smallsetminus M_1$. The set $V \smallsetminus M_1$ is open. There exists a compact neighbourhood K_1 of v_1 with $K_1 \subset V \smallsetminus M_1$. There exists a point $v_2 \in K_1^\circ \smallsetminus M_2$ and a compact neighbourhood $K_2 \subset K_1^\circ \smallsetminus M_2$ of v_2. Inductively, we find compact sets $K \supset K_1 \supset K_2 \supset \dots$ such that $K_n \cap M_j = \emptyset$ for $j \leq n$. The sets K_i are closed in the compact Hausdorff space K, hence $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$. A point in this intersection does not lie in any M_j, and this contradicts $M = \bigcup_j M_j$. □

(2.2.4) Proposition. *Let* $p\colon X \to Y$ *be a quotient map and* Z *locally compact. Then* $p \times \mathrm{id}\colon X \times Z \to Y \times Z$ *is a quotient map.*

Proof. Suppose $U \subset Y \times Z$ and $V = (p \times \mathrm{id})^{-1}(U)$ is open. We have to show that U is open. Let $(y,z) \in U$, $p(x) = y$, hence $(x,z) \in V$. Since Z is locally compact, each neighbourhood of z contains a compact neighbourhood. Hence z has a compact neighbourhood C such that $x \times C \subset V$. The set

$$W = \{x \in X \mid x \times C \subset V\} = \{x \in X \mid p(x) \times C \subset U\}$$

is open in X, see (1.10.4). The relation $p^{-1}p(W) = W$ holds, and therefore $p(W)$ is open in Y, by definition of the quotient topology. Hence U contains the neighbourhood $f(W) \times C$ of (y,z).

For a different proof which is conceptually more transparent see (2.9.6). □

Problems

1. The following assertions about a locally compact Hausdorff space are equivalent: (1) In the one-point compactification, ∞ has a countable neighbourhood basis. (2) The space is the union of a countable number of compact subsets.

2. The one-point compactification of $\mathbb{R}^n$ is S^n; see (1.3.5).

2.3 Real Valued Functions

We say that closed subsets of a space can be ***numerically separated*** if for any pair A, B of disjoint, closed, non-empty subsets of X there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Metric spaces have this property, see the proof of (**??**). If a function with the stated property exists, then also a function $g\colon X \to [a,b]$ such that $g(A) = \{a\}$ and $g(B) = \{b\}$. The next theorem is known as the ***extension theorem of Tietze.***

(2.3.1) Theorem. *Suppose closed subsets of* X *can be numerically separated. Let* $A \subset X$ *be closed . Then each continuous map* $f\colon A \to [0,1]$ *has a continuous extension* $f\colon X \to [0,1]$.

Proof. Let $0 < \varepsilon \le 1$. An ε-extension of f is a continuous map $g\colon X \to [0,1]$ such that:

(1) $|g(x)| \le 1 - \varepsilon$ for each $x \in X$.

(2) $|f(x) - g(x)| \le \varepsilon$ for each $x \in A$.

Given g, we construct an improved extension Vg as follows. Let

$$C = \{x \in A \mid f(x) - g(x) \ge \varepsilon/3\}, \qquad D = \{x \in A \mid f(x) - g(x) \le -\varepsilon/3\}.$$

We choose a continuous function $v\colon X \to [-\varepsilon/3, \varepsilon/3]$ with value $-\varepsilon/3$ on C and value $\varepsilon/3$ on D. The function $Vg = g - v$ has the properties:

(3) $|Vg(x)| \le 1 - 2\varepsilon/3$ for $x \in X$.
(4) $|f(a) - Vg(a)| \le 2\varepsilon/3$ for $a \in A$.
(5) $|g(x) - Vg(x)| \le \varepsilon/3$ for $x \in X$.

(3) and (5) hold by construction and (4) is verified separately on C, D, and the complement. We use this construction and find inductively a sequence of ε_n-extensions (g_n) with $g_0 = 0$, $g_{n+1} = Vg_n$, and $\varepsilon_n = (2/3)^n$. These functions have the further property $|g_m(x) - g_n(x)| \le (2/3)^p$ for $m, n \ge p$. The $(g_m(x) \mid m \in \mathbb{N}_0)$ are therefore a Cauchy-sequence; thus g_m converges point-wise to an extension F of f. Since the convergence is uniform, the limit it continuous. □

(2.3.2) Theorem. *Let X be as in* (2.3.1) *and $f\colon A \to \mathbb{R}^n$ a continuous map from a closed non-empty subset A of X. Then f has a continuous extension to X.*

Proof. It suffices to extend the n components of f. Since there exists a homeomorphism $\mathbb{R} \cong]-1,1[$, we can assume that $f\colon A \to]-1,1[$. Let $G\colon X \to [-1,1]$ be an extension according to (2.3.1). Let $u\colon X \to [0,1]$ be a continuous function which assumes the value 1 on A and the value 0 on $G^{-1}\{-1,1\}$. Then $F = G \cdot u$ is an extension of f with image contained in $]-1,1[$. □

(2.3.3) Lemma. *Let $D \subset [0,1]$ be a dense subset. Suppose that we are given for each $d \in D$ an open subset L_d of the space X . Suppose $\overline{L}_d \subset L_e$ whenever $d < e$. Then the function $f\colon X \to [0,1]$, $x \mapsto \inf\{d \in D \mid x \in L_d\}$ is continuous. In the case that $D(x) = \{d \in D \mid x \in L_d\}$ is empty, the infimum is, by definition, equal to* 1*; thus f assume the value* 1 *on the complement of the L_d.*

Proof. The sets of the form $[0,a[$ and $]a,1]$, $0 < a < 1$ form a subbasis for the topology of $[0,1]$. Thus it suffices to show that their pre-images under f are open. This follows from the set-theoretic relations

$$f^{-1}[0,a[= \{x | f(x) < a\} = \bigcup(L_d \mid d < a\}$$
$$f^{-1}]a,1] = \{x | f(x) > a\} = \bigcup(X \smallsetminus L_d \mid d > a) = \bigcup(X \smallsetminus \overline{L}_d \mid d > a).$$

For the proof of the last equality one uses the condition $\overline{L}_d \subset L_e$ and the denseness of D. □

We use this lemma in the proof of the following ***Urysohn existence theorem***.

(2.3.4) Theorem. *Let X be a T_4-space and suppose that A and B are disjoint closed subsets of X. Then there exists a continuous function $f\colon X \to [0,1]$ with $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. Hence* (2.3.1) *and* (2.3.2) *hold for T_4-spaces X.*

Proof. We apply the lemma with the set of rational numbers $D = \{m/2^n \mid 0 \leq m \leq 2^n, m \in \mathbb{Z}\}$. The sets L_d are chosen inductively according to the power of 2 in the denominator. We set $L_1 = X \smallsetminus B$ and choose $A \subset L_0 \subset \overline{L}_0 \subset L_1$; this is possible by the T_4-property X. In the next steps we shuffle $\overline{L_0} \subset L_{1/2} \subset \overline{L}_{1/2} \subset L_1$ and then

$$\overline{L}_0 \subset L_{1/4} \subset \overline{L}_{1/4} \subset L_{1/2} \subset \overline{L}_{1/2} \subset L_{3/4} \subset \overline{L}_{3/4} \subset L_1$$

and proceed similarly in the general induction step. □

2.4 The Theorem of Stone–Weierstraß

Let X be a compact Hausdorff space and $C(X)$ the algebra of continuous functions $f\colon X \to \mathbb{R}$. The space $C(X)$ with the norm $|f| = \sup\{|f(x)| \mid x \in X\}$ is a Banach space. A subalgebra A of $C(X)$ is said to ***separate points*** if for each pair x, y of different points in X there exists a function in A which assume different values on x and y.

(2.4.1) Theorem (Stone–Weierstraß). *Let $A \subset C(X)$ be a subalgebra which separates the points and contains the constant functions. Then the closure of A is $C(X)$.*

Proof. (1) We can assume that A is closed in $C(X)$. Let us assume, in addition, that with f and g also the functions $\max(f, g)$ and $\min(f, g)$ are contained in A. Let x_1 and x_2 be different points of X and let a_i be real numbers. Then there exists $h \in A$ such that $h(x_i) = a_i$. In order to see this, we take $g \in A$ such that $g(x_1) \neq g(x_2)$, and then

$$h(x) = x_1 + (x_2 - x_1)\frac{g(x) - g(x_1)}{g(x_2) - g(x_1)}$$

has the desired property.

(2) Let $f \in C(X)$ and $\varepsilon > 0$ be given. We show that there exists $g \in A$ such that the inequalities $f - \varepsilon < g < f + \varepsilon$ hold. By (1) we can choose for each pair $x, y \in X$ a function $h_{x,y} \in A$ such that $h_{x,y}(x) = f(x)$ and $h_{x,y}(y) = f(y)$. Each $y \in X$ has an open neighbourhood U_y such that for $z \in U_y$ the inequality $h_{x,y}(z) < f(z) + \varepsilon$ holds. Let $U_{y(1)}, \dots, U_{y(n)}$ be a covering of X. By our additional assumption, the minimum h_x of the $h_{x,y(j)}$ is contained in A and satisfies $h_x < f + \varepsilon$ as well as $h_x(x) = f(x)$. Each $x \in X$ has then an open neighbourhood V_x such that for $z \in V_x$ the inequality $f(z) - \varepsilon < h_x(z)$ holds. Let $V_{x(1)}, \dots, V_{x(m)}$ be a covering of X and denote by g the maximum of the $h_{x(j)}$. This function has the desired property.

(3) It remains to be shown that our additional assumption always holds. Since $2\max(f,g) = |f+g| + |f-g|$ and $2\min(f,g) = |f+g| - |f-g|$ it suffices to see that for each $f \in A$ also $|f| \in A$. Let P be a polynomial in the variable $t \in \mathbb{R}$ such that for $t \in [-a,a]$ the inequality $|P(t) - |t|| < \varepsilon$ holds. Then also $|P(f(x)) - |f(x)|| < \varepsilon$ holds and $x \mapsto P(f(x))$ is contained in A.

(4) In order to find suitable polynomials P we reduce the problem via the substitution $t \mapsto at$ to the case that $a = 1$. Let $s = t^2$ and $0 \le s \le 1$. Define inductively $P_1 = 0$ and $P_{n+1}(s) = P_n(s) + \frac{1}{2}[s - P_n(s)^2]$. These polynomials converge uniformly to $\sqrt{s}$. A proof uses the identity

$$|s| - P_{n+1}(s) = (|t| - P_n(s))\left(1 - \tfrac{1}{2}(|s| + P_n(s))\right),$$

which is a consequence of the recursion formula. It implies inductively

$$0 \le P_n(s) \le P_{n+1}(s) \le |s|.$$

A first consequence is that the sequence $(P_n(s) \mid n \in \mathbb{N})$ always converges. Passing to the limit in the recursion formula we see that the limit is $|s|$. The theorem of Dini tells us that the convergence is uniform. One can also verify inductively the inequality

$$|s| - P_n(s) \le |s|\left(1 - \tfrac{1}{2}|s|\right)^n < \tfrac{2}{n+1}.$$

Then it is not necessary to use the theorem of Dini. □

(2.4.2) Corollary. *Let $X \subset \mathbb{R}^n$ be compact. Then a continuous function $X \to \mathbb{R}$ is a uniform limit of polynomials in n variables.* □

(2.4.3) Theorem. *Let $C(X,\mathbb{C})$ be the space of continuous functions $X \to \mathbb{C}$ with sup-norm. Suppose A is a complex subalgebra which contains the constant functions, separates the points, and contains with f also the complex conjugate function. The the closure of A is $C(X,\mathbb{C})$.*

Proof. Let f be any function. It suffices to show that the real and imaginary part of f are contained in the closure of A. This is a consequence of (2.4.1) if we show that the subalgebra A_0 of real-valued functions in A satisfies the hypothesis of that theorem. But with g also the real part $\frac{1}{2}(f + \overline{f})$ and the imaginary part $\frac{1}{2i}(f - \overline{f})$ are contained in A, and hence in A_0. If g separates the points x and y, then either the real part or the imaginary part separates these points. The other hypotheses of (2.4.1) certainly hold. □

Problems

1. Let X be a compact Hausdorff space and $\varphi\colon C(X) \to \mathbb{R}$ a homomorphism of $\mathbb{R}$-algebras. Then there exists $x \in X$ such that $\varphi(f) = f(x)$.

2. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Suppose that $\int_a^b f(t)t^n\,dt = 0$ for each $n \in \mathbb{N}_0$. Then $f = 0$. In order to see this note that $\int_a^b f(t)P(t)\,dt = 0$ for each polynomial P. Choose P such that $\|f - P\| < \varepsilon$. Then $|\int_a^b f^2(t)\,dt| = |\int_a^b (f-P)f| \le \varepsilon \|f\|(b-a)$. But $\int f^2 = 0$ implies $f = 0$.
3. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous function of period 2π. Then for each $\varepsilon > 0$ there exists $T(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ such that $\|f - T\| < \varepsilon$.

2.5 Convergence. Filter

Sequences are too small or too short in order to build a convergence theory for general spaces. One needs longer index sets. This leads to the notion of a net.

A ***directed set*** $(I, \le)$ consists of a set I and a relation $\le$ on I such that:
(1) $i \le i$ for all $i \in I$.
(2) $i < j$, $j \in k$ implies $i \le k$.
(3) For each pair $i, j \in I$ there exists $k \in I$ such that $i \le k$, $j \le k$.
We also write $j \ge i$ for $i \le j$. The set $\mathbb{N}$ with the usual order is directed. The set $\mathcal{U}(x)$ of neighbourhoods of x is directed by $U \le V \Leftrightarrow V \subset U$.

A ***net*** with directed index set I in X is a map $I \to X$, $i \mapsto x_i$. We write $(x_i)_{i \in I}$ or just (x_i) for such a net. A net (x_i) in a topological space X ***converges*** to x, notation $x = \lim x_i$, provided for each neighbourhood U of x there exists $i \in I$ such that for $j \ge i$ we have $x_j \in U$. If one chooses from each $U \in \mathcal{U}(x)$ a point x_U, then the net (x_U) with index set $\mathcal{U}(x)$ converges to x.

(2.5.1) Theorem. *Let X and Y be topological spaces and $f\colon X \to Y$ a map. Let $A \subset X$. Then the following holds:*
(1) *A point x is contained in $\overline{A}$ if and only if there exists a net in A which converges to x.*
(2) *The map f is continuous in $x \in X$ if and only if for each net $(x_i)_{i \in I}$ with limit x the net $(f(x_i))_{i \in I}$ converges to $f(x)$.*

Proof. (1) Let $x \in \overline{A}$. Given $U \in \mathcal{U}(x)$ choose $x_U \in U \cap A$. Then (x_U) is a net in A which converges to x.

Let $(X_i)_{i \in I}$ be a net in A which converges to x. If $U \in \mathcal{U}(x)$, there exists $x_i \in U$, hence $U \cap A \neq \emptyset$, hence x is a touch point of A.

