On tensor representation of knot algebras

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The main purpose of this work is to communicate certain R-matrix relations. They are designed to yield tensor representations of (extended) Hecke algebras of Coxeter type B. We also define various Birman-Wenzl algebras of type B and derive the corresponding R-matrix identities. These identities are also the basis for representations of tangle categories of type B. The tensor representations yield, via quantum traces, Markov traces of the algebras and therefore B-type analogues of Jones polynomials, Kauffman polynomials and the like. The tangle theory will be the subject of another paper. We also collect some material for later use in this series of papers. This is Part V of the series Knot Algebra and Root Systems.

1. The four braid relation

Let W be a module over the integral domain $\mathfrak{K}$. We study automorphisms X and Y of W which satisfy the *four braid relation*

(1.1) $$XYXY = YXYX.$$

We are particularly interested in the following case:

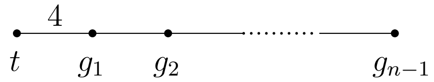
(1)The automorphism X is an R-matrix; this means $W = V \otimes V$ for a free module V with basis $v_1, \dots, v_n$ and X satisfies the Yang-Baxter equation

(1.2) $$(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X).$$

(2) The automorphism Y has the form $F \otimes 1$ for an automorphism F of V.

The interest in this case comes from the representation theory of braid groups. Recall that the braid group ZB_n associated to the Coxeter graph B_n

(1.3)



B_n

with n vertices has generators $t, g_1, \dots, g_{n-1}$ and relations:

(1.4)
$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & i > 1 \\ g_ig_j & = & g_jg_i & |i-j| \geq 2 \\ g_ig_jg_i & = & g_jg_ig_j & |i-j| = 1. \end{array}$$

Given automorphisms F and X as above, we obtain a tensor representation of ZB_n on the n-fold tensor power $V^{\otimes n}$ of V by setting:

(1.5)
$$\begin{aligned} t &= F \otimes 1 \otimes \cdots \otimes 1 \\ g_i &= 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \end{aligned}$$

The X in g_i acts on the factors i and $i+1$. Later we will study the cases when such representations of ZB_n induce representations of suitable Hecke algebras and Birman-Wenzl algebras. Moreover, we extend this to representations of tangle categories.

We point out that any two automorphisms $X, Y \colon W \to W$ which satisfy (1.1) yield a representation of ZB_2 on W.

We begin with a prototype computation which is used later on several occasions. We think of X and Y as given by $(2,2)$-matrices of the following type:

(1.6)
$$X = \begin{pmatrix} Z & 0 \\ 0 & qI \end{pmatrix}, \qquad Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $q \in \mathfrak{K}^*$ (the units of $\mathfrak{K}$) and I is the identity matrix. The matrices Z, A (and qI, D) are square matrices of the same size, respectively. A computation of (1.1) in block form yields:

(1.7) Proposition. *The relation* (1.1) *holds if and only if the following equalities hold:*
(I) $ZAZA + qZBC = AZAZ + qBCZ$
(II) $Z(AZB + qBD) = q(AZB + qBD)$
(III) $(CZA + qDC)Z = q(CZA + qDC)$. □

Equation (II) means that the columns of $AZB+qBD$ are eigenvectors of Z for the eigenvalue q (if they are nonzero). Equation (III) has a similar interpretation for the row vectors of $CZA + qDC$ (or consider the transpose).

(1.8) Corollary. *If A and D are zero, then* (1.1) *holds if and only if BC commutes with Z.* □

As a very special case, we think of X and Y in (1.6) as ordinary $(2,2)$-matrices over $\mathfrak{K}$. Then $Z \in \mathfrak{K}^*$ and $\det(Y) = AD - BC \in \mathfrak{K}^*$. From (1.7) we obtain:

(1.9) Proposition. *Let X and Y in* (1.6) *be* $(2,2)$*-matrices. Then* (1.1) *holds in the trivial cases $Z = q$ or $B = C = 0$. If $Z \neq q$ and B (or C) are nonzero, then* (1.1) *holds if and only if $AZ = -qD$.* □

Suppose we are still in the situation of (1.9). The element $\zeta = XYXY$ generates the center of ZB_2. Under the assumption $AZ = -qD$, the matrix of ζ is

(1.10)
$$\zeta = -\det(X)\det(Y)I.$$

If Y has the characteristic polynomial $(Y - b_1)(Y - b_2)$, then we compute (with $\mathrm{Tr}(Y) = b_1 + b_2$)

(1.11) $$A = -\frac{Z}{q-Z}\mathrm{Tr}(Y), \quad D = -\frac{q}{Z-q}\mathrm{Tr}(Y), \quad b_1b_2 = AD - BC.$$

