Temperley-Lieb algebras associated to the root system D

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Abstract In this note we define and compute the Temperley-Lieb algebras associated to the Coxeter–Dynkin graphs of type D_n. The computation relates these algebras to those corresponding to the root systems of type A and B. We also show the connection to braid theory and to the Kauffman bracket and describe a related graphical calculus.

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1. Hecke algebras and Temperley-Lieb algebras

This section collects some general results. Let S be a finite set. A *Coxeter matrix* is a symmetric mapping $m\colon S \times S \to \mathbb{N} \cup \{\infty\}$ such that $m(s,s) = 1$ and $m(s,t) \geq 2$ for $s \neq t$. A Coxeter matrix (S,m) is often specified by its *Coxeter graph* $\Gamma(S,m)$. It has S as its set of vertices and an edge with weight $m(s,t)$ whenever $m(s,t) \geq 3$. Usually, the weight $m(s,t) = 3$ is omitted from the notation.

The *standard Hecke algebra* $H_q(S,m)$ associated to a Coxeter matrix (S,m) is the associative algebra with 1 over the commutative ring $\mathcal{K}$ with generators $(x_s \mid s \in S)$ and relations

(1.1)
$$\begin{array}{rcll} x_s^2 &=& (q-1)x_s + q, & q \in K^* \\ x_s x_t x_s \ldots &=& x_t x_s x_t \ldots, & m(s,t) \geq 2 \end{array}$$

($m(s,t)$ factors on each side, alternating). Here $\mathcal{K}^*$ denotes the unit groups of $\mathcal{K}$.

Suppose m takes values in $\{1,2,3\}$. Then the *Temperley-Lieb algebra* $T_d(S,m)$ is the associative algebra with 1 over $\mathcal{K}$ with generators $(e_s \mid s \in S)$ and relations

(1.2)
$$\begin{array}{rcll} e_s^2 &=& d e_s & d \in \mathcal{K}^* \\ e_s e_t &=& e_t e_s & m(s,t) = 2 \\ e_s e_t e_s &=& e_s & m(s,t) = 3. \end{array}$$

We shall obtain the Temperley-Lieb algebra as a quotient of a Hecke algebra. For this purpose we assume

(1.3) $$p \in \mathcal{K}^*, \quad q = p^2, \quad d = p + p^{-1}.$$

(1.4) Proposition. *Under the hypothesis* (1.3) *the assignment* $x_s \mapsto pe_s - 1$ *defines a surjective homomorphism* φ: $H_q(S,m) \to T_d(S,m)$. *The kernel of* φ *is the twosided ideal generated by the elements* $x(s,t) = x_s x_t x_s + x_s x_t + x_t x_s + x_s + x_t + 1$; *here* (s,t) *runs over the pairs* (s,t) *with* $m(s,t) = 3$.

PROOF. (Compare [6, 2.11].) One verifies easily that φ respects the defining relations of the Hecke algebra. Certainly, φ is surjective. Let $I \subset H_q(S,m)$ denote the ideal generated by the $x(s,t)$ for (s,t) with $m(s,t) = 3$. We define a homomorphism ψ: $T_d(S,m) \to H_q(S,m)/I$ by $\psi(e_s) = p^{-1}(x_s + 1)$. One verifies that this is compatible with (1.2) and that $x(s,t)$ is contained in the kernel of φ. Hence φ induces φ: $H_q/I \to T_d$. By construction, φ and ψ are inverse homomorphisms.□

The preceding construction can, in particular, be applied to Coxeter matrices of ADE-type. The resulting algebras are then finite dimensional. The structure of TA_{n-1} associated to the linear graph A_{n-1} with n vertices is well known, see [6]; this is the classical Temperley-Lieb algebra. In the following sections 2 and 3 we study the algebras related to the graph D_n with n vertices ($n \geq 4$). In section 4 we briefly discuss D-tangles and the associated Kauffman functor.

But first we present one general result: By way of example we show that $T_d(S,m)$ is non-zero. This is done by constructing a standard module which arises from the reflection representation of the Hecke algebra.