(2) Let f be continuous in x and (x_i) a net which converges to x. Let V be a neighbourhood of $f(x)$. There exists i such that for $j \ge i$ we have $x_j \in f^{-1}(V)$, hence $f(x_j) \in V$ for $j \ge i$. This shows that $(f(x_i))$ converges to $f(x)$.

Suppose the convergence condition holds. If f is not continuous in x there exists a neighbourhood V of $f(x)$ such that no neighbourhood U of x is mapped under f into V. Choose $x_U \in U$ such that $f(x_U) \notin V$. The net (x_U) converges to x but $(f(x_U))$ does not converge. □

The directed sets $\mathcal{U}(x)$ are basic for convergence theory. Instead of choosing a point from each neighbourhood, one can right away work with $\mathcal{U}(x)$. This idea leads to the next definition.

A ***filter*** $\mathcal{F}$ on the set X is a set of subset with the properties:
(1) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$.
(2) $F_1, F_2 \in \mathcal{F} \quad \Rightarrow \quad F_1 \cap F_2 \in \mathcal{F}$.
(3) $F \in \mathcal{F}$, $G \supset F \quad \Rightarrow \quad G \in \mathcal{F}$.

A subset $\mathcal{F}_0 \subset \mathcal{F}$ is called a ***basis*** of $\mathcal{F}$ if each element of $\mathcal{F}$ contains an element of $\mathcal{F}_0$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filter and $\mathcal{F}_1 \supset \mathcal{F}_2$, then $\mathcal{F}_1$ is called ***finer*** than $\mathcal{F}_2$ and $\mathcal{F}_2$ ***coarser*** than $\mathcal{F}_1$. A filter which does not contain a different and finer one is called ***ultrafilter***.

A nonempty system $\mathcal{B}$ of nonempty subsets of X is a basis of a filter if and only if for each pair $B, C \in \mathcal{B}$ there exists $D \in \mathcal{B}$ such that $D \subset B \cap C$. The corresponding filter consists of the sets which contain a set of $\mathcal{B}$. The set $\mathcal{U}(x)$ is the ***neighbourhood filter*** of x.

(2.5.2) Theorem. *Each filter $\mathcal{F}$ is contained in an ultrafilter.*

Proof. The set of filters which are finer than $\mathcal{F}$ is ordered by inclusion. The union of a totally ordered subset in this set of filters is again a filter. Thus, by Zorn's lemma, this set contains a maximal element, and this is an ultrafilter. □

(2.5.3) Theorem. *A filter $\mathcal{F}$ is an ultrafilter if and only if for each $A \subset X$ either $A \in \mathcal{F}$ or $X \smallsetminus A \in \mathcal{F}$.*

Proof. Suppose always A or $X \smallsetminus A$ are in $\mathcal{F}$. Let $\mathcal{G} \supset \mathcal{F}$ be a filter. If $G \in \mathcal{G} \smallsetminus \mathcal{F}$, then $X \smallsetminus G \in \mathcal{F} \subset \mathcal{G}$. Since G and $X \smallsetminus G$ cannot be both elements of a filter, we reach a contradiction if $\mathcal{G} \neq \mathcal{F}$.

Let $\mathcal{F}$ be an ultrafilter and $A \subset X$. Suppose $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. Then $\{F \cap A \mid F \in \mathcal{F}\}$ is a basis for a filter which is finer than $\mathcal{F}$ and contains A. Since $\mathcal{F}$ is an ultrafilter, $A \in \mathcal{F}$. Similarly, if $(X \smallsetminus A) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. If both cases do not occur, then there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap A = \emptyset$ and $F_2 \cap (X \smallsetminus A) = \emptyset$. Then $F_1 \cap F_2 \subset (X \smallsetminus A) \cap A = \emptyset$, and this contradicts the definition of a filter. □

(2.5.4) Theorem. *The sets of an ultrafilter have a nonempty intersection if and only if the filter consists of all sets which contain a given point.*

Proof. By (2.5.3), the sets which contain a given point are an ultrafilter. If x is contained in all sets of a filter $\mathcal{F}$, then all sets which contain x form a filter which is finer than $\mathcal{F}$. □

We use filters as basic objects for convergence theory. A filter $\mathcal{F}$ on a topological space is said to ***converge*** to a point x if $\mathcal{F} \supset \mathcal{U}(x)$. Such points are called ***limit points*** or ***convergence points*** of the filter. A point x is called ***touch point*** of the filter $\mathcal{F}$ if each neighbourhood of x intersects each filter set. The set of touch points of a filter is the intersection of the closures of the filter sets. If $\mathcal{G} \supset \mathcal{F}$, then each touch point of $\mathcal{G}$ is a touch point of $\mathcal{F}$. A convergence point of a filter is a touch point.

(2.5.5) Theorem. *A point is a touch point of a filter if and only if there exists a finer filter which converges to this point.*

Proof. Let x be a touch point of the filter $\mathcal{F}$. Then $\{U \cap F \mid U \in \mathcal{U}(x),\ F \in \mathcal{F}\}$ is the basis of a filter which converges to x.

If $\mathcal{G} \supset \mathcal{F}$ and x is a convergence point of $\mathcal{G}$, then x is a touch point of $\mathcal{F}$. □

Let $f\colon X \to Y$ be a map and $\mathcal{F}$ a filter on X. Then $\{f(F) \mid F \in \mathcal{F}\}$ is the basis of a filter $f(\mathcal{F})$ on Y, called ***image filter*** of $\mathcal{F}$ under f.

(2.5.6) Theorem. *A map $f\colon X \to Y$ between topological spaces is continuous in x if and only if the image filter of each filter which converges to x converges to $f(x)$.*

Proof. Let f be continuous in x and let $\mathcal{F}$ converge to x. Let V be a neighbourhood of $f(x)$ and U a neighbourhood of x such that $f(U) \subset V$. Since $\mathcal{F}$ converges to x we have $U \in \mathcal{F}$ and hence $V \in f(\mathcal{F})$ since $f(U) \subset V$. Hence $f(\mathcal{F})$ is finer than $\mathcal{U}(f(x))$ and converges to $f(x)$.

Let $\mathcal{F} = \mathcal{U}(x)$. Each neighbourhood V of $f(x)$ belongs to $f(\mathcal{F})$ if $f(\mathcal{F})$ converges to $f(x)$. Thus there exists a neighbourhood U of x such that $f(U) \subset V$, since the $f(U)$ are a filter basis of $f(\mathcal{F})$. □

(2.5.7) Theorem. *Let X be a set, $(X_i \mid i \in I)$ be a family of topological spaces and $(f_i\colon X \to X_i \mid i \in I)$ a family of maps. Let X carry the coarsest topology such that each f_i is continuous. Then a filter $\mathcal{F}$ on X converges to x if and only if for each $i \in I$ the filter $f_i(\mathcal{F})$ converges to $f_i(x)$.*

Proof. The system of sets of the form $\bigcap_{k \in K} f_k^{-1}(U_k)$, $K \subset I$ finite, $U_k \in \mathcal{U}(f_k(x))$ is a neighbourhood basis of x. Suppose the $f_i(\mathcal{F})$ converge. Then there exists for each $U_k \in \mathcal{U}(f_k(x))$ an $F_k \in \mathcal{F}$ with $f_k(F_k) \subset U_k$. Then $F = \bigcap_{k \in K} F_k \in \mathcal{F}$, and F is contained in the basis set $f_k(F_k) \subset U_k$ of $\mathcal{U}(x)$.

The converse holds by (2.5.6). □

We can apply (2.5.7) to a topological product and the projections onto the factors.

(2.5.8) Theorem. *The following assertions about the topological space X are equivalent.*

(1) *X is compact.*
(2) *Each filter on X has a touch point.*
(3) *Each ultrafilter converges.*

Proof. (1) $\Rightarrow$ (2). Suppose $\mathcal{F}$ has no touch point. Then the intersection of the sets $\overline{F}$, $F \in \mathcal{F}$, is empty. Since X is compact, a finite intersection is empty. This contradicts the definition of a filter.
(2) $\Rightarrow$ (3). This is a consequence of (2.5.5).
(3) $\Rightarrow$ (1). Let $(U_j \mid j \in J)$ be an open covering without finite subcovering. For each finite $L \subset J$ the set $A_L = X \smallsetminus (\bigcup_{j\in L} U_j) = \bigcap_{j\in L}(X \smallsetminus U_j)$ is not empty. The system of the A_L is therefore a basis of a filter $\mathcal{F}$. Let $\mathcal{U} \supset \mathcal{F}$ be an ultrafilter. It converges to some point x. For some $j \in J$ we have $x \in U_j$. By convergence of $\mathcal{U}$, we have $U_j \in \mathcal{U}$; and by construction $X \smallsetminus U_j \in \mathcal{U}$. A contradiction. □

The following Theorem of Tychonoff is an important general result of topology.

(2.5.9) Theorem. *The product of compact spaces is compact.*

Proof. Let $(X_j \mid j \in J)$ be a family of compact spaces and $\mathcal{F}$ an ultrafilter on their product X. The image filter $\mathrm{pr}_j(\mathcal{F})$ on X_j is an ultrafilter: Suppose $\mathcal{G} \supset \mathrm{pr}_j(\mathcal{F})$; then the sets $\mathrm{pr}_i^{-1}(G)$, $G \in \mathcal{G}$ are a basis of a filter which contains $\mathcal{F}$. Since X_i is compact, by the previous theorem $p_i(\mathcal{F})$ converges to a point x_i, and by (2.5.7), $\mathcal{F}$ converges to (x_i). □

Problems

1. Let $f_i \colon X_{i+1} \to X_i$ for $i \in \mathbb{N}$ be continuous maps between non-empty compact Hausdorff-spaces. Then the set of sequences

$$\{(x_i) \mid x_i \in X_i,\ f_i(x_{i+1}) = x_i\}$$

is not empty. The set of these sequences, considered as a subspace of the product $\prod X_i$, is called ***(inverse) limit*** of the sequence f_i.

2.6 Proper Maps

The notion of a proper map codifies families of compact spaces. Among other things we characterize compact spaces without using coverings.

A continuous map $f \colon X \to Y$ is called **proper** if it is closed and the pre-images $f^{-1}(y), y \in Y$ are compact. From (1.10.8) we know:

(2.6.1) Proposition. *Let K be compact. Then* $\mathrm{pr}\colon X \times K \to X$ *is proper.* □

(2.6.2) Proposition. *If $f \colon X \to Y$ is proper and $K \subset Y$ compact, then $f^{-1}(K)$ is compact.*

Proof. Let $(U_j \mid j \in J)$ be an open covering of $f^{-1}(K)$. For each $c \in K$ there exists a finite $J_c \subset J$ such that $f^{-1}(c)$ is contained in the union U_c of the U_j, $j \in J_c$. The set $V_c = Y \smallsetminus f(X \smallsetminus U_c)$ is open, since f is closed. We have $c \in V_c$, $f^{-1}(V_c) \subset U_c$, and K is contained in a finite number of the V_c, in the union of $V_{c(1)}, \dots, V_{c(n)}$ say. We conclude that $f^{-1}(K)$ is contained in $U_{c(1)} \cup \dots \cup U_{c(n)}$. □

(2.6.3) Lemma. *A set map $f\colon X \to Y$ between topological spaces is closed if and only if for each neighbourhood U of $f^{-1}(y)$ there exists a neighbourhood V of y such that $f^{-1}(V) \subset U$.*

Proof. Let f be closed and U an open neighbourhood $f^{-1}(y)$. Then $X \smallsetminus U$ is closed and therefore also $f(X \smallsetminus U)$. The open neighbourhood $V = Y \smallsetminus f(X \smallsetminus U)$ has the desired property.

Suppose, conversely, that the condition holds. Let C be closed in X. Let $y \in Y \smallsetminus f(C)$. Then $f^{-1}(y)$ is contained in the open set $U = X \smallsetminus C$. Hence there exists an open neighbourhood V of y with $Y \subset Y \smallsetminus f(C)$. This shows that $Y \smallsetminus f(C)$ is open, hence $f(C)$ is closed. □

(2.6.4) Theorem. *The following assertion about a continuous map $f\colon X \to Y$ are equivalent:*
(1) *f is proper.*
(2) *For each space T the product $f \times \mathrm{id}\colon X \times T \to Y \times T$ is closed.*

Proof. (1) ⇒ (2). Let W be an open neighbourhood of $f^{-1}(y) \times \{t\}$ in $X \times T$. Since $f^{-1}(y)$ is compact, there exist a neighbourhoods U of $f^{-1}(y)$ and V of t, such that $U \times V \subset W$. Since f is closed, there exists, by (2.6.3), a neighbourhood N of y such that $f^{-1}(N) \subset U$. The set $N \times V$ is a neighbourhood of (y,t), and the inclusion

$$(f \times \mathrm{id})^{-1}(N \times V) \subset f^{-1}(N) \times V \subset U \times V \subset W$$

holds. The lemma then says that f is closed.

(2) ⇒ (1). If we use a point T, we see that f is closed. The hypothesis (2) shows that for each $y \in Y$ the map $\mathrm{pr}\colon f^{-1}(y) \times T \to T$ is closed, since in general with $f\colon X \to Y$ also $f\colon f^{-1}(B) \to B$ is closed. Therefore it remains to show the next theorem. □

(2.6.5) Theorem. *Suppose that for each T the projection $X \times T \to T$ is closed. Then X is compact.*

Proof. Let $\mathcal{W}$ be a set of open sets which cover X. Let $\mathcal{U}$ be the set of all finite unions of sets in $\mathcal{W}$. We have to show $X \in \mathcal{U}$.

Suppose this is not the case. We construct an auxiliary space $X' = X + \{*\}$. We furnish this set with a topology which has a basis $\mathcal{B}$ consisting of the sets:
(1) $X' \smallsetminus U$, $U \in \mathcal{U}$.

(2) $W \cap (X \smallsetminus U)$, $U \in \mathcal{U}$, $W \subset X$ open.

This is a basis of a topology, since the intersection of two of these sets is again a set of this form. Let $D = \{(x,x) \mid X\} \subset X \times X'$ and C the closure of D in $X \times X'$. By continuity of $\mathrm{pr}\colon X \times X' \to X'$ we have $\mathrm{pr}(C) = \mathrm{pr}(\overline{D}) \subset \overline{\mathrm{pr}(D)}$. Since, by hypothesis, pr is closed, we see that $\mathrm{pr}(\overline{D})$ is closed, hence also $\mathrm{pr}(\overline{D}) \supset \overline{\mathrm{pr}(D)}$. Altogether we see $\mathrm{pr}(C) = \overline{X}$.