The group ZB_2 has the presentation $\langle X, Y \mid XYXY = YXYX \rangle$. Another presentation is $Z'B_2 = \langle X, C \mid C^2X = XC^2 \rangle$. An isomorphism $Z' \to Z$ is given by $X \mapsto X$, $C \mapsto XY$. The element C generates the center. The computations above yield the following result:

(1.12) Proposition. *Let $\mathfrak{K}$ be a field. Suppose given $(a_1, a_2), (b_1, b_2) \in \mathfrak{K}^{*2}$ with $a_1 \neq a_2$ and $b_1 \neq b_2$. Then there exists an irreducible twodimensional representation of ZB_2, unique up to isomorphism, such that X, Y have the characteristic polynomial $(t-a_1)(t-a_2)$, $(t-b_1)(t-b_2)$, respectively.*

Proof. The characteristic polynomials are invariants of the isomorphism type. We have shown above the existence of a representation where

$$X = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and (1.11) determines A, D and BC. Within the isomorphism type we can change B, C to BU, CU^{-1}. Hence we can normalize to $B = 1$. □

2. R-matrices of type A_n

We now turn our attention to the standard R-matrices of quantum group theory. We use the basis $v_{ij} = v_i \otimes v_j$ with lexicographical ordering. In order to get some insight, we begin with the simplest non-trivial R-matrix

(2.1) $$X = \begin{pmatrix} q & & & \\ & \delta & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \qquad \delta = q - q^{-1} \neq 0.$$

As explained in section 1, we look for matrices

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that with $Y = F \otimes 1$ the four braid relation holds.

(2.2) Proposition. *Suppose F is not a multiple of the identity. Then* (1.1) *holds for the matrices X and F above if and only if $a = 0$.*

Proof. We reorder the basis $v_{12}, v_{21}, v_{11}, v_{22}$. Then X has the form (1.6) with

$$Z = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix},$$

and Y has the form (1.6) with

$$A = D = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad B = C = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

We check the conditions (1.7). Equation (I) holds if and only if $a^2\delta = ad\delta$. Hence either $a = 0$ or $a = d$. We compute $AZB + qBD$ to be

$$\begin{pmatrix} ac & ab\delta + qbd \\ qac & bd \end{pmatrix}.$$

Since $(q,1)^t$ is the eigenvector for the eigenvalue q of Z, we must have, by (II), that $ac = 0$ and $ab = 0$. In case $a \neq 0$, we arrive at a multiple of the identity. If $a = 0$, then (II), and dually (III), are satisfied. □

We now study the for braid relation for the standard R-matrix X associated to the root system A_{n-1}, see e. g. [21, p. 171]. It is given by the linear map

(2.3)
$$Xv_{ij} = \begin{cases} qv_{ij} & i = j \\ v_{ji} & i > j \\ v_{ji} + \delta v_{ij} & i < j, \end{cases}$$

where $1 \leq i, j \leq n$. (The case $n = 2$ is displayed in (2.1).)

Proposition (2.2) is a (weak) motivation for considering only matrices $F = (f_{ij})$ which are bottom-right triangular, i. e. $f_{ij} = 0$ for $i + j \leq n$. As a further motivation for later assumptions, we mention the following computational results.

(2.4) Proposition. *Suppose $F = (f_{ij})$ is bottom-right triangular. Then, in the cases $n = 3, 4$, the relation* (1.1) *for $Y = F \otimes 1$ and X as in* (2.3) *implies that all elements of F which do not lie on one of the diagonals are zero. Moreover, $f_{33} = f_{44}$ in the case $n = 4$.* □

If a reader wants to check this: It suffices to consider the matrix positions (5,6) in the case $n = 3$ and the positions (2,16) and (3,16) in the case $n = 4$. A similar inspection should yield such a result for all n.

Because of (2.2) and (2.4), we only consider maps F of the following form

(2.5)
$$\begin{array}{rcll} F(v_j) & = & \alpha_j v_j + \beta_j v_{n+1-j} & j \neq \frac{n+1}{2} \\ F(v_j) & = & av_j & j = \frac{n+1}{2} \end{array}$$

where $\alpha_j = 0$ for $1 \leq j < \frac{n+1}{2}$ and $\alpha_j = w$ for $\frac{n+1}{2} < j \leq n$. The a-term does not appear for odd n.