We work with a field $\mathcal{K}$. Let V denote the free $\mathcal{K}$-module with basis $\{v_s \mid s \in S\}$. We define a symmetric bilinear form B on V by

$$\begin{array}{ll} B(v_s, v_s) = q + 1 & \\ B(v_s, v_t) = p & m(s,t) = 3 \\ B(v_s, v_t) = 0 & m(s,t) = 2. \end{array}$$

We define a linear map X_s: $V \to V$ by $X_s(v) = qv_s - B(v_s, v)v$. Then $X_s(v_s) = -v_s$, and $X_s(v) = v$ for v in the orthogonal complement of v_s. We assume $q+1 \in \mathcal{K}^*$. Then V is the orthogonal direct sum of $\mathcal{K}v_s$ and $(\mathcal{K}v_s)^\perp$. On the latter, X_s acts as multiplication by q. Hence X_s satisfies the quadratic equation $X_s^2 = (q-1)X_s + q$ of the Hecke algebra.

The determinant $d_{s,t}$ of B on the submodule $\langle v_s, v_t \rangle$ generated by v_s and v_t equals

$$d_{s,t} = \begin{cases} (q+1)^2 & m(s,t) = 2 \\ q^2 + q + 1 & m(s,t) = 3. \end{cases}$$

We therefore also assume $q^2 + q + 1 \in \mathcal{K}^*$. Then V is the orthogonal direct sum of $\langle v_s, v_t \rangle$ and $\langle v_s, v_t \rangle^\perp$. On the latter subspace, X_s and X_t act as multiplication by q. The action of X_s and X_t on $\langle v_s, v_t \rangle$ in the basis v_s, v_t is given by

$$X_s = \begin{pmatrix} -1 & p \\ 0 & q \end{pmatrix}, \quad X_t = \begin{pmatrix} q & 0 \\ p & -1 \end{pmatrix},$$

in the case $m(s,t) = 3$. A simple computation shows

$$X_s X_t X_s = X_t X_s X_t = \begin{pmatrix} 0 & -pq \\ -pq & 0 \end{pmatrix}.$$

Thus we have constructed the reflection representation V of $H_q(S,m)$.

The assignment ω: $H_q(S,m) \to H_q(S,m)$, $x_s \mapsto -qx_s^{-1}$ is an involutive automorphism of the Hecke algebra. It transforms V into a new module $W = V^\omega$.

(1.5) Proposition. *The module W factors over the homomorphism φ of* (1.4).

Proof. We set $Y_s = -qX_s^{-1}$. We have to show that the operator

$$Y_{s,t} = Y_s Y_t Y_s + Y_s Y_t + Y_t Y_s + Y_s + Y_t + 1$$

acts on V as the zero map. We compute that $Y_s + 1$ and $Y_t + 1$ act on $\langle v_s, v_t \rangle$ in the basis v_s, v_t through the matrices

$$Z_s = \begin{pmatrix} q+1 & -p \\ 0 & 0 \end{pmatrix}, \quad Z_t = \begin{pmatrix} 0 & 0 \\ -p & q+1 \end{pmatrix}.$$

This is used to verify on $\langle v_s, v_t \rangle$ the relation $Z_s Z_t Z_s = q Z_s$. A formal calculation, using the quadratic equation for Y_s, yields

$$(Y_s + 1)(Y_t + 1)(Y_s + 1) - q(Y_s + 1) = Y_{s,t}.$$

Therefore $Y_{s,t}$ acts as zero on $\langle v_s, v_t \rangle$. Since X_s is multiplication by q on $\langle v_s, v_t \rangle^\perp$, we see that $-qX_s^{-1} + 1$ is the zero map. □

We give a more direct construction of a $T_d(S,m)$-module which does not use the reflection representation of the Hecke algebra. Let $A = (a_{st})$ denote a symmetric $S \times S$-matrix over $\mathcal{K}$. We consider the associative algebra $T(A)$ over $\mathcal{K}$ with generators $(Z_s \mid s \in S$ and relations

$$\begin{aligned} Z_s^2 &= a_{ss} Z_s \\ Z_s Z_t Z_s &= a_{st} a_{ts} Z_s. \end{aligned}$$

Then a simple verification from the definitions gives:

(1.6) Proposition. *Let V be the $\mathcal{K}$-module with basis $(v_s \mid s \in S$. The operators $Z_s(v_t) = a_{st} v_s$ make V into a $T(A)$-module. (Hence each Z_s has rank at most one on V.)* □

The matrix $A = (a_{st})$ is called *indecomposable*, if there is no partition $S = S_1 \coprod S_2$ with $a_{uv} = 0$ for $u \in S_1$, $v \in S_2$.