The set X is not closed in X'. For if this were the case, then $X' \smallsetminus X = \{*\}$ would be open, hence a union of sets of the form (1) and (2), and this is not the case, by our assumption $X \neq U$ for all $U \in \mathcal{U}$. Since $\mathrm{pr}(C) = \overline{X} \neq X$, there exists $x \in X$ such that $(x,*) \in C$. We show that this point x is not contained a set $U \in \mathcal{U}$, contrary to our assumption that $\mathcal{U}$ is a covering. Suppose $x \in U$. Then $U \times (X' \smallsetminus U)$ is a neighbourhood of $(x,*)$ in $X \times X'$. Since $C = \overline{D} \ni (x,*)$, this neighbourhood meets D, and this would mean $U \cap (X \smallsetminus U) \neq \emptyset$. Contradiction. □

The next three propositions are easily verified from the definitions.

(2.6.6) Proposition. *Let $f\colon X \to X'$ and $g\colon X' \to X''$ be continuous. Then:*

(1) *If f and g are proper, then $g \circ f$ is proper.*

(2) *If $g \circ f$ is proper and f surjective, then g is proper.*

(3) *If $g \circ f$ is proper and g injective, then f is proper.* □

(2.6.7) Proposition. *Let $f\colon X \to Y$ be injective. Then the following are equivalent:*

(1) *f is proper.*

(2) *f is closed.*

(3) *f is a homeomorphism onto a closed subspace.* □

(2.6.8) Proposition. *Let $f\colon X \to Y$ be continuous.*

(1) *If f is proper, then for each subset $B \subset Y$ the restriction $f = f_B\colon f^{-1}(B) \to B$ is proper.*

(2) *Let $(U_j \mid j \in J)$ be a covering of Y such that the canonical map $p\colon \coprod_{j\in J} U_j \to Y$ is a quotient map. If each restriction $f_j\colon f^{-1}(U_j) \to U_j$ is proper, then f is proper.* □

(2.6.9) Proposition. *Let f be a continuous map of a Hausdorff space X into a locally compact Hausdorff space Y. Then f is proper if and only if each compact set $K \subset Y$ has a compact pre-image. If f is proper, then X is locally compact.*

Proof. Is f is compact, then we know from (2.6.2) that pre-images of compact sets are compact. Conversely, let (U_j) be a covering of Y by relatively compact open sets. Then $f^{-1}(\overline{U}_j)$ is compact and f, restricted to these sets, is proper, for a continuous map of a compact Hausdorff space into a Hausdorff space is proper. By (2.6.8), f is proper. □

(2.6.10) Theorem. *Let $f\colon X \to X'$ and $g\colon X' \to X''$ be continuous and assume that gf is proper. If X' is a Hausdorff space, then f is proper.*

Proof. We consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "(\mathrm{id},f)"] \arrow[d, "f"] & X \times X' \arrow[d, "gf\times \mathrm{id}"] \\ X' \arrow[r, "(g,\mathrm{id})"] & X'' \times X'. \end{tikzcd}$$

The horizontal maps are homeomorphisms onto the graph of f and onto the interchanged graph of g. Since X' is a Hausdorff space, the graph of f is closed and hence (id, f) proper (2.6.7). By (2.6.13), $gf \times \mathrm{id}$ is proper. The commutativity then shows that $(g, \mathrm{id})\circ f$ is proper, and since (g, id) is injective, we see from (2.6.6) that f is proper. □

(2.6.11) Theorem. *The following statements about a continuous map f are equivalent:*

(1) *f is proper.*
(2) *If $\mathcal{F}$ is a filter on X and y a touch point of $f(\mathcal{F})$, then there exists a touch point x of $\mathcal{F}$ with $f(x) = y$.*
(3) *If $\mathcal{F}$ is an ultrafilter on X and if $f(\mathcal{F})$ converges to y, then there exists a convergence point x of $\mathcal{F}$ with $f(x) = y$.*

Proof. (1) $\Rightarrow$ (2). Let $M \in \mathcal{F}$. Since f is closed, the equality $f(\overline{M}) = \overline{f(M)}$ holds. Being a touch point of $f(\mathcal{F})$ the point y is contained in each $f(\overline{M})$, hence $\overline{M} \cap f^{-1}(y)$ is non-empty. Since $f^{-1}(y)$ is compact, there exists an $x \in f^{-1}(y)$ which is contained in each $\overline{M}, M \in \mathcal{F}$. But this means that x is a touch point of $\mathcal{F}$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). We begin by showing that f is closed. Let $\emptyset \neq A \subset X$ be closed. Let $\mathcal{F}$ be the filter of the sets which contain A. Then A is the set of touch points of $\mathcal{F}$. Let B be the set of touch points of $f(\mathcal{F})$. Then B is closed and contains $f(A)$. We show $B = f(A)$.

Suppose $y \in B$. Each neighbourhood V of y intersects $f(\mathcal{F})$, hence $f^{-1}(V)\cap F \neq \emptyset$ for each $F \supset A$. Therefore the $f^{-1}(V) \cap F$ for a filter basis. Let $\mathcal{U}$ be an ultrafilter which is finer. The ultrafilter $f(\mathcal{U})$ is finer than $\mathcal{U}(y)$, hence converges to y. By (3), there exists a convergence point x of $\mathcal{U}$ with $f(x) = y$. Since $\mathcal{U} \supset \mathcal{F}$, the element x is a touch point of $\mathcal{F}$ and therefore contained in A, hence $y \in f(A)$.

The next theorem is used to finish the proof. It implies that with f also the product $f \times \mathrm{id}(Z)$ is closed. □

(2.6.12) Theorem. *Suppose that condition* (3) *of the previous theorem holds for every map $f_i\colon X_i \to Y_i$. Then it also holds for the product $f = \prod_i f_i$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\prod_i X_i$ and $y = (y_i) \in \prod_i Y_i$ a convergence point of $f(\mathcal{U})$. By (2.6.11), $\mathrm{pr}_i(f(\mathcal{U})) = f_i(\mathrm{pr}_i(\mathcal{U}))$ converges to y_i. By condition (3) there exists then for each i a $x_i \in X_i$ such that $f_i(x_i) = y_i$ and $\mathrm{pr}_i(\mathcal{U})$ converges to x_i. Then, by (2.6.11), $\mathcal{U}$ converges to $x = (x_i)$, and the equality $f(x) = y$ holds. □

(2.6.13) Corollary. *Any product of proper maps is proper.* □

Problems

1. Let f and g be proper. Then $f \times g$ is proper. This is a special case of (2.6.12). Give a more elementary proof, using (**??**).
2. Let X and Y be locally compact Hausdorff spaces, let $f\colon X \to Y$ be continuous and $f^+\colon X^+ \to Y^+$ the extension to the one-point compactification. Then f^+ is continuous, if f is proper.
3. The restriction of a proper map to a closed subset is proper.
4. Let $f\colon X \to Y$ be proper and X a Hausdorff space. Then the subspace $f(X)$ of Y is a Hausdorff space.
5. Let $f\colon X \to Y$ be continuous. Let R be the equivalence relation on X induced by f, and denote by $p\colon X \to X/R$ the quotient map, by $h\colon X/R \to f(X)$ the canonical bijection, and let $i\colon f(X) \subset Y$. Then $f = i \circ h \circ p$ is the canonical decomposition of f. The map f is proper if and only if p is proper, h a homeomorphism, and $f(X) \subset Y$ closed.

2.7 Paracompact Spaces

Let $\mathcal{A} = (U_j \mid j \in J)$ be an open covering of the space X. An open covering $\mathcal{B} = (B_j \mid j \in J)$ is called a ***shrinking*** of $\mathcal{A}$ if for each $j \in J$ we have the inclusion $\overline{B}_j \subset U_j$. An open covering $\varphi = (C_j \mid j \in J)$ is called a ***partial shrinking*** with respect to $K \subset J$ if $\overline{C}_j \subset U_j$ for $j \in K$ and $C_j = U_j$ for $j \notin K$. Let (C_j) and (C'_j) be partial shrinkings with respect to K and K'; we define $C \le C'$ by $K \subset K'$, and $C_j = C'_j$ for $j \in K$.

(2.7.1) Lemma. *Let $(U_j \mid j \in J)$ be a point-finite open covering of X. Then the set of partial shrinkings is inductively ordered by $\le$.*

Proof. The following assertion is claimed: Let $\varphi^s = \big((C^s_j), K^s\big)$ be a totally ordered set of partial shrinkings ($s \in S$); then there exists a partial shrinking φ with $\varphi^s \le \varphi$ for all s. Let $K = \cup K^s$ and $C_j = C^s_j$ for $j \in K^s$, $C_j = U_j$ for $j \notin K$. This is a well-defined system of open sets C_j. We show: The C_j are a covering. Let $x \in X$ be given. The set $J(x) = \{j \in J \mid x \in U_j\}$ is finite, by the point finite assumption. If $j \in J(x) \cap (J \smallsetminus K)$, then $x \in U_j$. If $J(x) \subset K$, then there exists s with $J(x) \subset K^s$. Then $x \in C_l$ for an $l \in K^s \subset K$. □

(2.7.2) Proposition. *A point-finite open covering of a normal space has a shrinking.*

Proof. By (2.7.1) and Zorn's lemma, there exists a maximal partial shrinking $((C_j \mid j \in J), K)$ of the point-finite covering $(U_j \mid j \in J)$. Suppose $k \notin K$, and set $L = K \cup \{k\}$. Let D be the complement of $(\bigcup_{j \in K} C_j) \cup (\bigcup_{j \notin L} U_j)$. This is a closed subset, and is contained in U_k, since the (C_j) are a covering. We choose an open set C_k such that $D \subset C_k \subset \overline{C}_k \subset U_k$ and replace U_k by C_k. This is a larger partial shrinking, contradicting the maximality. Thus $K = J$. □

A space X is called ***paracompact*** if it is a Hausdorff space and if each open covering has an open, locally finite refinement. From this definition one verifies easily: A closed subset of a paracompact space is paracompact. A compact space is paracompact.

(2.7.3) Theorem. *A paracompact space is normal.*

Proof. Let A and B be closed disjoint sets of the Hausdorff space X. Let $(U_j \mid j \in J)$ be a locally finite family in X which covers A. Assume that for each $j \in J$ there exists an open neighbourhood V_j of B which is disjoint to U_j.

We claim that under these assumptions there exists an open neighbourhood W of B which is disjoint to $U = \bigcup_{\in J} U_j$.

Since (U_j) is locally finite, there exists for each $y \in B$ an open neighbourhood $W(y)$ such that $J(y) = \{j \in J \mid W(y) \cap U_j \neq \emptyset\}$ is finite. Then

$$W'(y) = W(y) \cap \bigcap\nolimits_{j \in J(y)} V_j$$

is an open neighbourhood of y which meets no U_j. Hence $W = \bigcup_{y \in B} W'(y)$ is an open neighbourhood of B, which is disjoint to the open neighbourhood $U = \bigcup_{j \in J} U_j$ of A.

Suppose now that X is paracompact. We consider disjoint closed sets A and $B = \{b\}$. Since X is separated, we can, by passing to a locally finite refinement, satisfy the hypothesis of the first paragraph. We therefore find for each $b \in B$ an open neighbourhood V_b and a disjoint open neighbourhood W_b of A. We refine the covering $(U_j \mid j \in J)$. Then we argue again as in the first paragraph, but now with the roles of A and B interchanged. □

(2.7.4) Theorem. *Suppose the locally compact Hausdorff space X is a countable union of compact sets. Then X is paracompact.*

Proof. Choose an open covering $(U_n \mid n \in \mathbb{N})$ of X properties as in (2.2.2). Let $(V_j \mid j \in J)$ be an open covering of X. For each $x \in \bar{U}_n \smallsetminus U_{n-1} = K_n$ there exists in $U_{n+1} \smallsetminus \bar{U}_{n-2}$ an open neighbourhood which is contained in one of the sets V_j. We choose a finite number of them which cover K_n. We do this for each n and obtain a locally finite refinement of (V_j). □

(2.7.5) Theorem. *Let X be paracompact and K be compact Hausdorff. Then $X \times K$ is paracompact.*

Proof. Let $(U_j \mid j \in J)$ be an open covering of $X \times K$. For each $(x,k) \in X \times K$ choose open neighbourhoods $V(x,k)$ of x and $W(x,k)$ of k such that $V(x,k) \times W(x,k)$ is contained in a set U_j. Suppose $W(x,k_1), \dots, W(x,k_n)$ cover K. We define $U(x)$ as the intersection of the $V(x,k_j)$. The $U(x)$ then form an open covering of X. Suppose it is is refined by the locally finite open covering $(C_a \mid a \in A)$. For each $a \in A$ choose x_a with $C_a \subset U(x_a)$ and consider the finite covering $W(x_a,k)$, $k \in J(x_a)$ which was chosen for x_a. Then the sets $C_a \times W(x_a,k)$, $a \in A$, $k \in J(x_a)$ form a locally finite refinement of (U_j). □

2.8 Partitions of Unity

Let $t\colon X \to \mathbb{R}$ be continuous. The closure of $t^{-1}(\mathbb{R} \smallsetminus 0)$ is the ***support*** $\operatorname{supp}(t)$ of t. A family $T = (t_j\colon X \to \mathbb{R} \mid j \in J)$ of continuous functions is said to be ***locally finite*** if the family of supports $(\operatorname{supp}(t_j) \mid j \in J)$ is locally finite. We call T a ***partition of unity*** if the t_j assume only non-negative values and if for each $x \in X$ we have $\sum_{j \in J} t_j(x) = 1$. A covering $\mathcal{U} = (U_j \mid j \in J)$ is ***numerable*** if there exists a partition of unity T such that $\operatorname{supp}(t_j) \subset U_j$ holds for each $j \in J$; the family T is then called a ***numeration*** of $\mathcal{U}$.