(2.6) Theorem. *Suppose F has the form* (2.5) *and X is given as in* (2.3)*. Then the four braid relation* (1.1) *holds in the case $n = 2m$. If $n = 2m - 1$, then* (1.1) *holds if and only if $a^2 = \beta_j\beta_{n+1-j} + aw$ for all $j \neq m$.*

Proof. For the proof we need a bit of organisation. We take advantage of the fact that X and F have many zeros and repetitions. We have two involutions σ

and τ on the set of indices $J = \{(i,j) \mid 1 \le i,j \le n\}$, namely $\sigma(i,j) = (n+1-j, j)$ and $\tau(i,j) = (j,i)$. Since $\sigma\tau\sigma\tau = \tau\sigma\tau\sigma$, they formally generate the dihedral group D_8 of order 8. We decompose J into the orbits under this D_8-action. We have to consider 4 orbit types. Set $n+1-i = i'$.

Let $n = 2m$. Then we have orbits of type $(i,i), (i',i), (i,i'), (i',i')$ of length 4. This is the orbit of (i,j) if $i = j$ or $i = j'$. If $i \ne j, j'$, then the orbit of (i,j) has length 8.

Let $n = 2m-1$. Then we have the fixed point (m,m). There is another orbit type of length 4, namely $(m,j), (j,m), (j',m), (m,j')$ for $j \ne m$. The subspace spanned by an orbit is invariant under X and Y. Therefore it suffices to verify the four braid relation on these subspaces. The matrices involved depend only on the isomorphism type of the orbit. Therefore we need only consider the cases $n = 3$ and $n = 4$.

We present some details of the computation.

Let $n = 3$. We consider the subspace generated by $v_{12}, v_{21}, v_{23}, v_{32}$. The corresponding matrices have the following form (we set $\beta_1 = v$, $\beta_3 = u$):

$$X = \begin{pmatrix} \delta & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \delta & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 & u \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ v & 0 & 0 & w \end{pmatrix}.$$

We compute the product

$$XY = \begin{pmatrix} 0 & a & 0 & \delta u \\ 0 & 0 & 0 & u \\ v & 0 & a\delta & w \\ 0 & 0 & a & 0 \end{pmatrix}.$$

and its square

$$(XY)^2 = \begin{pmatrix} 0 & 0 & a\delta u & au \\ 0 & 0 & au & 0 \\ va\delta & va & a^2\delta^2 + aw & \delta vu + a\delta w \\ va & 0 & a^2\delta & aw \end{pmatrix}.$$

If we transpose this matrix and interchange u and v we obtain $(YX)^2$. On the other hand, we obtain by this procedure the same matrix if and only if a satisfies $a^2 = aw + vu$.

Let $n = 4$. We use the subspace generated by $v_{12}, v_{21}, v_{13}, v_{31}, v_{24}, v_{42}, v_{34}, v_{43}$. Then X has the block diagonal matrix

$$X = \begin{pmatrix} H & & & \\ & H & & \\ & & H & \\ & & & H \end{pmatrix}, \quad H = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}$$

and Y the matrix (with notation $\beta_1 = v_1$, $\beta_2 = v_2$, $\beta_3 = u_2$, $\beta_4 = u_1$)

$$Y = \left(\begin{array}{cc|cc|cc|cc} & & 0 & 0 & 0 & u_1 & & \\ & & 0 & u_2 & 0 & 0 & & \\ \hline 0 & 0 & 0 & 0 & & & 0 & u_1 \\ 0 & v_2 & 0 & w & & & 0 & 0 \\ \hline 0 & 0 & & & 0 & 0 & u_2 & 0 \\ v_1 & 0 & & & 0 & w & 0 & 0 \\ \hline & & 0 & 0 & v_2 & 0 & w & 0 \\ & & v_1 & 0 & 0 & 0 & 0 & w \end{array}\right).$$

Empty places contain, as always, a zero. These data yield the product:

$$XY = \left(\begin{array}{cc|cc|cc|cc} & & 0 & u_2 & 0 & \delta u_1 & & \\ & & 0 & 0 & 0 & u_1 & & \\ \hline 0 & v_2 & 0 & w & & & 0 & \delta u_1 \\ 0 & 0 & 0 & 0 & & & 0 & u_1 \\ \hline v_1 & 0 & & & 0 & w & \delta u_2 & 0 \\ 0 & 0 & & & 0 & 0 & u_2 & 0 \\ \hline & & v_1 & 0 & \delta v_2 & 0 & w\delta & w \\ & & 0 & 0 & v_2 & 0 & w & 0 \end{array}\right)$$

and its square

$$\begin{pmatrix} 0 & 0 & 0 & u_1 u_2 H \\ 0 & 0 & v_2 u_1 H & u_1 w H \\ 0 & v_1 u_2 H & z\delta H & u_2 w(\delta H + I) \\ v_1 v_2 H & v_1 w H & v_2 w(\delta H + I) & z\delta H + w^2(\delta H + I) \end{pmatrix}.$$

This matrix does not change if we transpose it and interchange u and v. This finishes the case $n = 4$. $\square$