(1.7) Proposition. *Let $\mathcal{K}$ be a field. Suppose A is indecomposable and* $\det(A) \neq 0$. *The the module V of the previous proposition is a simple $T(A)$-module.*

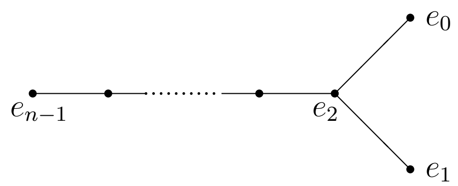
Proof. We have $Z_s(\sum_j a_j v_j = (\sum_j a_j a_{sj})v_s$. Suppose $v = \sum_j a_j v_j \neq 0$. Since $\det(A) \neq 0$, not all $Z_s v$ are zero. If $0 \neq M \subset V$ is a $T(A)$-submodule, then there exists $s \in S$ with $v_s \in M$. Suppose $v_t \notin M$. Since $Z_t v_s = a_{ts} v_t \in M$, we must have $a_{ts} = 0$. This contradicts the indecomposability of A. Hence all v_t are containes in M. $\square$

In the case of a Coxeter graph, we set $a_{ss} = d$, $a_{st} = 1$ for $m(s,t) = 3$, and $a_{st} = 0$ for $m(s,t) = 2$. Then V becomes a module over $T_d(S,m)$. Also, $\det(A)$ is a non-trivial monic polynomial in d, hence in general not zero.

2. The structure of TD_n

The algebra TD_n with parameter $d \in \mathcal{K}$ is associated to the graph D_n. In the following figure we have specified the names of the generators.

D_n :



The algebra TD_n will be decomposed into an algebra which belongs to the linear graph A_{n-1} and another algebra which is related to the graph B_n. Here A_{n-1} is the linear Coxeter graph with $n-1$ vertices $e_1, \dots, e_{n-1}$ and $m(e_j, e_{j+1}) = 3$. We use the same notation for the generators of TD_n and TA_{n-1}. The following is easily verified.

(2.1) Proposition. *The assignment $e_0 \mapsto e_1$ and $e_j \mapsto e_j$ $(j \geq 1)$ defines a surjective homomorphism $\alpha \colon TD_n \to TA_{n-1}$.* $\square$

We remark that the automorphism of the graph D_n which interchanges e_0 and e_1 and fixes e_j for $j \geq 2$ induces an involution $\tau \colon TD_n \to TD_n$. We have $\alpha\tau = \alpha$.

(2.2) Proposition. *The kernel of α is the twosided ideal I generated by the difference $e_0 - e_1$. The homomorphism $\alpha_1 \colon TA_{n-1} \to TD_n$, $e_1 \mapsto e_j$ is right inverse to α. We therefore have a splitting of modules $TD_n = I \oplus TA_{n-1}$.*

Proof. The inclusion $I \subset \operatorname{kernel} \alpha$ follows from the definitions. The relation $\alpha\alpha_1 = \mathrm{id}$ is obvious. The composition $\alpha_1 \circ \alpha$ is easily seen to be be the identity on generators. Hence $I = \operatorname{kernel} \alpha$. $\square$

We now use the following algebra $T'D_n$ of Temperley-Lieb type: It has generators $\varepsilon_0, \dots, \varepsilon_{n-1}$ and relations

$$
\text{(2.3)}\qquad
\begin{array}{rcll}
\varepsilon_j^2 & = & d\varepsilon_j & j \geq 1 \\
\varepsilon_0^2 & = & 2\varepsilon_0 & \\
\varepsilon_i\varepsilon_j & = & \varepsilon_j\varepsilon_i & |i-j| \geq 2 \\
\varepsilon_1\varepsilon_0\varepsilon_1 & = & d\varepsilon_1 & \\
\varepsilon_i\varepsilon_j\varepsilon_i & = & \varepsilon_i & |i-j| = 1; i,j \geq 1.
\end{array}
$$

This is a variant of the algebra of B_n type which has been studied in [3],[4].