(2.8.1) Theorem. *A locally finite open covering of a normal space is numerable.*

Proof. Let $U = (U_j \mid j \in J)$ be a locally finite covering of the normal space X and $V = (V_j \mid j \in J)$ a shrinking of U and $W = (W_j \mid j \in J)$ a shrinking of V. By the theorem of Urysohn there exist continuous functions $\tau_j\colon X \to [0,1]$ which assume the value 1 on W_j and the value 0 on the complement of V_j. The function $\tau = \sum_{j \in J} \tau_j\colon X \to [0,1]$ is well-defined and continuous, since by local finiteness of V, in a suitable neighbourhood of a point only a finite number of τ_j are non-zero. We set $f_j(x) = \tau_j(x) \cdot \tau^{-1}(x)$. The functions $(f_j \mid j \in J)$ are a numeration of U. □

(2.8.2) Lemma. *Let the covering $V = (V_k \mid k \in K)$ be a refinement of the covering $U = (U_j \mid j \in J)$. If V is numerable, then also U is numerable.*

Proof. Let $(f_k \mid k \in K)$ be a numeration of V. For each $k \in K$ choose $a(k) \in J$ with $V_k \subset U_{a(k)}$. This defines a map $a\colon K \to J$. We set $g_j(x) =$

$\sum_{k,a(k)=j} f_k(x)$; this is the zero function if the sum is empty. Then g_j is continuous; the support of g_j is the union of the supports of the f_k with $a(k) = j$ and is therefore contained in U_j. Moreover, the sum of the g_j is one. The family $(g_j \mid j \in J)$ is locally finite: If W is an open neighbourhood of x which meets only a finite number of supports $\mathrm{supp}(f_k)$, $k \in E \subset J$, E finite, then W meets only the supports of the g_j with $j \in a(E)$. □

(2.8.3) Theorem. *Each open covering of a paracompact space is numerable.*

Proof. Let $U = (U_j \mid j \in J)$ be an open covering of the paracompact space X and let $V = (V_k \mid k \in K)$ be a locally finite refinement. since X is normal, there exists a numeration $(f_k \mid k \in K)$ of V. Now apply the previous lemma. □

(2.8.4) Lemma. *Let $(f_j \colon X \to [0, \infty[\mid j \in J)$ be a family of continuous functions such that $U = (f^{-1}]0, \infty[\mid j \in J)$ is a locally finite covering of X. Then U is numerable and has, in particular a shrinking.*

Proof. Since U is locally finite, $f \colon x \mapsto \max(f_j(x) \mid j \in J)$ is continuous and nowhere zero. We set $g_j(x) = f_j(x)f(x)^{-1}$. Then

$$t_j \colon X \to [0,1], \quad x \mapsto \max(2g_j(x) - 1, 0)$$

is continuous. Since $t_j(x) > 0 \Leftrightarrow g_j(x) > 1/2$, we have the inclusions $\mathrm{supp}(t_j) \subset g_j^{-1}[1/4, \infty[\subset f^{-1}]0, \infty[$. For $x \in X$ and $i \in J$ with $f_i(x) = \max(f_j(x))$ we have $t_i(x) = 1$. Hence the supports of the t_j form a locally finite covering of X, and the functions $x \mapsto t_i(x)/t(x)$, $t(x) = \sum_{j \in J} t_j(x)$ are a numeration of U. □

(2.8.5) Theorem. *Let $\mathcal{U} = (U_j | j \in J)$ be a covering of the space X. The following assertions are equivalent:*

(1) *$\mathcal{U}$ is numerable.*

(2) *There exists a family $(s_{a,n} \colon X \to [0, \infty[\mid a \in A,\ n \in \mathbb{N}) = S$ of continuous functions $s_{a,n}$ with the properties:*
   - (a) *S, i.e., $(s_{a,n}^{-1}]\,0, \infty\,[)$, refines $\mathcal{U}$.*
   - (b) *For each n the family $(s_{a,n}^{-1}]0, \infty[\mid a \in A)$ is locally finite.*
   - (c) *For each $x \in X$ there exists (a, n) such that $s_{a,n}(x) > 0$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). $(s_{a,n})$ is, by assumption, a countable union of locally finite families. From these data we construct a locally finite family. By replacing $s_{a,n}$ with $s_{a,n}/(1 + s_{a,n})$ we can assume that $s_{a,n}$ has an image contained in $[0, 1]$. Let

$$q_r(x) = \sum_{a \in A,\, i<r} s_{a,i}(x), \qquad r \geq 1$$

and $q_r(x) = 0$ for $r = 0$. (The sum is finite for each $x \in X$.) Then q_r and

$$p_{a,r}(x) = \max(0, s_{a,r}(x) - rq_r(x))$$

are continuous. Let $x \in X$; Then there exists $s_{a,k}$ with $s_{a,k}(x) \neq 0$; we choose such a function with minimal k; then $q_k(x) = 0$, $p_{a,k}(x) = s_{a,k}(x)$. Therefore the sets $p_{a,k}^{-1}\,]0,1]$ also cover X. Choose $N \in \mathbb{N}$ such that $N > k$ and $s_{a,k}(x) > \frac{1}{N}$. Then $q_N(x) > \frac{1}{N}$, and therefore $Nq_N(y) > 1$ for all y in a suitable neighbourhood of x. In this neighbourhood, all $p_{a,r}$ with $r \geq N$ vanish. Hence

$$(p_{a,n}^{-1}\,]0,1] \mid a \in A,\ n \in \mathbb{N})$$

is a locally finite covering of X which refines $\mathcal{U}$. We finish the proof by an application of the previous lemma. □

(2.8.6) Theorem. *A metrizable space is paracompact.*

Proof. Let $(U_j \mid j \in J)$ be an open covering of the metric space (X, d). Suppose the index set J is well-ordered. For $i \in J$ and $n \in \mathbb{N}$ consider

$$B_{i,n} = \{x \in X \mid d(x, X \smallsetminus U_i) \geq 2^{-n}; d(x, X \smallsetminus U_j) \leq 2^{-n-1} \text{ for all } j < i\}$$

and the function

$$s_{i,n}(x) = \max(0, 2^{-n-3} - d(x, B_{i,n})).$$

Fix $x \in X$. Let i be the minimal index such that $x \in U_i$; it exists since J is well-ordered. There exists an n such that $d(x, X \smallsetminus U_i) > 2^{-n}$, since $X \smallsetminus U_i$ is closed. For $j < i$ then $x \in X \smallsetminus U_j$ such that, altogether, $x \in B_{i,n}$ and $s_{i,n}(x) > 0$.

We now show: For $j < i$ the sets $s_{i,n}^{-1}\,]0,\infty[$ and $s_{j,n}^{-1}\,]0,\infty[$ are disjoint. The inequality $s_{i,n}(x) > 0$ implies $d(x, B_{i,n}) < 2^{-n-3}$. Therefore there exists $y \in B_{i,n}$ with $d(x,y) < 2^{-n-3}$. We now use the definition of the $B_{i,n}$ and obtain

$$d(x, X \smallsetminus U_i) \geq 2^{-n} - 2^{-n-3} \qquad d(x, X \smallsetminus U_j) \leq 2^{-n-1} + 2^{-n-3}.$$

If $s_{j,n} > 0$, we conclude similarly

$$d(x, X \smallsetminus U_j) \geq 2^{-n} - 2^{-n-3}.$$

Since $2^{-n-1} + 2^{-n-3} < 2^{-n} - 2^{-n-3}$, both inequalities cannot hold simultaneously for the index j. We also see that $s_{i,n}(x) > 0$ implies $x \in U_i$.

We thus have verified the hypotheses of (2.8.5). □

(2.8.7) Theorem. *Let $(U_j \mid j \in J)$ be a numerable covering of $B \times [0,1]$. Then there exists a numerable covering$(V_k \mid k \in K)$ of B and a family $(\epsilon(k) \mid k \in K)$ of positive real numbers such that for $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and $|t_1 - t_2| < \epsilon(k)$ there exist a $j \in J$ with $V_k \times [t_1, t_2] \subset U_j$.*

Proof. Let $(t_j \mid j \in J)$ be a numeration of (U_j). For each r-tuple $k = (j_1, \ldots, j_r) \in J^r$ define a continuous map

$$v_k \colon B \to I,\ x \mapsto \prod_{i=1}^{r} \min\left(t_{j_i}(x,s) \mid s \in \left[\frac{i-1}{r+1}, \frac{i+1}{r+1}\right]\right).$$

Let $K = \bigcup_{r=1}^{\infty} J^r$. We show that the $V_k = v_k^{-1}]0,1]$ and $\epsilon(k) = \frac{1}{2r}$ for $k = (j_1, \ldots, j_r)$ satisfy the requirements of the theorem. Namely if $|t_1 - t_2| < \frac{1}{2r}$, there exists i with $[t_1, t_2] \subset [\frac{i-1}{r+1}, \frac{i+1}{r+1}]$ and hence $V_k \times [t_1, t_2] \subset U_{j_i}$.

We show that (V_k) is a covering. Let $x \in B$ be given. Each point (x,t) has an open neighbourhood of the form $U(x,t) \times V(x,t)$ which is contained in a suitable set $W(i) = t_i^{-1}]0,1]$ and meets only a finite number of the $W(j)$. Suppose $V(x,t_1), \ldots, V(x,t_n)$ cover the interval $I = [0,1]$; let $\frac{2}{r+1}$ be a Lebesgue number of this covering. We set $U = U(x,t_1) \cap \ldots \cap U(x,t_n)$. Each set $U \times [\frac{i-1}{r+1}, \frac{i+1}{r+1}]$ is then contained in a suitable $W(j_i)$. Hence x is contained in V_k, $k = (j_1, \ldots, j_r)$.

There are only a finite number of $j \in J$ for which $W(j) \cap (U \times I) \neq \emptyset$. Since $v_k(x) \neq 0$ implies the relation $W(j_i) \cap \{x\} \times I \neq \emptyset$, the family $(V_k \mid k \in J^r)$ is locally finite for r fixed. The existence of a numeration for $(V_k \mid k \in K)$ follows now from theorem (2.8.5). □

A family of continuous maps $(t_j \colon X \to [0,1] \mid j \in J)$ is called a ***generalized partition of unity*** if for each $x \in X$ the family $(t_j(x) \mid j \in J)$ is summable with sum 1.

(2.8.8) Lemma. *Let $(t_j \mid j \in J)$ be a generalized partition of unity. Then $(t_j^{-1}]0,1] \mid j \in J)$ is a numerable covering.*

Proof. Summability of $(t_j(a))$ means: For each $\varepsilon > 0$ there exists a finite set $E \subset J$ such that for all finite sets $F \supset E$ the inequality $|1 - \sum_{j \in F} t_j(a)| > 1 - \varepsilon$ holds. In that case $V = \{x \mid \sum_{j \in E} t_j(x) > 1 - \varepsilon\}$ is an open neighbourhood of a. If $k \notin E$, $x \in V$ and $t_k(x) > \varepsilon$, then $t_k(x) + \sum_{j \in E} t_j(x) > 1$. This is impossible. Hence for each $a \in X$ there exists an open neighbourhood $V(a)$ such that only a finite number of functions t_j have a value greater then ε on $V(a)$. Let $s_{j,n}(x) = \max(t_j(x) - n^{-1}, 0)$ for $j \in J$ and $n \in \mathbb{N}$. By what we have just shown, the $s_{j,n}$ are locally finite for fixed n. The claim now follows from (2.8.5). □

(2.8.9) Theorem. *Let $\mathcal{U} = (U_j \mid j \in J)$ be a numerable covering of $X \times K$ and K a compact Hausdorff space. Then there exists a numerable covering $(V_i \mid i \in I)$ of X and for each $i \in I$ a finite numerable covering $(W_\ell \mid \ell \in L_i)$ of K such that the $V_i \times W_\ell$, $\ell \in L_i$ form a numerable covering of $X \times K$ which refines (U_j).*

Proof. We use notion which will be introduced later. Let $B(\mathcal{U})$ be the geometric realization of the nerve $N(\mathcal{U})$ of $\mathcal{U}$ with the metric topology. Let $(t_j \mid j \in J)$ be a numeration of $\mathcal{U}$. We obtain a continuous map into the mapping space with sup-metric $\tau : X \to B(\mathcal{U})^K$ by setting $\tau(x)(k) = \sum_{j \in J} t_j(x,k)[j]$. (Note: the space $B(\mathcal{U})$ is a set of functions $f \colon J \to [0,1]$ which we write as $\sum f(j)[j]$.)

To begin with, we show that $B(\mathcal{U})^K \times K$ has a suitable covering which we then pull back to $X \times B$ via $\tau \times \mathrm{id}$. For this purpose, we consider the set A of all functions $a : \mathcal{K}(a) \to J$, where $\mathcal{K}(a)$ is a finite numerable covering of K by compact sets. For each function we consider

$$V_a = \{x \in X \mid x \times C \subset U_{a(C)},\, C \in \mathcal{K}_a\}.$$

We claim:

(1) $(V_a \mid a \in A)$ is a numerable covering of X.

(2) $(V_a \times C \mid a \in A,\, c \in \mathcal{K}_a)$ is a numerable covering of $X \times K$ which refines $\mathcal{U}$.

We use for this purpose analogous sets in $B(\mathcal{U})^K \times K$. Let $\lambda_j \colon B(\mathcal{U}) \to [0,1]$ be the barycentric coordinate which belongs to the index j. Let

$$B_a = \{f \in B(\mathcal{U})^K \mid \lambda_{a(K)} \circ f(c) \neq 0,\, c \in C,\, C \in \mathcal{K}_a\}.$$

Then

$$V_a \supset \tau^{-1}(B_a);$$

for if $\tau(x) \in B_a$, then $t_{a(K)}(x,c) \neq 0$, $c \in C$, $C \in \mathcal{K}_a$, i.e.,

$$x \times C \subset t^{-1}_{a(C)}]0,1] \subset U_{a(C)}.$$

In order to see that (V_a) is a numerable covering it suffices to show that (B_a) is a numerable covering of $B(\mathcal{U})^K$. But the latter space is metric hence paracompact. Therefore it suffices to show that the interiors (B_a°) are a covering.

Let $f \in B(\mathcal{U})^K$ be given. The sets $\lambda_j^{-1}]0,1]$ form an open covering of $B(\mathcal{U})$. Therefore there exists a function $a \in A$ with $f(C) \subset \lambda^{-1}_{a(C)}]0,1]$ for all $C \in \mathcal{K}_a$, and this means $f \in B_a$.

Since also $B(\mathcal{U})^K \times K$ is paracompact, by (2.7.5), the family $(B_a \times C \mid a \in A,\, C \in \mathcal{K}_a)$ is numerable. This shows (2). □

(2.8.10) Theorem. *Let X be a metric space, E a normed vector space and A a non-empty closed set in X. A continuous map $f \colon A \to E$ has a continuous extension $F \colon X \to E$. One can choose F such that $F(X)$ is contained in the convex hull of $f(A)$.*

Proof. Let $p \in X \smallsetminus A$ and set

$$U_p = \{x \in X \mid 2d(x,p) < d(p,A)\}.$$

Let (φ_p) be a partition of unity which is subordinate to the open covering (U_p) of $X \smallsetminus A$. We then define

$$F(x) = \begin{cases} f(x), & x \in A \\ \sum_{p \in X \smallsetminus A} \varphi_p(x) f(a(p)), & x \in X \smallsetminus A \end{cases}$$

with a point $a(p) \in A$ which satisfies $d(p,a(p)) < 2d(p,A)$. The map F is continuous on A and on $X \smallsetminus A$. The continuity is only a problem at points x_0 in the boundary of A. For $x \in U_p$ we have

$$d(x_0,p) \le d(x_0,x) + d(x,p) < d(x_0,x) + \frac{1}{2}d(p,A) \le d(x_0,x) + \frac{1}{2}d(p,x_0),$$

hence $d(x_0,p) < 2d(x_0,x)$ for $x \in U_p$. Since $d(p,a(p)) < 2d(p,A) \le 2d(p,x_0)$, we conclude for $x \in U_p$

$$d(x_0,a(p)) \le d(x_0,p) + d(p,a(p)) < 3d(p,x_0) < 6d(x_0,x).$$

For $x \in X \smallsetminus A$ we have

$$\|F(x) - F(x_0)\| \le \textstyle\sum_p \varphi_p(x) \|f(a(p)) - f(x_0)\|$$

with a sum over the $p \in X \smallsetminus A$ with $x \in U_p$. Given $\varepsilon > 0$ we choose $\delta > 0$, by continuity of f, such that $\|f(y) - f(x_0)\| < \varepsilon$, provided $y \in A$ and $d(x_0,y) < 6\delta$. For $x \in X \smallsetminus A$ and $d(x,x_0) < \delta$ we conclude for p with $x \in U_p$ by the inequality above $d(x_0,a(p)) < 6\delta$, hence $\|f(a(p)) - f(x_0)\| < \varepsilon$, and altogether we arrive at $\|F(x) - F(x_0)\| \le \sum \varphi_p(x)\varepsilon = \varepsilon$. □

2.9 Mapping Spaces and Homotopy

It is customary to endow sets of continuous maps with a topology. In this section we study the compact-open topology on mapping spaces. This topology enables us to consider a homotopy $H \colon X \times I \to Y$ as a family of paths in Y, parametrised by X.