(2.7) Remark. For $n = 2m - 1$, the matrix Y satisfies the equation $Y^2 = wY + z$ with $z = \beta_j \beta_{n+1-j}$ for $j \neq m$. For $n = 2m$ the matrix Y satisfies $\prod_{j=1}^{m}(Y^2 - wY - \beta_j \beta_{n+1-j}) = 0$. Thus, in the generic case, the minimal polynomial of Y has degree n. A similar result as (2.6) holds for the slightly more general R-matrices in [21, p. 171]. Suppose $q + q^{-1}$ is invertible in $\mathfrak{K}$. Then the eigenspace $S^2(V)$ of X for the eigenvalue q has the basis $v_{ij} + q^{-1} v_{ji}$ for $i < j$ and v_{ii}; and the eigenspace $\wedge^2(V)$ for the eigenvalue $-q^{-1}$ has the basis $v_{ij} - q v_{ji}$, $i < j$. $\heartsuit$

3. A three-dimensional example

We aim at results for the root systems B_n and C_n which are analogous to those in section two. The case $\dim V = 3$ seems to be exceptional and we study it in detail. In any case, explicit computations in higher dimensions are difficult.

We use the following R-matrix X.

$$
\left|\begin{array}{ccc|ccc|ccc}
q & & & & & & & & \\
 & \delta & & 1 & & & & & \\
 & & \mu & & \lambda & & q^{-1} & & \\
\hline
 & 1 & & 0 & & & & & \\
 & & \lambda & & 1 & & & & \\
 & & & & & \delta & & 1 & \\
\hline
 & & q^{-1} & & & & 0 & & \\
 & & & & & 1 & & 0 & \\
 & & & & & & & & q
\end{array}\right|
$$

It uses

$$\delta = q - q^{-1}, \quad \mu = \delta(1 - q^{-1}), \quad \lambda = -p^{-1}\delta, \quad q^{1/2} = p.$$

The inverse of X is obtained by reflection in the skew diagonal and replacement of p by p^{-1}. From this, the matrix

$$E_1 = \frac{X - X^{-1}}{\delta} - 1$$

is computed to be:

$$
\left|\begin{array}{ccc|ccc|ccc}
 & & & & & & & & \\
 & & & & & & & & \\
 & & -q^{-1} & & -p^{-1} & & -1 & & \\
\hline
 & & & & & & & & \\
 & & -p^{-1} & & -1 & & -p & & \\
 & & & & & & & & \\
\hline
 & & -1 & & -p & & -q & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{array}\right|
$$

We see that E_1 has rank one. We have

(3.1) $$E_1^2 = -(q + 1 + q^{-1})E_1.$$

As in the previous section, we look for a (3,3)-matrix F such that with $Y = F \otimes 1$ the four braid relation (1.1) holds. We specify F in bottom-right triangular form

$$F = \begin{pmatrix} 0 & 0 & b_2 \\ 0 & a & a_2 \\ b_1 & a_1 & b \end{pmatrix}.$$

The product YX is computed to be the following matrix.

$$
\left|\begin{array}{ccc|ccc|ccc}
\hline
 & & q^{-1}b_2 & & & & & & \\
 & & & & & b_2 & & & \\
 & & & & & & & & b_2q \\
\hline
 & a & a_2q^{-1} & & & & & & \\
 & & a\lambda & & a & a_2 & & & \\
 & & & & & a\delta & & a & a_2q \\
\hline
b_1 & a_1 & bq^{-1} & & & & & & \\
 & b_1\delta & a_1\lambda & b_1 & a_1 & b & & & \\
 & & b_1\mu & & b_1\lambda & a_1\delta & b_1q^{-1} & a_1 & bq \\
\hline
\end{array}\right|
$$

In order to test the braid relation we note: Since X is symmetric, $XYXY$ is obtained from $YXYX$ by transposition and interchange of the indices $\{1,2\}$. We decompose the matrix $YXYX$ into $(3,3)$-blocks. It turns out that each block is a bottom-right triangular matrix. We display the blocks B_{ij}, $1 \leq i,j \leq 3$.