(2.4) Proposition. *The assignment $\beta(e_0) = (\varepsilon_0 - 1)\varepsilon_1(\varepsilon_0 - 1)$ and $\beta(e_j) = \varepsilon_j$ for $j \geq 1$ defines a homomorphism $\beta\colon TD_n \to T'D_n$.*

Proof. For $j \geq 1$, the e_j and ε_j satisfy the same relations; we consider the remaining ones. We use $(\varepsilon_0 - 1)^2 = 1$, $\varepsilon_1(\varepsilon_0 - 1)\varepsilon_1 = 0$, and verify easily $\beta(e_0^2) = d\beta(e_0)$ and $\beta(e_0e_2e_0) = \beta(e_2e_0e_2)$. Moreover $\beta(e_0e_1) = 0 = \beta(e_1e_0)$. □

The proof of the previous proposition shows that the twosided ideal J generated by e_0e_1 is contained in the kernel of β. The image of β will turn out to be half of $T'D_n$, and J is equal to the kernel of β. In order to prove these statements we introduce the crossed product of $TD_n/J =: \mathcal{A}$ with the algebra $\mathcal{K}[\tau]/(\tau^2 - 1)$ where τ acts via the previously defined involution τ on $\mathcal{A}$. Formally, this crossed product $\mathcal{B}$ is defined as the free $\mathcal{A}$-module with basis $1, \tau$ and multiplication

$$(a + b\tau) \cdot (c + d\tau) := (ac + bd^\tau) + (bc^\tau + ad)\tau,$$

where x^τ denotes the action of τ on x. Note that the ideal J is τ-stable.

(2.5) Proposition. *The assignment $\varepsilon_j \mapsto \varepsilon_j$ $(j \geq 1)$ and $\varepsilon_0 \mapsto 1 + \tau$ defines an isomorphism $\beta_1\colon T'D_n \to \mathcal{B}$. The image of β corresponds to the subalgebra $\mathcal{A}$. The kernel of β is equal to J.*

Proof. The relation $\varepsilon_0^2 = 2\varepsilon_0$ corresponds to $(1+\tau)^2 = 1 + 2\tau + 1 = 2(1+\tau)$. The element $\varepsilon_1\varepsilon_0\varepsilon_1$ is mapped to $e_1 \cdot (1+\tau) \cdot e_1 = (e_1 + e_1\tau) \cdot e_1 = e_1^2 + e_1e_0\tau$ and this equals de_1 modulo J. We see that β_1 is well-defined and surjective. The inverse homomorphism is given by β and $\tau \mapsto \varepsilon_0 - 1$. □

The algebras $T'D_n$ and TA_{n-1} have augmentation homomorphisms to $\mathcal{K}$ which map the generators ε_j and e_j to zero. Let $T''D_n$ denote the image of β. We have a sequence

$$\text{(2.6)}\qquad 0 \to TD_n \xrightarrow{\alpha,\beta} TA_{n-1} \oplus T''D_n \to \mathcal{K} \to 0;$$

the map to $\mathcal{K}$ is the difference of the augmentations. The structure of TD_n will be obtained from the next result.

(2.7) Theorem. *The sequence* (2.6) *is exact.*

Proof. We show that α maps the kernel J of β isomorphically onto the kernel of the augmentation. For this purpose we recall a basis of TA_{n-1}, see [6]. We set

$$e(i,j) = e_i e_{i-1} \dots e_j, \quad i \geq j.$$

Then a basis of TA_{n-1} is given by the products

(2.8)
$$(i_1, j_1) \cdot \ldots \cdot e(i_p, j_p)$$

with

$$\begin{array}{cc} 1 \leq i_1 < \ldots < i_p \leq n-1, & 1 \leq j_1 < \ldots < j_p \leq n-1 \\ j_s \leq i_s, & 0 \leq p \leq n-1. \end{array}$$

We exhibit a basis of J which is mapped onto this basis. We use the notation

$$\bar{e}(i,j) = e_i e_{i-1} \dots e_1 e_0 e_2 \dots e_j, \quad j \geq 2.$$

This element is mapped under α to $de(i,j)$. Recall that $d \in \mathcal{K}^*$ is a unit.

We show that J is spanned by the elements of the form (2.8) where $e(i_1, j_1)$ is replaced by $\bar{e}(i_1, j_1)$; this finishes the proof of (2.7).

We consider words in the symbols $e_0, \dots, e_{n-1}$. An *elementary reduction* of a word is one of the following replacements: $e_j e_j$ by de_j, $e_i e_j e_i$ by e_i, $e_i e_j$ by $e_j e_i$. Note that the length of the word is not increased. A coefficient of the form d^r may appear. A word is in *reduced form*, if it cannot be shortened by an elementary reduction. The words (2.8) are reduced. Certainly, J is generated by reduced words.

We claim that J is generated by reduced words of the form ae_0e_1b in which a and b do not involve e_0 and e_1.