We denote by Y^X or $F(X,Y)$ the set of continuous maps $X \to Y$. For $K \subset X$ and $U \subset Y$ we set $W(K,U) = \{f \in Y^X \mid f(K) \subset U\}$. The ***compact-open topology*** (CO-topology) on Y^X is the topology which has as a subbasis the sets of the form $W(K,U)$ for compact $K \subset X$ and open $U \subset Y$. In the sequel the set Y^X always carries the CO-topology. A continuous map $f \colon X \to Y$ induces continuous maps $f^Z \colon X^Z \to Y^Z$, $g \mapsto fg$ and $Z^f \colon Z^Y \to Z^X$, $g \mapsto gf$.

Recall: A space X is ***locally compact*** if each point has a neighbourhood basis consisting of compact sets. If X is a Hausdorff space and if each point has a compact neighbourhood, then X is locally compact. Thus a compact Hausdorff space is locally compact. If X and Y are locally compact so is their product $X \times Y$.

(2.9.1) Proposition. *Let X be locally compact. Then the* **evaluation** $e_{X,Y} = e\colon Y^X \times X \to Y$, $(f,x) \mapsto f(x)$ *is continuous.*

Proof. Let U be an open neighbourhood of $f(x)$. Since f is continuous and X locally compact, there exists a compact neighbourhood K of x such that $f(K) \subset U$. The neighbourhood $W(K,U) \times K$ of (f,x) is therefore mapped under e into U. This shows the continuity of e at (f,x). □

(2.9.2) Proposition. *Let $f\colon X \times Y \to Z$ be continuous. Then the* **adjoint map** $f^\wedge\colon X \to Z^Y$, $f^\wedge(x)(y) = f(x,y)$ *is continuous.*

Proof. Let $K \subset Y$ be compact and $U \subset Z$ open. It suffices to show that $W(K,U)$ has an open pre-image under $f^\wedge$. Let $f^\wedge(x) \in W(K,U)$ and hence $f(\{x\} \times K) \subset U$. Since K is compact, there exists a neighbourhood V of x in X such that $V \times K \subset f^{-1}(U)$ and hence $f^\wedge(V) \subset W(K,U)$. □

From (2.9.2) we obtain a set map $\alpha\colon Z^{X\times Y} \to (Z^Y)^X$, $f \mapsto f^\wedge$. Let $e_{Y,Z}$ be continuous. A continuous map $\varphi\colon X \to Z^Y$ induces a continuous map $\varphi^\vee = e_{Y,Z} \circ (\varphi \times \mathrm{id}_Y)\colon X \times Y \to Z^Y \times Y \to Z$. Hence we obtain a set map $\beta\colon (Z^Y)^X \to Z^{X\times Y}, \varphi \mapsto \varphi^\vee$.

(2.9.3) Proposition. *Let $e_{Y,Z}$ be continuous. Then α and β are inverse bijections. Thus $\varphi\colon X \to Z^Y$ is continuous if $\varphi^\vee\colon X \times Y \to Z$ is continuous, and $f\colon X \times Y \to Z$ is continuous if $f^\wedge\colon X \to Z^Y$ is continuous.* □

(2.9.4) Corollary. *If $h\colon X \times Y \times I \to Z$ is a homotopy, then $h^\wedge\colon X \times I \to Z^Y$ is a homotopy* (see (2.9.2)). *Hence $[X \times Y, Z] \to [X, Z^Y]$, $[f] \mapsto [f^\wedge]$ is well-defined. If, moreover, $e_{Y,Z}$ is continuous, e.g., Y locally compact, then this map is bijective* (see (2.9.3)). □

(2.9.5) Definition (Dual version of homotopy). We have the continuous evaluation $e_t\colon Y^I \to Y$, $w \mapsto w(t)$. A ***homotopy*** from $f_0\colon X \to Y$ to $f_1\colon X \to Y$ is a continuous map $h\colon X \to Y^I$ such that $e_\varepsilon \circ h = f_\varepsilon$ for $\varepsilon = 0, 1$. The equivalence with our original definition follows from (2.9.3): Since I is locally compact, continuous maps $X \times I \to Y$ correspond bijectively to continuous maps $X \to Y^I$. ◇

(2.9.6) Theorem. *Let Z be locally compact. Suppose $p\colon X \to Y$ is a quotient map. Then $p \times \mathrm{id}(Z)\colon X \times Z \to Y \times Z$ is a quotient map.*

Proof. We verify for $p \times \mathrm{id}$ the universal property of a quotient map: If $h\colon Y \times Z \to C$ is a set map and $h \circ (p \times \mathrm{id})$ continuous, then h is continuous. The adjoint of $h \circ (p \times \mathrm{id})$ is $h^\wedge \circ p$. By (2.9.2), it is continuous. Since p is a quotient map, $h^\wedge$ is continuous. Since Z is locally compact, h is continuous, by (2.9.3). □

(2.9.7) Theorem (Exponential law). *Let X and Y be locally compact. Then the adjunction map $\alpha\colon Z^{X\times Y} \to (Z^Y)^X$ is a homeomorphism.*

Proof. By (2.9.3), α is continuous, if $\alpha_1 = e_{X,Z^Y} \circ (\alpha \times \mathrm{id})$ is continuous. And this map is continuous, if $\alpha_2 = e_{Y,Z} \circ (\alpha_1 \times \mathrm{id})$ is continuous. One verifies that $\alpha_2 = e_{X\times Y,Z}$. The evaluations which appear are continuous by (2.9.1).

The inverse α^{-1} is continuous, if $e_{X\times Y,Z} \circ (\alpha^{-1} \times \mathrm{id})$ is continuous, and this map equals $e_{Y,Z} \circ (e_{X,Z^Y} \times \mathrm{id})$. □

Let (X,x) and (Y,y) be pointed spaces. We denote by $F^0(X,Y)$ the space of pointed maps with CO-topology as a subspace of $F(X,Y)$. In $F^0(X,Y)$ we use the constant map as a base point. The adjoint $f^\wedge\colon X \to F(Y,Z)$ of $f\colon X \times Y \to Z$ is a pointed map into $F^0(Y,Z)$ if and only if $X \times y \cup x \times Y$ is send under f to the base point of Z. Let $p\colon X \times Y \to X \wedge Y = X \times Y/(X \times y \cup x \times Y)$ be the quotient map.

Let (A,a) and (B,b) be pointed spaces. Their ***smash product*** is

$$A \wedge B = A \times B/A \times b \cup a \times B = A \times B/A \vee B.$$

(This is not a categorical product. It is rather analogous to the tensor product.) The smash product is a functor in two variables and also compatible with homotopies: Given $f\colon A \to C, g\colon B \to D$ we have the induced map $f \wedge g\colon A \wedge B \to C \wedge D, (a,b) \mapsto (f(a), g(b))$, and homotopies f_t, g_t induce a homotopy $f_t \wedge g_t$.

If $g\colon X \wedge Y \to Z$ is given, we denote the adjoint of $g \circ p\colon X \times Y \to X \wedge Y \to Z$ by $\alpha^0(g)$ and consider it as an element of $F^0X, F^0(Y,Z))$. In this manner we obtain a set map $\alpha^0\colon F^0(X \wedge Y, Z) \to F^0(X, F^0(Y,Z))$.

The evaluation $F^0(X,Y) \times X \to Y, (f,x) \mapsto f(x)$ factors over the quotient space $F^0(X,Y) \wedge X$ and induces $e^0 = e^0_{X,Y}\colon F^0(X,Y) \wedge X \to Y$. From (2.9.1) we conclude:

(2.9.8) Proposition. *Let X be locally compact. Then $e^0_{X,Y}$ is continuous.* □

Let $e^0_{Y,Z}$ be continuous. From a pointed map $\varphi\colon X \to F^0(Y,Z)$ we obtain $\varphi^\vee = \beta^0(\varphi) = e^0_{Y,Z} \circ (\varphi \wedge \mathrm{id})\colon X \wedge Y \to Z$, and hence a set map $\beta^0\colon F^0(X, F^0(Y,Z)) \to F^0(X \wedge Y, Z)$.

(2.9.9) Proposition. *Let $e^0_{Y,Z}$ be continuous. Then α^0 and β^0 are inverse bijections.* □

(2.9.10) Corollary. *Let $h\colon (X \wedge Y) \times I \to Z$ be a pointed homotopy. Then $\alpha^0(h_t)\colon X \to F^0(Y,Z)$ is a pointed homotopy and therefore*

$$[X \wedge Y, Z]^0 \to [X, F^0(Y,Z)]^0, \quad [f] \mapsto [\alpha^0(f)]$$

well-defined. If, moreover, $e^0_{Y,Z}$ is continuous, then this map is bijective. □

By a proof, formally similar to the proof of (2.9.7), we obtain the pointed version of the exponential law.

(2.9.11) Theorem (Exponential law). *Let X and Y be are locally compact. Then the pointed adjunction map $\alpha^0\colon F^0(X \wedge Y, Z) \to F^0(X, F^0(Y,Z))$ is a homeomorphism.* □

(2.9.12) Lemma. *Let $k_a\colon Z \to A$ denote the constant map with value a. Then $\psi\colon X^Z \times A \to (X \times A)^Z$, $(\varphi, a) \mapsto (\varphi, k_a)$ is continuous.*

Proof. Let $\psi(f,a) \in W(K,U)$. This means: For $x \in K$ we have $(f(x),a) \in U$. There exists open neighbourhoods V_1 of $f(K)$ in X and V_2 of a in A such that $V_1 \times V_2 \subset U$. The inclusion $\psi(W(K,V_1) \times V_2) \subset W(K,U)$ shows the continuity of ψ at (f,a). □

(2.9.13) Proposition. *A homotopy $H_t\colon X \to Y$ induces homotopies H_t^Z and Z^{H_t}.*

Proof. In the first case we obtain, with a map ψ from (2.9.12), a continuous map

$$H^Z \circ \psi\colon X^Z \times I \to (X \times I)^Z \to Y^Z.$$

In the second case we use the composition

$$e \circ (\alpha \times \mathrm{id}) \circ (Z^H \times \mathrm{id})\colon Z^Y \times I \to Z^{X \times I} \times I \to (Z^X)^I \times I \to Z^X$$

which is continuous. □

(2.9.14) Corollary. *Let f be a homotopy equivalence. Then the induced maps $F(Z,X) \to F(Z,Y)$ and $F(Y,Z) \to F(X,Z)$ are h-equivalences. If f is a pointed h-equivalence, the induced maps $F^0(Z,X) \to F^0(Z,Y)$ and $F^0(Y,Z) \to F^0(X,Z)$ are pointed h-equivalences.* □

Problems

1. Verify that f^Z and Z^f are continuous.
2. An inclusion $i\colon Z \subset Y$ induces an embedding $i^X\colon Z^X \to Y^X$.
3. The canonical map $F(\coprod_j X_j, Y) \to \prod_j F(X_j, Y)$ is always a homeomorphism.
4. The canonical map $F(X, \prod_j Y_j) \to \prod_j F(X, Y_j)$, $f \mapsto (\mathrm{pr}_j\, f)$ is always bijective

and continuous. If X is locally compact, it is a homeomorphism.
5. Let $p\colon X \to Y$ be a surjective continuous map. Suppose the pre-image of a compact set is compact. Then $Z^p\colon Z^Y \to Z^X$ is an embedding.
6. We have a canonical bijective map $F^0(\bigvee_{j\in J} X_j, Y) \to \prod_{j\in J} F^0(X_j, Y)$, since $\bigvee_j X_j$ is the sum in TOP^0. If J is finite, it is a homeomorphism.
7. Let $\mathcal{S}$ be a subbasis for the topology on Y and let X be a Hausdorff space. Then the sets $W(K,U)$, $K \subset X$ compact, $U \in \mathcal{S}$, are a subbasis of the CO-topology on Y^X.
8. Let X and Y be Hausdorff spaces. Then the sets of the form $W(K \times L, U)$, $K \subset X$ compact, $L \subset Y$ compact, $U \subset Z$ open, form a subbasis for the KO-topology on $Z^{X\times Y}$.
9. The map α has these properties:
(1) If X is a Hausdorff space, then α is continuous.
(2) If Y is locally compact, then α is surjective.
(3) If X and Y are Hausdorff spaces, then α is an embedding.
(4) If X and Y are Hausdorff spaces and Y is locally compact, then α is a homeomorphism.
10. Let X, Y, U, and V be spaces. Cartesian product of maps gives a map

$$\pi\colon U^X \times V^Y \to (U\times V)^{X\times Y}, \quad (f,g) \mapsto f \times g.$$

Let X and Y be Hausdorff spaces. Then the map π is continuous.
11. By definition of a product, a map $X \to Y \times Z$ is essentially the same thing as a pair of maps $X \to Y$, $X \to Z$. In this sense, we obtain a tautological bijection $\tau\colon (Y\times Z)^X \to Y^X \times Z^X$. Let X be a Hausdorff space. Then the tautological map τ is a homeomorphism.
12. Let X and Y be locally compact. Then composition of maps $Z^Y \times Y^X \to Z^X$, $(g,f) \mapsto g \circ f$ is continuous.
13. Let $(Y,*)$ be a pointed space, (X,A) a pair of spaces and $p\colon X \to X/A$ the quotient map. The space X/A is pointed with base point $\{A\}$. Let $F((X,A),(Y,*))$ be the subspace of $F(X,Y)$ of the maps which send A to the base point. Composition with p induces a bijective continuous map $\gamma\colon F^0(X/A,Y) \to F((X,A),(Y,*))$; and a bijection of homotopy sets $[X/A,Y]^0 \to [(X,A),(Y,*)]$. If p has compact pre-images of compact sets, then γ is a homeomorphism.
14. Consider diagrams where the right one is obtained by multiplying the left one

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} A\times X \arrow[r] \arrow[d] & B\times X \arrow[d] \\ C\times X \arrow[r] & D\times X \end{tikzcd}$$

with X. If the left diagram is a pushout in TOP and X locally compact, then the right diagram is a pushout in TOP.
15. The CO-topology on the set of linear maps $\mathbb{R}^n \to \mathbb{R}$ is the standard topology.
16. Let X be compact space and Y a metric space. Then the CO-topology on Y^X is induced by the supremum-metric.
17. Let X be a compact Hausdorff space and $H(X)$ the group of homeomorphism.