$$
B_{11} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & b_1b_2\mu q \end{pmatrix}
$$

$$
B_{12} = \begin{pmatrix} 0 & & \\ & 0 & a\delta b_2 \\ & b_1b_2q\lambda & a_1b_2q\delta \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & & \\ & 0 & a^2\lambda \\ & ab_1\delta & a_2qb_1\mu + aa_1\lambda \end{pmatrix}
$$

$$
B_{22} = \begin{pmatrix} 0 & & ab_2 \\ & 0 & aa_2 + aa_2\delta \\ ab_1 & a_2b_1q\lambda + aa_1 & a_1a_2q\delta + ab + a^2\delta^2 \end{pmatrix}
$$

$$
B_{13} = \begin{pmatrix} 0 & & b_2^2 \\ & ab_2 & a_2b_2q \\ b_1b_2 & a_1b_1q & b_2bq^2 \end{pmatrix}
$$

$$
B_{31} = \begin{pmatrix} 0 & & b_1b_2 \\ & b_1a & b_1a_2q^{-1} + a_1a\lambda \\ b_1^2 & b_1a_1q^{-1} + a_1b_1\delta & ab_1\lambda^2 + b_1bq^{-2} + a_1^2\lambda + b_1bq\mu \end{pmatrix}
$$

$$
B_{23} = \begin{pmatrix} 0 & & a_2b_2 \\ & aa_2 & ab_2q\lambda + a_2^2a \\ a_2b_1 & a_1a_2q + a^2\delta & a_2bq^2 + aa_2q\delta \end{pmatrix}
$$

$$
B_{32} = \begin{pmatrix} 0 & & a_1b_2 \\ & aa_1 & b_1b_2\delta + a_1a_2 + ba\delta \\ a_1b_1 & ab_1\lambda + a_1^2 + b_1bq\lambda & a_2b_1\lambda + aa_1\delta^2 + a_1b + a_1bq\delta \end{pmatrix}
$$

$$
B_{33} = \begin{pmatrix} 0 & & bb_2 \\ & ab & a_2bq + a_1b_2q\lambda \\ b_1b & aa_1\delta + a_1bq & b_1b_2q\mu + a_1a_2q\delta + b^2q^2 \end{pmatrix}
$$

In order to test the four braid relation, we note the following simplifications: In most places the corresponding matrix entries coincide right away. The other places appear pairwise: interchange the indices. We display only one member of each such pair, the one arising from B_{ij} with $i \leq j$. Altogether, we obtain 9 conditions:

$$\begin{array}{lll}
(1) & a^2 = b_1 b_2 q & B_{12} \\
(2) & a_1 b_2 q\delta = a_1 b_2 q\mu + a a_2 \lambda & B_{12} \\
(3) & a a_2 (1+\delta) = a a_2 + a_1 b_2 q \lambda & B_{22} \\
(4) & a_1 b_2 q = b_2 a_1 q^{-1} + a_2 a \lambda & B_{13} \\
(5) & b_2 b q^2 = a b_2 \lambda^2 + b_2 b q^{-2} + a_2^2 \lambda + b_2 b q \mu & B_{13} \\
(6) & a_1 a_2 q + a^2 \delta = b_1 b_2 \delta + a_1 a_2 + \delta b a & B_{23} \\
(7) & a b_2 q \lambda + a_2^2 q = a b_2 \lambda + a_2^2 + b_2 b q \lambda & B_{23} \\
(8) & a_2 b q^2 + a a_2 q \delta = b_2 a_1 \lambda + a a_2 \delta^2 + a_2 b + a_2 b q \delta & B_{23} \\
(9) & a_2 q b + a_1 b_2 q \lambda = a a_2 \delta + a_2 b q & B_{33}
\end{array}$$

We discuss the conditions. (1) is a condition on the diagonal elements of F. We can multiply F by a scalar without changing the four braid relation; we thus could normalize to $a = 1$. (Since F is required to be invertible, we have $ab_1b_2 \neq 0$.) Condition (2) can be rewritten as

$$p a_1 b_2 = -a a_2. \tag{3.2}$$

Therefore a_1, a_2 are both zero or non-zero. Inspection shows that (3), (4), (8), and (9) lead to the same condition (3.2). Equation (5) can be rewritten as

$$a_2^2 = b_2 (p + p^{-1})(a p^{-1}(p - p^{-1}) - b). \tag{3.3}$$

Again (6) and (7) lead to (3.3). It is remarkable that from the potentially 81 conditions only 2 remain.

We normalize $a = 1$. It is seen that, without essentially restricting the generality, we can assume F to be symmetric. Then $b_1 = b_2 = p^{-1}$. The characteristic polynomial of F is

$$(t - q^{-1})(t^2 + (1 + q^{-1} - b)t + 1). \tag{3.4}$$

We formally split the quadratic factor $(t + \rho)(t + \rho^{-1})$. Then

$$\rho + \rho^{-1} - 2 = q^{-1} - 1 - b.$$

In the case $\rho = 1$ we have $a_1 a_2 = 0$ and F satisfies a quadratic equation. If we write $\rho = \gamma^2$ and introduce $\alpha = (p + p^{-1})^{1/2}(\gamma - \gamma^{-1})$, then F obtaines the form

$$F = \begin{pmatrix} 0 & 0 & p^{-1} \\ 0 & -1 & p^{-1/2}\alpha \\ p^{-1} & p^{-1/2}\alpha & 1 + q^{-1} - (\rho + \rho^{-1}) \end{pmatrix}. \tag{3.5}$$