We know already that J is the ideal generated by words ce_0e_1d. If d contains e_1, say, then the word contains a string of the form $e_1 x e_1$ in which x involves only e_j, $j \geq 2$. A word of this type is never reduced; this follows easily by using (2.8) for x. This shows the claim.

We next consider normal forms of reduced words in J by induction on n. Suppose a reduced word contains two factors e_{n-1}, say a string $e_{n-1} y e_{n-1}$ with y not involving e_{n-1} and of shortest length. Then, by induction, this string must equal $\bar{e}(n-1, n-1)$. If a word contains $z = \bar{e}(n-1, n-1)$, it is not reduced, unless it is equal to z. Therefore z is the only reduced word in J with two appearances of e_{n-1}. Next, consider reduced words which have the form $w = xe_{n-1}y$. By interchanging elements, if necessary, we assume that y has minimal length. Then y necessarily has the form $e(n-2, j)$ or $\bar{e}(n-2, j)$. Since x does not contain e_{n-1}, we can apply the induction hypothesis to x. Since w is reduced, it is easily seen that w has the form (2.8) with $e(i_1, j_1)$ replaced by $\bar{e}(i_1, j_1)$. □

We assume known the structure of TA_{n-1} in the generic case (q not a root of unity) [6]. It remains to study the algebra $T''D_n$. This is the subject of the next section.

We conclude this section with some remarks concerning the algebra of the graph D_n: braid groups and Hecke algebras.

Each Coxeter matrix (S, m) has associated to it a *braid group* $Z(S, m)$ with generators $(x_s \mid s \in S)$ and relations $x_s x_t x_s \ldots = x_t x_s x_t \ldots$ with $m(s, t)$ factors on each side. For the graph D_n we define another braid group $Z'D_n$ with generators $\kappa_0, \ldots, \kappa_{n-1}$ and relations

(2.9)
$$\begin{array}{rclc} \kappa_i\kappa_j\kappa_i & = & \kappa_j\kappa_i\kappa_j & |i-j| = 1; i, j \geq 1 \\ \kappa_0\kappa_1\kappa_0\kappa_1 & = & \kappa_1\kappa_0\kappa_1\kappa_0 & \\ \kappa_i\kappa_j & = & \kappa_j\kappa_i & |i-j| \geq 2 \\ \kappa_0^2 & = & 1. & \end{array}$$

This is a quotient of the group ZB_n for which the last relation is not present.

(2.10) Proposition. *The group $Z'D_n$ is the semidirect product of ZD_n with $\mathbb{Z}/2$. The generator τ of $\mathbb{Z}/2$ acts on ZD_n by the automorphism induced by the graph automorphism.*

Proof. Let G denote the semi-direct product. We define inverse homomorphisms $f\colon G \to Z'D_n$ and $g\colon Z'D_n \to G$ by

$$f\colon \tau, x_0, x_1, \ldots, x_{n-1} \mapsto \kappa_0, \kappa_1\kappa_0\kappa_1, \kappa_1, \ldots, \kappa_{n-1}$$

$$g\colon \kappa_0, \kappa_1, \ldots, \kappa_{n-1} \mapsto \tau, x_1, \ldots, x_{n-1}.$$

□

We remark that conjugation by κ_0 corresponds to τ.

We define the Hecke algebra $H'D_n$ as the associative algebra with 1 generated by $\kappa_0, \ldots, \kappa_{n-1}$ with braid relations as above and quadratic relations $\kappa_0^2 = 1$ and $\kappa_j^2 = (q-1)\kappa_j + q$ for $j \geq 1$. This is a Hecke algebra of B_n-type where the parameter Q belonging to κ_0 has been specialized to 1. We have an embedding $\tilde{\alpha}\colon HD_n \to H'D_n$, $x_0 \mapsto \kappa_0\kappa_1\kappa_0$, $x_j \mapsto \kappa_j$ for $j \geq 1$. As in the case of the Temperley-Lieb algebras we see:

(2.11) Proposition. *The algebra $H'D_n$ is the crossed product of HD_n with $\mathcal{K}[\tau]/(\tau^2 - 1)$.* □

There is a connection between Hecke algebras and Temperley-Lieb algebras as follows.

(2.12) Proposition. *The algebra $T'D_n$ is a quotient of $H'D_n$ under the homomorphism $\varphi'\colon \kappa_0 \mapsto \varepsilon_0 - 1$, $\kappa_j \mapsto p\varepsilon_j - 1$ $(j \geq 1)$. Moreover $\alpha \circ \varphi = \varphi' \circ \tilde{\alpha}$.* □

3. The reduced Temperley-Lieb algebra

This section presents the structure of $T'D_n$ and $T''D_n$ for generic parameters (p not a root of unity). The algebra $T'D_n$ is of the type B_n but not exactly the same. Therefore we have to extend some of results in [3] to the present situation.