Then $H(X)$ together with the CO-topology is a topological group and $H(X) \times X \to X$, $(f, x) \mapsto f(x)$ a continuous group action.

2.10 Compactly Generated Spaces

A compact Hausdorff space will be called a ch-space. For the purpose of the following investigations we also call a ch-space a ***test space*** and a continuous map $f\colon C \to X$ of a test space C a ***test map***. A space X is called ***weakly hausdorff*** or ***wh-space***, if the image of each test map is closed.

(2.10.1) Proposition. *A Hausdorff space is a wh-space. A wh-space is a T_1-space.*

Proof. If X is Hausdorff and $f\colon K \to X$ a test map, then $f(K)$ is compact therefore closed in X. If X is a wh-space, then a one-point space has a closed image in X. □

(2.10.2) Proposition. *A space X is a wh-space if and only if each test map $f\colon K \to X$ is proper. If X is a wh-space, then the image of each test map is a Hausdorff space.*

Proof. Let X be a wh-space. A closed set $L \subset K$ is compact Hausdorff; therefore $f|L$ is a test map and hence has a closed image $f(L)$. This means: f is closed. For each $x \in X$ the pre-image $f^{-1}(x)$ is closed in K, hence compact.

Since proper maps have closed images, we see that the condition is also necessary.

A proper image of a ch-space is a ch-space (??). Therefore the image of a test map is Hausdorff. □

(2.10.3) Proposition. *A subspace of a wh-space is a wh-space. Products of wh-spaces are wh-spaces.*

Proof. Let $B \subset X$ and $f\colon K \to B$ a test map. Then the image of f is closed in X, provided X is a wh-space, and this image is then also closed in B.

Let $(X_j \mid j \in J)$ be wh-spaces and let $f\colon K \to \prod_j X_j$ be a test map with components $f_j\colon K \to X_j$. We write f as composition of the diagonal $\Delta\colon K \to \prod_j K$ with the product $\prod_j f_j$. By (2.10.1), the f_j are proper. Then (2.6.12) tells us that the product $\prod_j f_j$ is proper. Since K is Hausdorff, $\Delta(K)$ is closed in $\prod_j K$. Hence $f(K)$ is closed, being the image of a closed set under a proper map. □

A subset A of a topological space $(X, \mathcal{T})$ is said to be ***k-closed*** (***k-open***) if for each test map $f\colon K \to X$ the pre-image $f^{-1}(A)$ is closed (open) in

K. The k-open sets in $(X, \mathcal{T})$ form a topology $k\mathcal{T}$ on X. A closed (open) subset is also k-closed (k-open). Therefore $k\mathcal{T}$ is finer than $\mathcal{T}$ and the identity $\iota = \iota_X : kX \to X$ is continuous. We set $kX = k(X) = (X, k\mathcal{T})$. Let $f\colon K \to X$ be a test map. The same set map $f\colon K \to kX$ is then also continuous. For if $U \subset kX$ is open, then $U \subset X$ is k-open, hence $f^{-1}(U) \subset K$ open. Therefore ι_X induces for each ch-space K a bijection.

$$\mathrm{TOP}(K, kX) \xrightarrow{\cong} \mathrm{TOP}(K, X), \quad f \mapsto \iota_X \circ f.$$

Hence X and kX have the same k-open sets, i.e., $k(kX) = kX$. A topological space X is called ***k-space***, if the k-closed sets are closed, i.e., if $X = kX$. Because of $k(kX) = kX$ the space kX is always a k-space. A k-space is also called ***compactly generated***. We let k-TOP be the full subcategory of TOP with objects the k-spaces. A whk-space is a space which is a wh-space and a k-space.

The next proposition explains the definition of a k-space. We call a topology $\mathcal{S}$ on X ch-definable, if there exists a family $(f_j\colon K_j \to X \mid j \in J)$ of test maps such that: $A \subset X$ is $\mathcal{S}$-closed $\Leftrightarrow$ for each $j \in J$ the pre-image $f_j^{-1}(A)$ is closed in K_j. We can rephrase this condition: The canonical map $\langle f_j \rangle \colon \coprod_j K_j \to (X, \mathcal{S})$ is a quotient map. A ch-definable topology is finer than $\mathcal{T}$. We define a partial ordering on the set of ch-definable topologies by $\mathcal{S}_1 \le \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1 \supset \mathcal{S}_2$.

(2.10.4) Proposition. *The topology $k\mathcal{T}$ is the maximal ch-definable topology with respect to the partial ordering.*

Proof. By Zorn's Lemma there exists a maximal ch-definable topology $\mathcal{S}$. If this topology is different from $k\mathcal{T}$, then there exists an $\mathcal{S}$-open set U, which is not k-open. Hence there exists a test map $t\colon K \to X$ such that $t^{-1}(U)$ is not open. If we adjoin this test map to the defining family of $\mathcal{S}$, we see that $\mathcal{S}$ is not maximal. □

(2.10.5) Corollary. *The k-spaces are the spaces which are quotients of a topological sum of ch-spaces.* □

(2.10.6) Proposition. *The following are equivalent:*
(1) *X is a k-space.*
(2) *A set map $f\colon X \to Y$ is continuous if and only if for each test map $t\colon K \to X$ the composition ft is continuous.*

Proof. (1) $\Rightarrow$ (2). Let $U \subset Y$ be open. In order to see that $f^{-1}(U)$ is open it suffices to show that this set is k-open, since X is a k-space. Let $t\colon K \to X$ be a test map and ft continuous. Then $k^{-1}(f^{-1}(U))$ is open, and this shows what we want.

(2) $\Rightarrow$ (1). We show that the identity $X \to kX$ is continuous. This holds by (2) and because X and kX have the same test maps. □

(2.10.7) Proposition. *Let $f\colon X \to Y$ be continuous. Then the same set map $kf\colon kX \to kY$ is continuous.*

Proof. By (2.10.6) it suffices to show that for each test map $t\colon K \to kX$ the composition $kf \circ t$ is continuous. But this is a consequence of (**??**). □

The assignments $X \mapsto kX$, $f \mapsto kf$ yield a functor k; moreover, we have the inclusion functor i

$$k\colon \mathrm{TOP} \to \text{k-}\mathrm{TOP}, \qquad i\colon \text{k-}\mathrm{TOP} \to \mathrm{TOP}\,.$$

(2.10.8) Proposition. *The functor k is right adjoint to the functor i.*

Proof. A natural bijection is $\text{k-}\mathrm{TOP}(Y, kX) \cong \mathrm{TOP}(iY, X), f \mapsto \iota \circ f$. This map is certainly injective. If Y is a k-space and $f\colon Y \to X$ continuous, then $kf\colon Y = kY \to kX$ is continuous; this is used to show surjectivity. □

(2.10.9) Proposition. *Let X be a wh-space. Then is $A \subset X$ is k-closed if and only if for each ch-space $K \subset X$ the set $A \cap K$ is closed in K. In particular a wh-space X is a k-space if and only if: $A \subset X$ closed $\Leftrightarrow$ for each ch-space $K \subset X$ the intersection $A \cap K$ is closed in K.*

Proof. Let A be k-closed. The inclusion $K \subset X$ of a ch-space is a test map. Hence $A \cap K$ is closed in K.

Conversely, suppose that A satisfies the stated condition and let $f\colon L \to X$ be a test map. Since X is a wh-space, $f(L)$ is a ch-space and therefore $f(L) \cap A$ closed in $f(L)$. Then $f^{-1}(A) = f^{-1}(f(L) \cap A)$ is closed in $L = f^{-1}f(L)$. This shows: A is k-closed. □

Thus we see that wh-spaces have an internal characterization of their k-closed sets. We have already used this earlier in the context of Hausdorff spaces. For wh-spaces therefore $k(X)$ can be defined from internal properties of X. If X is a wh-space, so is kX.

(2.10.10) Theorem. *X is a k-space under one of the following conditions:*

(1) *X is metrizable.*
(2) *Each point of X has a countable neighbourhood basis.*
(3) *Each point of X has a neighbourhood which is a ch-space.*
(4) *For $Q \subset X$ and $x \in \overline{Q}$ there exists a ch-subspace $K \subset X$ such x is contained in the closure of $Q \cap K$ in K.*
(5) *For each $Q \subset X$ the following holds: $Q \cap K$ open (closed) in K for each test space $K \subset X$ implies Q open (closed) in X.*

Proof. (1) is a special case of (2).

(2) Let $Q \subset X$ and suppose that $f^{-1}(Q)$ is closed for each test map $f\colon C \to X$. We have to show that Q is closed. Thus let $a \in \overline{Q}$ and let $(U_n \mid n \in \mathbb{N})$ be

a neighbourhood basis of a. For each n choose $a_n \in Q \cap U_1 \cap \ldots \cap U_n$. Then the sequence (a_n) converges to a. The subspace $K = \{0, 1, 2^{-1}, 3^{-1}, \ldots\}$ of $\mathbb{R}$ is compact. The map $f\colon K \to X$, $f(0) = a$, $f(n^{-1}) = a_n$ is continuous, and $n^{-1} \in f^{-1}(Q)$. By assumption, $f^{-1}(Q)$ is closed in K, hence $0 \in f^{-1}(Q)$, and therefore $a = f(0) \in Q$.

(3) $\Rightarrow$ (4). Let $Q \subset X$ and suppose $a \in \overline{Q}$. We choose a ch-neighbourhood K of a and show that a is contained in the closure of $Q \cap K$ in K. Thus let U be a neighbourhood of a in K. Then there exists a neighbourhood U' of a in X such that $U' \cap K \subset U$. Since $U' \cap K$ is a neighbourhood of a in X and $a \in \overline{Q}$, we conclude

$$U \cap (Q \cap K) \supset (U' \cap K) \cap (Q \cap K) = (U' \cap K) \cap Q \neq \emptyset.$$

Hence a is contained in the closure of $Q \cap K$ in K.

(4) $\Rightarrow$ (5). Suppose $Q \cap K$ is closed in K for every test subspace $K \subset X$. Let $a \in \overline{Q}$. By (4), there exists a test subspace K_0 of X, such that a is contained in the closure of $Q \cap K_0$ in K_0. By the assumption (5), $Q \cap K_0$ is closed in K_0; and hence $a \in Q \cap K_0 \subset Q$.

(5) Let $f^{-1}(Q)$ be closed in K for each test map $f\colon K \to X$. Then, in particular, for each test subspace $L \subset X$ the set $Q \cap L$ is closed in L. The assumption (5) then says that Q is closed in X. This shows that X is a k-space. □

(2.10.11) Theorem. *Let $p\colon Y \to X$ be a quotient map and Y a k-space. Then X is a k-space.*

Proof. Let $B \subset X$ be k-closed. We have to show that B is closed, hence, since p is a quotient map, that $p^{-1}(B)$ is closed in Y. Let $g\colon D \to Y$ be a test map. Then $g^{-1}(p^{-1}(B)) = (pg)^{-1}(B)$ is closed in D, because B is k-closed. Since Y is a k-space, $p^{-1}(B)$ is closed in Y. □

(2.10.12) Proposition. *A closed (open) subspace of a k-space is a k-space. The same holds for whk-spaces.*

Proof. Let A be closed and $B \subset A$ a subset such that $f^{-1}(B)$ is closed in C for test maps $f\colon C \to A$. We have to show: B is closed in A or, equivalently, in X.

If $g\colon D \to X$ is a test map, then $g^{-1}(A)$ is closed in D and hence compact, since D is compact. The restriction of g yields a continuous map $h\colon g^{-1}(A) \to A$. The set $h^{-1}(B) = g^{-1}(B)$ is closed in $g^{-1}(A)$ and therefore in D, and this shows that B is closed in X.

Let U be open in the k-space X. We write X as quotient $q\colon Z \to X$ according to (2.10.5). Then $q\colon q^{-1}(U) \to U$ is a quotient map and $q^{-1}(U)$ as topological sum of locally compact Hausdorff spaces a k-space. Therefore the quotient U is a k-space.

The second assertion follows, if we take (2.10.1) into account. □

In general, a subspace of a k-space is not a k-space (see (2.10.25)). Let X be a k-space and $i\colon A \subset X$ the inclusion. Then the map $k(i)\colon k(A) \to X = k(X)$ is continuous. The next proposition shows that $k(i)$ has in the category k- TOP the formal property of a subspace.

(2.10.13) Proposition. *A map $h\colon Z \to k(A)$ from a k-space Z into $k(A)$ is continuous if and only if $k(i) \circ h$ is continuous.*

Proof. If h is continuous then also $k(i)\circ h$. Conversely, let $k(i)\circ h$ be continuous. We have $k(i) = i \circ \iota_A$. Since i is the inclusion of a subspace, $\iota_A \circ h$ is continuous; (2.10.8) now shows that h is continuous. □

(2.10.14) Theorem. *The product in* TOP *of a k-space X with a locally compact Hausdorff space Y is a k-space.*

Proof. By (2.10.10), a locally compact Hausdorff space is a k-space. We write X as quotient of $q\colon Z \to X$, where Z is a sum of ch-spaces (2.10.5). Since the product of a quotient map with a locally compact space is again a quotient map, we see that $X \times Y$ is a quotient of the locally compact Hausdorff space, hence k-space, $Z \times Y$, and therefore a k-space (2.10.11). □

A product of k-spaces is not always a k-space (see (2.10.25)). Therefore one is looking for a categorical product in the category k- TOP. Let $(X_j \mid j \in J)$ be a family of k-spaces and $\prod_j X_j$ its product in the category TOP, i.e., the ordinary topological product. We have a continuous map

$$p_j = k(\mathrm{pr}_j)\colon k(\textstyle\prod_j X_j) \to k(X_j) = X_j.$$

The next theorem is a special case of the fact that a right adjoint functor respects limits.

(2.10.15) Theorem. *$(p_j\colon k(\prod_j X_j) \to X_j \mid j \in J)$ is a product of $(X_j \mid j \in J)$ in the category* k- TOP*.*

Proof. We use (2.10.8) and the universal property of the topological product and obtain, in short-hand notation, for a k-space B the canonical bijection

$$\text{k- TOP}(B, k(\textstyle\prod X_j)) = \mathrm{TOP}(B, \prod X_j) \cong \prod \mathrm{TOP}(B, X_j) = \prod \text{k- TOP}(B, X_j),$$

and this is the claim. □

In the case of two factors, we use the notation $X \times_k Y$ for the product in k- TOP just defined. The next result shows that the wh-spaces are the formally hausdorff spaces in the category k- TOP.