The inverse of this matrix F is obtained by reflection in the skew diagonal and application of the involution $p \mapsto p^{-1}$, $\gamma \mapsto -\gamma^{-1}$. The matrix

$$\widetilde{E} = F + F^{-1} + (\rho + \rho^{-1})I$$

is computed to be

$$\textbf{(3.6)} \qquad \widetilde{E} = \begin{pmatrix} 1+p^2 & p^{1/2}\alpha & p+p^{-1} \\ p^{1/2}\alpha & (\gamma-\gamma^{-1})^2 & p^{-1/2}\alpha \\ p+p^{-1} & p^{-1/2}\alpha & 1+p^{-2} \end{pmatrix}.$$

It satisfies the relation

$$\textbf{(3.7)} \qquad \widetilde{E}F = F\widetilde{E} = q^{-1}\widetilde{E}.$$

We set $E = (\rho + \rho^{-1})\widetilde{E}$ and compute

$$\textbf{(3.8)} \qquad E^2 = \left(\frac{q+q^{-1}}{\rho+\rho^{-1}} + 1 \right) E.$$

(Since $\widetilde{E}$ has rank one, the coefficient ν in the equality $\widetilde{E}^2 = \nu\widetilde{E}$ is computed to be the scalar product of the first row with itself.) Later we also need to know the following identity. It is used for the representation theory of tangle categories.

$$\textbf{(3.9)} \qquad E_1 YXY = E_1.$$

Since E_1 has rank one, there must hold identities of the form $E_1 Y^{\pm 1} E_1 = \lambda_\pm E_1$. The coefficients are determined by applying both sides to a suitable vector, e. g. $(0,0,1,0,0,0,0,0,0)$. The result is as follows:

$$\textbf{(3.10)} \qquad \lambda_\pm = q^{\pm 1}(\rho + \rho^{-1} - 1).$$

From this, one obtaines finally

$$\textbf{(3.11)} \qquad E_1 E E_1 = (q + 1 + q^{-1})E_1.$$

(3.12) Remark. The equation (3.9) also determines the matrix Y. Since E_1 has many zeros, this simplifies the computation. Also, since E_1 has rank one, it suffices to determine a single row. Since (3.9) is inhomogeneous in Y, the normalization of Y is determined up to sign, once the other data are fixed. ♡

For tangle theory it is important to decompose E

$$E\colon V \xrightarrow{\varphi} \mathfrak{K} \xrightarrow{\kappa} V, \qquad V = \mathfrak{K}^3.$$

We aim at a symmetric decomposition and set (using the standard basis e_i)

$$\textbf{(3.13)} \qquad \begin{aligned} &\varphi\colon V \to \mathfrak{K}, \quad (e_1,e_2,e_3) \mapsto (p^{1/2}\sqrt{p+p^{-1}}, \gamma-\gamma^{-1}, p^{-1/2}\sqrt{p+p^{-1}}) \\ &\kappa\colon \mathfrak{K} \to V, \quad 1 \mapsto p^{1/2}\sqrt{p+p^{-1}}e_1 + (\gamma-\gamma^{-1})e_2 + p^{-1/2}\sqrt{p+p^{-1}}e_3. \end{aligned}$$

These maps satisfy$E = \kappa\varphi$. Moreover, we have

(3.14) $$F\kappa(1) = q^{-1}\kappa(1).$$

4. R-matrices of type B_n and C_n

In this section we establish a four braid relation for R-matrices of Coxeter-Dynkin type B_n and C_n. We begin by specifying the relevant R-matrices. See [28] and [29] for the use of these matrices in knot and tangle theory.

The matrix $X_n = X(B_n)$ describes an automorphism of $V \otimes V$ in the lexicographical basis v_{ij}, $1 \le i, j \le 2n+1 = m$. If $i + j \ne m+1$, then X_n coincides with a matrix of A-type, as specified in (2.3). The subspace of $V \otimes V$ generated by v_{ij}, $i + j = m+1$ is invariant under X_n. The corresponding matrix block will be denoted by Z_n. We describe Z_n inductively. Again we use $\delta = q - q^{-1}$ and $p = q^{1/2}$. We let Z_0 denote the unit matrix of size 1. The matrix Z_n is a symmetric matrix with central matrix Z_{n-1}, i. e. we adjoin to Z_{n-1} new rows and columns in the positions 1 and $2n+1$. The $(2n+1)$-row is $(q^{-1}, 0, \dots, 0)$. The first row is

$$-\delta(q^{-(2n-1)} - 1, q^{-(2n-3)}, \dots, q^{-n}, p^{-(2n-1)}, q^{-n+1}, \dots, q^{-1}, 1) + (0, \dots, 0, q).$$

We need information about the eigenspaces of Z_n. We set $e_j = v_{j,m+1-j}$. (For (4.1) and (4.2), we assume that $(q+q^{-1})(q - q^{-m+1})(q^{-1} + q^{-m+1})$ is invertible in $\mathfrak{K}$. There analogous assumptions in (4.4) and (4.5). See section 6 for these conditions.)