There exists idempotent elements f_k and g_k in $T'D_n$ with the following properties:

$$\begin{array}{ll}
f_0 = 1 - \frac{1}{2}\varepsilon_0 & \\
f_k = f_{k-1} + \frac{p^{k-1}+p^{-k+1}}{p^k+p^{-k}} f_{k-1} v e_k f_{k-1}, & 1 \le k \le n-1 \\
g_0 = \frac{1}{2}\varepsilon_0 & \\
g_k = g_{k-1} + \frac{p^{k-1}+p^{-k+1}}{p^k+p^{-k}} g_{k-1}\varepsilon_k g_{k-1}, & 1 \le k \le n-1 \\
\varepsilon_j f_k = f_k \varepsilon_j = \varepsilon_j g_k = g_k \varepsilon_j = 0, & 1 \le j \le k \\
\varepsilon_0 g_k = g_k \varepsilon_0, & 0 \le k \le n-1 \\
g_k f_k = f_k g_k = 0, & 0 \le k \le n-1 \\
\eta(f_k) = 1 - \frac{1}{2}\varepsilon_0,\ \eta(g_k) = \frac{1}{2}\varepsilon_0. &
\end{array}$$

The map η is the augmentation which sends e_j, $j \ge 1$, to zero.

The proof for these assertions is as for [3], Satz 5.2, by induction on k. With the help of the central orthogonal idempotents f_{n-1} and g_{n-1} it is shown as in [3], Satz (7.1), that the Bratteli diagram of the inclusion $T'D_{n-1} \subset T'D_n$ is the same as for the inclusion $TB_{n-1} \subset TB_n$. In particular, $T'D_n$ has $n+1$ simple modules $M_0(n), M_1(n), \dots, M_n(n)$ with $M_j(n) = N_j$ of dimension $\binom{n}{j}$.

The simple modules of $T''D_n$ are determined via restriction from $T'D_n$.

(3.1) Theorem. *The algebra $T''D_n$ has the following irreducible modules:*

(1) *Suppose $n = 2k+1$. The restrictions* $\mathrm{res}M_j$ *for $j \le k$. Moreover* $\mathrm{res}M_j \cong \mathrm{res}M_{n-j}$.

(2) *Suppose $n = 2k$. The restrictions* $\mathrm{res}M_j$, $j < k$. *In this case* $\mathrm{res}M_j \cong \mathrm{res}M_{n-j}$. *The module* $\mathrm{res}M_k$ *is the direct sum of two simple $T''D_n$-module of the same dimension.*

The proof of (3.1) is by induction on n. One uses the structure of the Bratteli-diagram for $T'D_{n-1} \subset T'D_n$ and the following general fact about the crossed product construction of $T'D_n$ from $T''D_n$.

Let $\mathcal{A}$ be a semi-simple algebra with an involutive automorphism τ over the field $\mathcal{K}$ of characteristic zero and let $\mathcal{B}$ denote the crossed product algebra as described in section 2. If U is an $\mathcal{A}$-module, let U^τ denote the same vector space with the $\mathcal{A}$-action twisted by τ. The map $a + b\tau \mapsto a - b\tau$ is an automorphism of $\mathcal{B}$. If V is a $\mathcal{B}$-module, then $\overline{V}$ is obtained from V by twisting with this automorphism (conjugate module). A simple $\mathcal{A}$-module U is called of type I (resp. type II) if $U \cong U^\tau$ (resp. $U \ncong U^\tau$). A simple $\mathcal{B}$-module V is called of type I (resp. type II) if $V \ncong \overline{V}$ (resp. $V \cong \overline{V}$). If U is an $\mathcal{A}$-module, we call $\mathcal{B} \otimes_{\mathcal{A}} U$ the induced $\mathcal{B}$-module $\mathrm{ind}\,U$. These notations are used in the statement of the following result.

(3.2) Proposition.