(2.10.16) Proposition. *A k-space X is a wh-space if and only if the diagonal D_X of the product $X \times_k X$ is closed.*

Proof. Let X be a wh-space. In order to verify that D_X is closed, we have to show that for each test map $f\colon K \to X \times_k X$ the pre-image $f^{-1}(D_X)$ is closed. Let $f_j\colon K \to X$ be the j-th component of f. Then $L_j = f_j(K)$ is a ch-space, since X is wh-space. Hence $L = L_1 \cup L_2 \subset X$ is a ch-space. The relation $f^{-1}D_X = f^{-1}((L \times L) \cap D_X)$ shows that this set is closed.

Let D_X be closed in $X \times_k X$ and $f\colon K \to X$ a test map. We have to show that $f(K) \subset X$ is closed. Let $g\colon L \to X$ be another test map. Since X is a k-space, we have to show that $g^{-1}f(K) \subset L$ is closed. We use the relation

$$g^{-1}f(K) = \mathrm{pr}_2((f \times g)^{-1}D_X).$$

Since D_X is closed, the pre-image under $f \times g$ is closed and therefore also pr_2 as a compact set in a Hausdorff space. □

Recall the mapping space $F(X,Y)$ with compact-open topology.

(2.10.17) Theorem. *Let X and Y be k-spaces, and let $f\colon X \times_k Y \to Z$ be continuous. The adjoint map $f^\wedge\colon X \to kF(Y,Z)$, which exists as a set map, is continuous.*

Proof. The map $f^\wedge\colon X \to kF(Y,Z)$ is continuous, if for each test map $t\colon C \to X$ the composition $f^\wedge \circ t$ is continuous. We use $f^\wedge \circ t = (f \circ (t \times \mathrm{id}_Y))^\wedge$. Therefore is suffices to assume that X is a ch-space. But then, by (2.10.14), $X \times_k Y = X \times Y$ and therefore $f^\wedge\colon X \to F(Y,Z)$ is continuous and hence also $f^\wedge\colon X \to kF(Y,Z)$, by (2.10.6). □

(2.10.18) Theorem. *Let Y be a k-space. Then the evaluation*

$$e_{Y,Z}\colon kF(Y,Z) \times_k Y \to Z, \quad (f,y) \mapsto f(y)$$

is continuous.

Proof. Let $t\colon C \to kF(Y,Z) \times_k Y$ be a test map. We have to show the continuity of $e_{Y,Z} \circ t$. Let $t_1^\wedge\colon C \to F(Y,Z)$ and $t_2\colon C \to Y$ be the continuous components of t. We show first: The adjoint $t_1\colon C \times Y \to Z$ of $t_1^\wedge$ is continuous. By (?? 1.4), this continuity is equivalent to the continuity of the second adjoint map $t_1^\vee\colon Y \to F(C,Z)$. In order to show its continuity, we compose with a test map $s\colon D \to Y$. But $t_1^\vee \circ s = F(s,Z) \circ t_1^\wedge$ is continuous. Moreover we have $e_{Y,Z} \circ t = t_1 \circ (\mathrm{id}, t_2)$, and the right hand side is continuous. □

A combination of (2.10.17) and (2.10.18) now yields the ***universal property of the evaluation*** $e_{Y,Z}$ for k-spaces:

(2.10.19) Proposition. *Let X and Y be k-spaces. The assignments $f \mapsto f^\wedge$ and $g \mapsto e_{Y,Z} \circ (g \times_k \mathrm{id}_Y) = g^\sim$ are inverse bijections*

$$\mathrm{TOP}(X \times_k Y, Z) \cong \mathrm{TOP}(X, kF(Y,Z))$$

between these sets. □

(2.10.20) Theorem. *Let X, Y and Z be k-spaces. Since $e_{Y,Z}$ is continuous, we have an induced set map*

$$\lambda\colon kF(X, kF(Y,Z)) \to kF(X \times_k Y, Z), \quad f \mapsto e_{Y,Z} \circ (f \times_k \mathrm{id}_Y) = f^\sim.$$

The map λ is a homeomorphism.

Proof. We use the commutative diagram

$$\begin{tikzcd} kF(X, kF(Y,Z)) \times_k X \times_k Y \arrow[r, "e_1 \times \mathrm{id}"] \arrow[d, "\lambda \times \mathrm{id} \times \mathrm{id}"] & kF(Y,Z) \times_k Y \arrow[d, "e_2"] \\ kF(X \times_k Y, Z) \times_k X \times_k Y \arrow[r, "e_3"] & Z \end{tikzcd}$$

with $e_1 = e_{X,kF(Y,Z)}$, $e_2 = e_{Y,Z}$, and $e_3 = e_{X \times_k Y, Z}$. Since $e_1 \times \mathrm{id}$ and e_2 are continuous, the universal property of e_3 shows that λ is continuous; namely, using the notation from (2.10.19), we have $e_2 \circ (e_1 \times \mathrm{id}) = \lambda^\sim$. The universal property of e_1 provides us with a unique continuous map

$$\mu\colon kF(X \times_k Y, Z) \to kF(X, kF(Y,Z)), \quad f \mapsto f^\wedge,$$

such that $e_1 \circ (\mu \times \mathrm{id}(X)) = e_3^\wedge$, where $e_3^\wedge\colon kF(X \times_k Y, Z) \times_k X \to kF(Y,Z)$ is the adjoint of e_3 with respect to the variable Y. One checks that λ and μ are inverse to each other, hence homeomorphisms. □

(2.10.21) Theorem. *Let X and Y be k-spaces, and $f\colon X \to X'$ and $g\colon Y \to Y'$ be quotient maps. Then $f \times g\colon X \times_k Y \to X' \times_k Y'$ is a quotient map.*

Proof. It suffices to treat the case $g = \mathrm{id}$, since a composition of quotient maps is a quotient map. Using (2.10.20), the proof is now analogous to (**??**). □

(2.10.22) Proposition. *Let $f\colon X \to Y$ be a quotient map and X a whk-space. Then Y is a whk-space if and only if $R = \{(x_1, x_) \mid f(x_1) = f(x_2)\}$ is closed in $X \times_k X$.*

Proof. The set R is the pre-image of D_Y under $f \times f$. Since $f \times_k f$ is a quotient map (2.10.21), D_Y is closed if and only if R is closed. Now apply (2.10.11) and (2.10.16). □

(2.10.23) Proposition. *Let Y and Z be k-spaces and assume that Z is a wh-space. Then the mapping space $kF(Y,Z)$ is a wh-space. In particular, if Y and Z are whk-spaces, then $kF(Y,Z)$ is a whk-space.*

Proof. Let $f^\wedge\colon K \to kF(Y,Z)$ be a test map. We have to show that it has a closed image hence a k-closed. For this purpose let $g^\wedge\colon L \to kF(Y,Z)$ be another test map. It remains to show that the pre-image M of $f^\wedge(K)$ under

$g^\wedge$ is closed. We use the adjoint maps $f\colon K \times Y \to Z$ and $g\colon L \times Y \to Z$. For $y \in Y$ let $i_y\colon K \times L \to (K \times Y) \times_k (L \times Y)$, $(k,l) \mapsto (k,y,l,y)$. Then $M = \mathrm{pr}_2(\bigcap_{y \in Y}((f \times g)i_y)^{-1}D_Z)$. Since Z is a wh-space and therefore the diagonal D_Z closed (see (2.10.16)), we see that M is closed. □

We now consider pointed spaces. Let $(X_j \mid j \in J)$ be a family of pointed k-spaces. Let $\prod_j^k X_j$ be its product in k-TOP. Let $W_J X_j$ be the subset of the product of those points for which at least one component equals the base point. The ***smash product*** $\bigwedge_j^k X_j$ is the quotient space $(\prod_j^k X_j)/W_J X_j$. In the case that $J = \{1,\dots,n\}$ we denote this space by $X_1 \wedge_k \dots \wedge_k X_n$. A family of pointed maps $f_j\colon X_j \to Y_j$ induces a pointed map $\bigwedge^k f_j\colon \bigwedge_j^k X_j \to \bigwedge_j^k Y_j$.

Let X and Y be pointed k-spaces. Let $F^0(X,Y) \subset F(X,Y)$ be the subspace of pointed maps. We compose a pointed map $f\colon X \wedge_k Y \to Z$ with the projections $p\colon X \times_k Y \to X \wedge_k Y$. The adjoint $(fp)^\wedge\colon X \to kF(Y,Z)$ is continuous and has an image contained in $kF^0(Y,Z)$. We obtain a continuous map $X \to kF^0(Y,Z)$ which will be denoted by $f^\wedge$.

The evaluation $e_{Y,Z}$ induces $e^0_{Y,Z}$ which makes the following diagram commutative.

$$\begin{tikzcd}
kF^0(Y,Z) \times_k X \arrow[r, "k(i)\times \mathrm{id}"] \arrow[d, "p"] & kF(Y,Z) \times_k X \arrow[d, "e_{Y,Z}"] \\
kF^0(Y,Z) \wedge_k X \arrow[r, "e^0_{Y,Z}"] & Y
\end{tikzcd}$$

i is the inclusion and p the quotient map. The continuity of $k(i)$ and $e_{Y,Z}$ implies the continuity of the pointed evaluation $e^0_{Y,Z}$. In analogy to (2.10.20) one proves:

(2.10.24) Theorem. *Let X, Y and Z be pointed k-spaces. The assignment*

$$\mu^0\colon kF^0(X \wedge_k Y, Z) \to kF^0(X, kF^0(Y,Z)), \quad f \mapsto f^\wedge$$

is a homeomorphism. □

(2.10.25) Example. Let $\mathbb{R}/\mathbb{Z}$ be obtained from $\mathbb{R}$ by identifying the subset $\mathbb{Z}$ to a point (so this is not the factor group!). We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the quotient map.

(1) The product $p \times \mathrm{id}\colon \mathbb{R} \times \mathbb{Q} \to \mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ of quotient maps is not a quotient map.

(2) The product $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ is not a k-space, but the factors are k-spaces (see (2.10.6)).

(3) The product $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ is a k-space (see (2.10.11) and (2.10.14)), but the subspace $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ is not a k-space by (2).

If $K \subset \mathbb{R}/\mathbb{Z}$ is compact, then there exists $l \in \mathbb{N}$ such that $K \subset p[-l, l]$.

Let $(r_n \mid n \in \mathbb{N})$ be a strictly decreasing sequence of rational numbers with limit $\sqrt{2}$. The set $F = \{(m + \frac{1}{2n}, \frac{r_n}{m}) \mid n, m \in \mathbb{N}\} \subset \mathbb{R} \times \mathbb{Q}$ is saturated with respect to $p \times \mathrm{id}$ and closed in $\mathbb{R} \times \mathbb{Q}$.

The set $G = (p \times \mathrm{id})(F)$ is not closed in $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$. Note that $z = (p(0), 0) \notin G$; but we show that $z \in \overline{G}$. Let U be a neighbourhood of z. Then there exists a neighbourhood V of $p(0)$ in $\mathbb{R}/\mathbb{Z}$ and $\varepsilon > 0$ such that $V \times (\,]-\varepsilon, \varepsilon[\, \cap \mathbb{Q}) \subset U$. Choose $m \in \mathbb{N}$ such that $m^{-1}\sqrt{2} < 2^{-1}\varepsilon$. The set $p^{-1}(V)$ is then a neighbourhood of m in $\mathbb{R}$, since $m \in p^{-1}p(0) \subset p^{-1}(V)$. Hence there exists $\delta > 0$ such that $]m - \delta, m + \delta[\, \subset p^{-1}(V)$. Now choose $n \in \mathbb{N}$ such that $\frac{1}{2n} < \delta$ and $r_n - \sqrt{2} < m\frac{\varepsilon}{2}$. Then $(p \times \mathrm{id})(m + \frac{1}{2n}, \frac{r_n}{m}) \in V \times (\,]-\varepsilon, \varepsilon[\, \cap \mathbb{Q}) \subset U$ holds, because $m + \frac{1}{2n} \in\,]m - \delta, m + \delta[\, \subset p^{-1}(V)$ and $0 < \frac{r_n}{m} = \frac{\sqrt{2}}{m} + \frac{r_n - \sqrt{2}}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. We see that $U \cap G \neq \emptyset$. This finishes the proof that $z \in \overline{G}$.

We now see that $p \times \mathrm{id}$ is not a quotient map, since there exists a saturated closed set F with non-closed image G.

The space $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ is not a k-space. Let $s\colon K \to \mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ be an arbitrary test map. We show that $s^{-1}(G)$ is closed in K although G is not closed (this could not occur in a k-space). The two projections $\mathrm{pr}_i\, s(K)$ are compact and Hausdorff. Hence there exists $l \in \mathbb{N}$ such that $\mathrm{pr}_1\, s(K) \subset p[-l, l]$. The inclusion

$$s(K) \subset \mathrm{pr}_1\, s(K) \times \mathrm{pr}_2\, s(K) \subset p[-l - l] \times \mathrm{pr}_2\, s(K)$$

then shows $s^{-1}(G) = s^{-1}(G \cap p[-l, l] \times \mathrm{pr}_2\, s(K))$. But the set $G \cap p[-l, l] \times \mathrm{pr}_2\, s(K)$ is finite: By construction, F is a closed discrete subspace of $\mathbb{R} \times \mathbb{Q}$; moreover, $F \cap [-l, l] \times \mathrm{pr}_2\, s(K)$ is finite as closed discrete subspace of the compact space $[-l, l] \times \mathrm{pr}_2\, s(K)$; therefore also

$$(p \times \mathrm{id})(F \cap [-l, l] \times \mathrm{pr}_2\, s(K)) = G \cap p[-l, l] \times \mathrm{pr}_2\, s(K)$$

is finite. A finite set in a Hausdorff space is closed, and therefore $s^{-1}(G)$ as pre-image of a closed set closed itself. ◇

(2.10.26) Example. It is already stated in [**?**, p. 336] that $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}_0$ and $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N}_0)$ are not homoeomorphic. In [**?**, p. 26] it is proved that the canonical continuous bijection from the first to the second space is not a homeomorphism. ◇

Problems

1. A space is a k-space if and only if it is a quotient of a locally compact Hausdorff space.

2. Let $X_1 \subset X_2 \subset \ldots$, let X_j be a whk-space and let $X_j \subset X_{j+1}$ be closed. Then $X = \cup_j X_j$, with colimit topology, is an whk-space. If the X_i are k-spaces, then X

is a k-space, being a quotient of the k-space $\coprod_i X_i$. If the X_i are wh-spaces, hence T_1-spaces, then each test map $f\colon K \to X$ has an image which is contained in some X_i and therefore closed. If each inclusion is $X_i \subset X_{i+1}$ closed, the image is also closed in X and therefore X a wh-space.
3. Let X and Y be k-spaces. Passage to adjoint maps induces bijections of homotopy sets $[X \times_k Y, Z] \cong [X, kF(Y, Z)]$ and $[X \wedge Y, Z]^0 \cong [X, kF^0(Y, Z)]^0$.
4. A map $f\colon X \to Y$ between topological spaces is said to be ***quasi-continuous***, if the composition with each test map $K \to X$ is continuous. Continuous maps are certainly quasi-continuous. The composition of quasi-continuous maps is quasi-continuous. We obtain the category QU of topological spaces and quasi-continuous maps. TOP is a subcategory of QU; but in QU there may exist more morphisms between two topological spaces than in TOP. We can rephrase (2.10.6): X is a k-space, if each quasi-continuous map $X \to Y$ is continuous.
5. Let $(X_j \mid j \in J)$ be a family of k-spaces. Then the topological sum $\sum_{j \in J} X_j$ is a k-space. The product in k-TOP is compatible with sums.
6. Let a pushout of topological spaces with closed $j\colon A \subset X$ be given.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

Let X and B be whk-spaces. Then Y is a whk-space.