(4.1) Proposition. *The matrix Z_n has eigenvalues $q, -q^{-1}, q^{-m+1}$.*

(1) *The q-eigenspace has the basis*

$$\begin{aligned} z_j &= qe_j + q^{-1}e_{m+1-j} - e_{j+1} - e_{m+1-(j+1)}, \qquad 1 \le j \le n-1 \\ z_n &= qe_{n-1} - (p + p^{-1})e_n + q^{-1}e_{n+1}. \end{aligned}$$

(2) *The $(-q^{-1})$-eigenspace has the basis*

$$\begin{aligned} y_j &= (e_j - e_{m+1-j}) - (q^{-1}e_{j+1} - qe_{m+1-(j+1)}) \qquad 1 \le j \le n-1 \\ y_n &= e_{n-1} + (p - p^{-1})e_n - e_{n+1}. \end{aligned}$$

(3) *An eigenvector for q^{-m+1} is*

$$(1, q, \dots, q^{n-1}, p^{2n-1}, q^n, \dots, q^{2n-1}).$$

Proof. We prove (1) by induction on n. The case $n = 1$ is a simple verification. For the induction step it remains to check:

(1) z_1 is an eigenvector of Z_n.
(2) The scalar product of the first row of Z_n with $z_2, \dots, z_n$ is zero.

For (1), we compute the scalar product of z_1 with the first row to be

$$\delta(1-q^{-2n+1})q+\delta q^{-2n+2}+\delta q^{-1}+q^{-1}q^{-1}=q^2,$$

and with the second row to be

$$-\delta q^{-2n+2}q-\delta(1-q^{-2n+3})-q^{-1}=q.$$

These values are correct. The scalar product with rows 3 to m gives trivially the correct result. For (2), we compute the scalar product with z_2 to be

$$-\delta q^{-2n+2}q+\delta q^{-2n+3}+\delta p^{-2}-\delta q^{-2}=0$$

and similarly for $z_3,\ldots,z_{n-1}$. For z_n we have

$$-\delta q^{-n}q+\delta(p+p^{-1})p^{-2n+1}-\delta q^{-n+1}q^{-1}=0.$$

The verification for the other eigenspaces is similar. □

Let now F_n denote a $(2n+1,2n+1)$-matrix as in (2.5) with $w=p^{-1}-p$, $a=-p$, $\beta_j=1$ and set $Y_n=F_n\otimes 1$.

(4.2) Theorem. *The matrices $X(B_n)$ and Y_n satisfy the four braid relation.*

Proof. As before, we decompose $X(B_n)$ and Y_n into suitable blocks. The subspace W of $V\otimes V$ generated by v_{ii}, $v_{i,m+1-i}$, $1\le i\le m$ is invariant under X and Y. The remaining basis elements generate a subspace where the four braid relation is satisfied by the results of section 2. We order the basis of W as follows:

$$v_{1,m},v_{2,m-1},\ldots,v_{m,1},v_{11},\ldots,v_{mm}.$$

We assume that $v_{n,n}$ occurs among the $v_{j,m+1-j}$. In that case, we are in the formal situation of (1.6) with $Z=Z_n$ and $A=\mathrm{Dia}(0,\ldots,0,-p,w,\ldots,w)$, $D=\mathrm{Dia}(0,\ldots,0,w,\ldots,w)$ with w appearing n times in A and D. Moreover $B^t=C$ and

$$B=\left(\begin{array}{c|c} 0 & J \\ \hline 0 & 0 \\ \hline J & 0 \end{array}\right),\qquad J=\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix},$$

i. e. J is the co-unit matrix. In order to check the conditions of (1.7), we use:

(4.3) Proposition. *The column vectors of $AZB+qBD$ generate the q-eigenspace of Z.*

Let us assume this for the moment. Then (1.7, II) holds. Since Z,A,D are symmetric and $B^t=C$, the condition (1.7, III) follows by transposition. The matrix BC is the unit matrix, but with the central diagonal element replaced by zero. In order to verify (1.7, I), one notes the following facts which are simple consequences of the general structure of the matrix blocks:

$qBCZ$ is qZ with the n-th column replaced by zero;