(1) *Suppose V is a simple $\mathcal{B}$-module of type I. Then* $\mathrm{res}V = U$ *is simple of type I and* $\mathrm{ind}\, U \cong V \oplus \overline{V}$.
(2) *Suppose V is a simple $\mathcal{B}$-module of type II. Then* $\mathrm{res}V \cong U \oplus U^\tau$ *and* $V \cong \mathrm{ind}\, U \cong \mathrm{ind}\, U^\tau$. *Moreover, U, U^τ of type II.*
(3) *Suppose U is a simple $\mathcal{A}$-module of type I. Then* $\mathrm{ind}\, U \cong V \oplus \overline{V}$. *Moreover,* $\mathrm{res}V \cong \mathrm{res}\overline{V} \cong U$.
(4) *Suppose U is a simple $\mathcal{A}$-module of type II. Then* $\mathrm{ind}\, U = V$ *is a simple $\mathcal{B}$-module of type II and* $\mathrm{res}V \cong U \oplus U^\tau$.

Proof. The proof of this proposition is by an adaption of the argument in [2], Ch. VI for the proof of Theorem (7.3). □

Proof of (3.1). By (3.2) it suffices to determine the modules $\overline{M}_j(n)$. Let res_{n-1} denote the restriction via $T'D_{n-1} \subset T'D_n$. From the Bratteli diagram we know

(3.3)
$$\begin{aligned} &\mathrm{res}_{n-1} M_j(n) = M_{j-1}(n-1) \oplus M_j(n), \quad 1 \le j \le n-1 \\ &\mathrm{res}_{n-1} M_0(n) = M_0(n-1),\ \mathrm{res}_{n-1} M_n(n) = M_{n-1}(n-1). \end{aligned}$$

The isomorphism type of a simple $T'D_n$-module M is therefore determined by $\mathrm{res}_{n-1}M$. We show by induction on n that $\overline{M}_j(n) = M_{n-j}(n)$. Since restriction is compatible with conjugation, the induction step follows from (3.3). The induction starts with the irreducible representations of the group $\mathbb{Z}/2$ generated by τ. □

4. Braids and tangles of type D

The permutations σ of $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$ with the property $\sigma(-i) = -\sigma(i)$ form the Weyl group WB_n of the root system B_n. The subgroup of even permutations in WB_n is the Weyl group WD_n of the root system D_n. The reflection representation of WD_n on $\mathbb{C}^n$ is given as follows: The subgroup S_n acts by permutation of coordinates and $(\mathbb{Z}/2)^{n-1}$ by sign changes $(z_j) \mapsto (\pm z_j)$ with an even number of minus signs. The reflection hyperplanes are given by $z_i = z_j$ and $z_i = -z_j$ for all pairs (i,j) with $i \neq j$. Let X be the complement of the reflection hyperplanes and X/W the orbit space of the free $W = WD_n$ action. Brieskorn [1] has shown that the fundamental group $\pi_1(X/W)$ is the braid group ZD_n.

We translate this result and obtain a description of ZD_n by planar braid pictures.

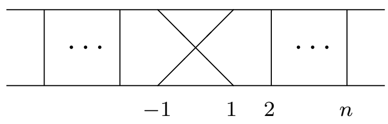
A loop $[w]$ in X/W with base point $(1, \dots, n)$ can be lifted to X with $(1, \dots, n)$ as starting point. Let w: $[0,1] \to X$, $\quad t \mapsto (w_j(t))$ be the resulting path from $(1, \dots, n)$ to $(\pm\sigma(1), \dots, \pm\sigma(n))$. Here $\sigma \in S_n$, and the number of minus signs is even. We consider the braid in $\mathbb{C} \times [0,1]$ with $2n$ strings given by

$$t \mapsto \{-w_n(t), \dots, -w_1(t), w_1(t), \dots, w_n(t)\} \times \{t\}.$$

The braid is symmetric with respect to the symmetry $\mathbb{C} \to \mathbb{C}$, $z \mapsto -z$. the strings are $\zeta_{\pm j}$: $t \mapsto (\pm w_j(t), t)$. Since w maps into X, the strings have the following property: The string pairs (ζ_j, ζ_{-j}) and (ζ_k, ζ_{-k}) never meet for $j \neq k$; an intersection would correspond to a point $w_j(t) = \pm w_k(t)$ on a reflection hyperplane. A value $w_j(t) = 0$ is not excluded, though. In this case, the strings ζ_j and ζ_{-j} intersect. Therefore we are not dealing with a braid in the usual sense. Of course, we can always choose representing paths w such that no intersection of ζ_j with ζ_{-j} occurs.