Proof. As a quotient of the k-space $X + B$, the space Y is a k-space. As a closed subspace of X the space A is a whk-space. One verifies that the relation for the definition of Y is closed in $(X + B) \times_k (X + B)$. Now use (2.10.22). □

2.11 Interval and Circle

We characterize the unit interval and the unit circle by topological properties. The proofs are a remarkable example of an axiomatic deduction. The method goes back to Hilbert and aims at a topological foundation of plane geometry.

In order to formulate the main theorems we introduce the notion of a cut point. A point of x in a connected space X is said to be a ***cut point*** of X if $X \smallsetminus x$ is disconnected. A point which is not a cut point will be called in this context *end point* of X. A decomposition $Y = U \cup V$ of a space Y in the sense of section (**??**) will be expressed by the symbol $Y = U|V$. We assume that X contains more than one point.

(2.11.1) Theorem (Interval theorem). *A compact connected metric space whose point with the exception of at most two are cut points is homeomorphic to the unit interval.*

(2.11.2) Theorem (Circle theorem). *A compact connected metric space which becomes disconnected when we remove any two point set is homeomorphic to the unit circle.*

The idea for proving (2.11.1) is: By studying connected subsets of X we construct a total ordering of X. This ordering allows to construct a monotonic map $X \to 0,1]$ which will turn out to be a homeomorphism. We know that the topology of $[0,1]$ is defined from the ordering.) We try to imitate this situation for X.

(2.11.3) Proposition. *Let X be a connected Hausdorff space, $x \in X$ a cup point and $X \smallsetminus x = U|V$. Then:*

(1) $\overline{U} = U \cup x$, $\overline{V} = V \cup x$. *In particular U and V are open in X.*
(2) $\overline{U}$ *and* $\overline{V}$ *are connected.*
(3) *If $y \in U$ and $X \smallsetminus y = A|B$, then A or B is contained in U.*

Proof. (1) U is closed in $X \smallsetminus x$, hence $U = \overline{U} \cap (X \smallsetminus x) = \overline{U} \smallsetminus x$ and therefore $\overline{U} \subset U \cup x$. If U were equal to $\overline{U}$, then U and $\overline{V} \cup x$ would be closed sets in X, and $U \cap (\overline{V} \cup x) = U \cap \overline{V} \subset U \cap (V \cup x) = \emptyset$ would show that they are a decomposition of X. As complement of $\overline{V}$ the set U is open.

(2) Let $\overline{U} = A|B$ and $x \in A$, say. Then we have $B \cap \overline{V} = B \cap (V \cup x) = B \cap V = \emptyset$ and therefore $X = B|(A \cup \overline{V})$, since A and B ate closed in $\overline{U}$ and hence in X.

(3) $\overline{V} = V \cup x$ is contained in $X \smallsetminus y$, and, being connected, contained in A or in B. If $V \cup x \subset A$, say, then $B \subset X \smallsetminus \overline{V} = U$. □

(2.11.4) Proposition. *Assume in addition to* (2.11.3) *that X is compact metric. Then U and V each contain an end point of X.*

Proof. Suppose each point of U, and hence of $\overline{U} = U \cup x$, is a cut point of X. Since $\overline{U}$ is, by (2.11.3), a connected compact metric space, which contains more than one point, there exists in $\overline{U}$ and hence in U a countably infinite dense subset. Let $\{x(1), x(2), \dots\} \subset U$ be dense. We show inductively: There exists a subsequence $(x(n_1), x(n_2), \dots)$ and decompositions $X \smallsetminus x(n_r) = U_r|V_r$ with these properties:

(α) $X \smallsetminus x(n_r) = U_r|V_r$,
(β) n_{r+1} is the smallest element of $\{j \in \mathbb{N} \mid x(j) \in U_r\}$,
(γ) $U \supset U_1 \supset U_2 \supset \cdots$.

Suppose $n_1 = 1$ and $X \smallsetminus x(1) = U_1|V_1$, where U_1 the part lying in U (2.11.3) (3). Suppose $x(j)$, U_j, V_j are given with the stated properties $1 \le j \le t$. Then U_t is a non-empty, and by (2.11.3) (1) open, subset of U. Therefore U_t contains points of the form $x(n)$. We define n_{t+1} as smallest integer in $\{j \in \mathbb{N} \mid x(j) \in U_t\}$. By assumption, $X \smallsetminus x(n_{t+1})$ is decomposable; we choose a decomposition

$$X \smallsetminus x(n_{t+1}) = U_{t+1}|V_{t+1}, \quad U_{t+1} \subset U_t.$$

Since $x(n_1), \dots, x(n_t) \notin U_{t+1}$, the index n_{t+1} is different from $n_1, \dots, n_t$.

The compact sets $\overline{U} \supset \overline{U_1} \supset \overline{U_2} \supset \dots$ have a non-empty intersection U_∞. Since $\overline{U}_{t+1} = U_{t+1} \cup x(n_{t+1}) \subset U_t$, the $U \supset U_1 \supset U_2 \supset \dots$ have the same intersection. Let $z \in U_\infty \subset U$ and $X \smallsetminus z = A|B$. By (2.11.3), each $U_m$contains either A or B. Say A is contained in infinitely many U_m, hence in U_∞. Since A is open, we can select a point $x(i) \in A$. Let n_{t+1} the smallest of the integers $n_2, n_3, \dots$, which is larger than i. Then $x(i) \in A \subset U_t$, and this contradicts (β); therefore not every point of U is a cut pint. Similarly for V. □

(2.11.5) Proposition. *Suppose X satisfies the hypotheses of* (2.11.1). *Let x be a cup point and $X \smallsetminus x = U|V$. Then U and V are connected.*

Proof. Since X has at most two end point it has exactly two, by (2.11.4), say $a \in U$ and $b \in V$. Suppose $U = A|B$ and $a \in A$. Then

$$X \smallsetminus x = B|((X \smallsetminus x) \smallsetminus B),$$

but B does not contain a and b, and this contradicts (2.11.4) applied now to this decomposition. □

Now we are able to define a total order $<$ on a space X satisfying the hypotheses of (2.11.1). Recall that a ***total order*** is a relation $<$ which satisfies:
(o_1) For no x the relation $x < x$ holds.
(o_2) If $x \neq y$, then either $x < y$ or $y < x$.
(o_3) $x < y$ and $y < z$ implies $x < z$.

For $x \in X$ let $L_x = \emptyset$ in the case that $x = a$, and other the component of $X \smallsetminus x$ which contains a. Similarly let $R_x = \emptyset$ in the case that $x = b$ ist, and otherwise the component of $X \smallsetminus x$ which contains b. Here a and b are the two end points, as in the proof of (2.11.5). For each x we now have a disjoint union $X = L_x \cup x \cup R_x$.

We now postulate:

$$x < y \quad \Leftrightarrow \quad L_x \subset L_y, \ L_x \neq L_y.$$

(2.11.6) Lemma. $x < y \Leftrightarrow x \in L_y$.

Proof. Let $x \in L_y$. Then $L_x \neq L_y$, since $x \notin L_x$. We have $y \neq a$, since $L_a = \emptyset$ but $x \in L_y$. If $y = b$, then $L_x \subset X \smallsetminus b = L_y$. Finally, if $y \notin \{a, b\}$, then, by (2.11.3) (3), $L_x \subset L_y$ oder $R_x \subset L_y$; and because of $a \in L_x \cap L_y$ the first relation holds.

Let $x < y$. Then $y \neq a$, since $L_y \neq \emptyset$. If $x = a$, then $x \in L_y$ by definition of L_y. If $x \neq a$, then $L_x \cup x = \overline{L}_x \subset \overline{L}_y = L_y \cup y$. The relation $L_x \neq L_y$ implies $x \neq y$, and hence $x \in L_y$. □

In a similar manner one shows: $x \in R_y \Leftrightarrow R_x \subset R_y \ R_x \neq R_y$.

(2.11.7) Proposition. *The relation $<$ is a total order on X.*

Proof. (o_1) and (o_3) are direct consequences of the definition.

(o_2): Let $x \neq y$. Then $x \in L_y$ or $x \in R_y$. If $x \in L_y$, then $x < y$ holds by (11.7). If $x \in R_y$, then we conclude $R_x \subset R_y$; $L_x \cup x \supset L_y \cup y$; $L_x \supset L_y$; $y < x$. □

The ***order topology*** on X has as basis the sets of the $U(< q) = \{x \in X \mid x < q\}$ and $U(p <) = \{y \in X \mid p < y\}$. For $p < q$ we then have the open sets $U(p < q) = \{x \in X \mid p < x < q\} = U(p <) \cap U(< q)$. These sets are non-empty, because otherwise $L_p \cup p \mid R_q \cup q$ would be a decomposition of X. Therefore there exists a further element between two given elements. These sets are also open in the original topology, for, by (2.11.7), we have

$$U(< q) = L_q, \;\; U(p <) = R_p;$$

and these sets are open by (2.11.3).

The order topology is hausdorff: Suppose $x < y$, and choose z with $x < z < y$; then L_z and R_z are disjoint open neighbourhoods of x and y. The identity map from X with the original topology to X with order topology is therfore continuous, and therefore a homeomorphism since X is compact.

Let $E \subset X \smallsetminus \{a, b\}$ be a countable dense subset of X. The induced order on E has, as we have seen the property that there exists between any two elements a further element, more over there do not exist maximal and minimal elements. Under these conditions there exists an order preserving bijection $f\colon E \to \mathbb{Q} \cap]0,1[$ from E to the rational numbers in $]0,1[$. We want to extend f to a homeomorphism $X \to [0,1]$. for this purpose we show that (X, E) satisfies the Dedekind axiom:

(2.11.8) Proposition (Dedekind cuts). *Let $A \subset E$ be a subset without largest element. Moreover assume: If $x \in A$, $y \in E$ and $y < x$, then $y \in A$. Then the set $K = \{s \in X \mid x < s \, for \, all x \in A\}$ of upper bounds of A has a minimum s_A, and $A = U(< s_A) \cap E$.*

Proof. Since $b \in K$ the set K is non-empty. If $K = X$, then a is the smallest element of K. The set $X \smallsetminus K$ is open: Let $x \in X \smallsetminus K$. There exists $y \in A$ with $x < y$. The set $U(a < y)$ is then open, contains in x and is contained in $X \smallsetminus K$. If K does not have a minimum, then K is open too and $K | (X \smallsetminus K)$ a decomposition of X. □

A subset $A \subset E$ which satisfies the hypotheses of (2.11.8) is called a *cut* of E. (2.11.8) implies that the assignment $x \mapsto U(< x) \cap E = A_x$ is a bijection of X with the set of cuts of E. This bijection is order preserving in the following sense: $x < y \Leftrightarrow A_x \subset A_y$. We assign to a cut A of E the cut $f(A)$ of $R = \mathbb{Q} \cap]0,1[$. The cuts of R are the sets $\mathbb{Q} \cap [0,t]$ for $t \in [0,1]$. The bijection of cuts from E to R induces in this manner a bijective order preserving map $X \to [0,1]$. It is continuous with respect to the order topologies and, by compactness of X, a homeomorphism. This finishes the proof of theorem (2.11.1).

PROOF of (2.11.2). We reduce the proof of (2.11.2) to (2.11.1): The space is seen to be the union of two intervals with the same end points. We divide the proof into five steps.

(1) No point is a cut point. If $X \smallsetminus x = U \cup V$ were a decomposition, then there would exist, by (2.11.4), a point $y \in U$ which does not separate $\overline{U} = U \cup x$ and a point $z \in V$, which does not separate $V \cup x = \overline{V}$. Then $X \smallsetminus \{y,z\} = (\overline{U} \smallsetminus y) \cup (\overline{V} \smallsetminus z)$ would be the union of connected sets which contain x. Hence $X \smallsetminus \{y,x\}$ would be connected, and this contradicts the assumption.

(2) Suppose here and in the sequel $X \smallsetminus \{a,b\} = U|V$, $a \neq b$. Then $U \cup \{a,b\} = \overline{U}$, because $\overline{U} \subset U \cup \{a,b\}$ as in the proof of (2.11.3) (1). The set $X \smallsetminus a$ is connected, by (1), and $b \in$ is cut point of this space. If we apply (2.11.3) (1) to $(X \smallsetminus a, b)$ instead of (X,x), we see that $b \in \overline{U}$, and analogously $a \in \overline{U}$.

(3) $U \cup \{a,b\}$ is connected. Suppose we have a decomposition $U \cup a = A|B$ with $a \in A$. From (2) we see $U \cup a = \overline{U} \smallsetminus b$, hence this set is closed in $X \smallsetminus b$. Similarly, $V \cup a$ is closed in $X \smallsetminus b$. From

$$X \smallsetminus b = U \cup a \cup V = B \cup A \cup (V \cup a)$$

we obtain $X \smallsetminus b = B|(A \cup a \cup V)$, and this contradicts (1).

(4) Since $U \cup a$ is connected we also have $U \cup \{a,b\} = \overline{U \cup a}$.

(5) We show that $\overline{U}$ and $\overline{V}$ satisfy the hypotheses of (11.1), and that $\{a,b\}$ are the end points. We have already seen in (3) that a and b are not cut points. Let $u \in U$ and suppose that $\overline{U} \smallsetminus u$ is connected. Then for no $v \in V$ the set $\overline{V} \smallsetminus v$ is connected, for otherwise $X \smallsetminus \{u,v\} = \overline{U} \smallsetminus u \cup \overline{V} \smallsetminus v$ and $a \in (\overline{U} \smallsetminus u) \cap (\overline{V} \smallsetminus v)$ the space $X \smallsetminus \{u,v\}$ would be connected, in contrast to the assumption. By (2.11.1), $\overline{V}$ is a simple arc with end points a and b. Therefore the sets $\overline{V} \smallsetminus v$, $v \in V$ has two components $a \in C_a$ and $b \in C_b$ and hence $C_a \cup (\overline{U} \smallsetminus u) \cup C_b$ is connected. But this set equals $X \smallsetminus \{u,v\}$. The assumption that $\overline{U} \smallsetminus u$, $u \in U$ is connected thus leads to a contradiction. From (2.11.1) we now see that $\overline{U}$ is a simple arc with end points a,b; the same holds for $\overline{V}$. Therefore X is the union of two simple arcs with the same endpoint and therefore homeomorphic to S^1. $\square$