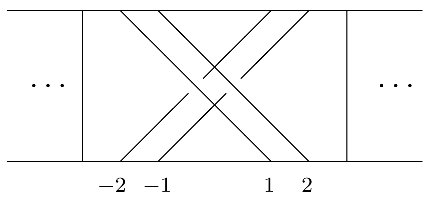
As usual, we consider planar generic projections of braids in the strip $\mathbb{R} \times [0,1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$ which are symmetric with respect to the axis $0 \times [0,1]$. The transverse intersections on the axis are ordinary crossings, and the other crossings are over- and undercrossings which appear in symmetric pairs.

In the geometric picture, the extended braid group $Z'D_n$ has generators $\kappa_0, \dots, \kappa_{n-1}$ with κ_0 given by



κ_0

and κ_j $(j \geq 1)$ given by the symmetrized crossing of the j-th and $1+j$-th string (see the next figure for κ_1). The braid group ZD_n has generators $x_j = \kappa_j$ $(j \geq 1)$ and $x_0 = \kappa_0 \kappa_1 \kappa_0$ represented by



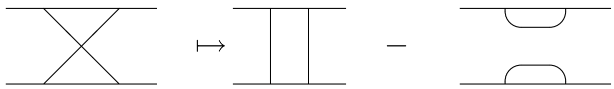
x_0

The relation $\kappa_0^2 = 1$ corresponds to a standard Reidemeister move of type II. It would also be possible to use over- and under-crossing on the axis, but then allow for an interchange of over-crossing and under-crossing on the axis.

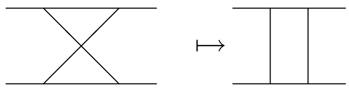
Elements in the subgroup ZD_n have in their geometric picture an even number of crossings on the axis.

The geometric braid groups $Z'D_n$ and ZD_n are included in tangle categories $S'D$ and SD (in the sense of [8], [9]). The category $S'D$ has objects $[\pm n]$, $n \in \mathbb{N}_0$. The morphisms from $[\pm m]$ to $[\pm n]$ are tangle pictures in $\mathbb{R} \times [0,1]$ from $[\pm m] \times 0$ to $[\pm n] \times 1$ which are symmetric with respect to the axis $0 \times [0,1]$. The crossings on the axis are ordinary crossings. Composition is defined by placing one tangle above the other and shrinking of $[0,2]$ to $[0,1]$. The subcategory SD of $S'D$ consists of tangles with an even number of points on the axis. There are similar categories S_0D and $S_0'D$ of oriented tangles. Also, one may consider banded (framed) tangles by not allowing Reidemeister type I moves. The D-tangle categories are analogous to the B-tangle categories [4], except for the special treatment of the crossings on the axis. The categories are tensor module categories over the appropriate categories of ordinary tangles. Ordinary tangles are included by symmetrizing.

There is a Kauffman functor from $S'D$ to the category $T'D$ of bridges. In this context one chooses a parameter A with $p = -A^2$. The Kauffman functor resolves a symmetrized ordinary crossing as usual in the definition of the Kauffman bracket [7]. A crossing on the axis is treated as in the following figure.



There is also a forgetful functor to A-tangles which maps



and takes the $\mathbb{Z}/2$-quotient of the resulting tangle. These two functors correspond to the splitting of the algebra TD_n in section 2.

References

1. Brieskorn, E.: Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Invent. Math. 12, 57 – 61 (1971).

2. Bröcker, Th., and T. tom Dieck: Representations of compact Lie groups. New York – Heidelberg, Springer 1985.

3. tom Dieck, T.: Symmetrische Brücken und Knotentheorie zu den Dynkin-Diagrammen vom Typ B. J. reine angew. Math. 451, 71 – 88 (1994).

4. tom Dieck, T.: Knotentheorien und Wurzelsysteme. Teil II. Mathematica Gottingensis 44 (1993).

5. tom Dieck, T.: Knot theories and root systems – Part III. Mathematica Gottingensis 22 (1994).

6. Goodman, F. M., de la Harpe, P., and V. F. R. Jones: Coxeter graphs and towers of algebras. New York – Heidelberg, Springer 1989.

7. Kauffman, L. H.: State models and the Jones polynomial. Topology 26, 395 – 407 (1987).

8. Reshetikhin, N., and V. G. Tuarev: Ribbon graphs and their invariants derived from quantum groups. Commun. Math. Phys. 127, 1 – 26 (1990).

9. Turaev, V. G.: Operator invariants of tangles and R-matrices. USSR Izvestia 35, 411 – 444 (1990).