$qZBC$ is qZ with the n-th row replaced by zero;
$AZAZ$ is the n-th row of Z, multiplied by q, and the rest replaced by zero;
$ZAZA$ is the n-th column of Z, multiplied by q, and the rest replaced by zero.
(1.7, I) is a direct consequence of these facts. □

Proof of (4.3). By inspection, one sees that the non-zero columns $S_1, \ldots, S_n$ of $AZB + qBD$ are given in terms of the eigenvectors (4.1) as follows: $wz_j = S_j - qS_{j+1}$ for $1 \le j \le n-1$ and $wz_n = S_n$. □

We now consider the R-matrices $X(C_n) = X'_n$ which act on $V \otimes V$ with $\dim V = 2n = m$. Again, the v_{ij} for $i+j \ne 2n+1$ are mapped as for $X(B_n)$. The subspace of $V \otimes V$ generated by the v_{ij} with $i+j = 2n+1$ is invariant under X'_n and the corresponding matrix block Z'_n is defined inductively, beginning with

$$Z'_1 = \begin{pmatrix} \delta(1+q^{-2}) & q^{-1} \\ q^{-1} & 0 \end{pmatrix}.$$

The matrix Z'_n is a symmetric matrix with central matrix Z'_{n-1}. Its $2n$-th row is $(q^{-1}, 0, \ldots, 0)$. The first row is

$$\delta(1+q^{-2n}, q^{-2n+1}, \ldots, q^{-n-1}, -q^{-n+1}, \ldots, -q^{-1}, -1) + (0, \ldots, 0, q).$$

Again we need the eigenspace structure of Z'_n.

(4.4) Proposition. *Z'_n has eigenvalues q, $-q^{-1}$, $-q^{-m-1}$.*

(1) *The q-eigenspace has the basis*

$$\begin{aligned} z'_j &= qe_j + q^{-1}e_{m+1-j} - e_{j+1} - e_{m+1-(j+1)} \qquad 1 \le j \le n-1 \\ z'_n &= qe_n + q^{-1}e_{n+1}. \end{aligned}$$

(2) *The $(-q^{-1})$-eigenspace has the basis*

$$y'_j = (e_j - e_{m+1-j}) - (q^{-1}e_{j+1} - qe_{m+1-(j+1)}), \qquad 1 \le j \le n-1.$$

(3) *An eigenvector for the eigenvalue $-q^{-m-1}$ is*

$$(1, q, \ldots, q^{n-1}, -q^{n+1}, \ldots, -q^{2n}).$$

Proof. The proof is by induction on n as for (4.1). □

Let now F'_n be a $(2n, 2n)$-matrix as in (2.5) with $\beta_j = \beta$ and set $Y'_n = F'_n \otimes 1$.

(4.5) Theorem. *The matrices $X(C_n)$ and Y'_n satisfy the four braid relation.*

Proof. We use the same method as for Theorem (4.2). We have $A = D = \mathrm{Dia}(0, \ldots, 0, w, \ldots, w)$ with w appearing n times and $B = C = \beta J$. It is easy to see that the non-zero columns $S'_1, \ldots, S'_n$ of $AZ'B + qBD$ have the form $S'_j - q^{-1}S'_{j+1} = \beta w z'_j$ for $1 \le j \le n-1$ and $S'_n = \beta w z'_n$. Therefore (1.7, II and

III) hold. The matrix BC is a multiple of the identity, hence commutes with Z'. The matrix $Z'AZ'A$ is zero. Therefore (1.7, I) is satisfied too. $\square$

The inverse of the matrices $X(B_n)$ and $X(C_n)$ is obtained by reflection in the skew diagonal and replacement of p by p^{-1}. The matrix $X = X(B_n)$ satisfies

$$0 = (X-q)(X+q^{-1})(X-q^{-m+1}), \qquad m = 2n+1$$

and the matrix $X = X(C_n)$

$$0 = (X-q)(X+q^{-1})(X+q^{-2n-1}).$$

The matrices E, defined by $(q-q^{-1})(I+E) = X - X^{-1}$ are in both cases of rank one. Their entries are nonzero only in places (ij, kl) with $i+j = k+l = m+1$. This essential part of E is symmetric and therefore determined by its first row. This row is in the case B_n

$$-(q^{-2n+1}, q^{-2n+2}, \dots, q^{-n}, p^{-2n+1}, q^{-n+1}, \dots, 1)$$

and in the case C_n

$$(q^{-2n}, \dots, q^{-n-1}, -q^{-n+1}, \dots, -1).$$

The matrices E satisfy the following equations

$$\textbf{(4.6)} \qquad \begin{aligned} &X(B_n)E(B_n) = E(B_n)X(B_n) = q^{-2n}E(B_n) \\ &X(C_n)E(C_n) = E(C_n)X(C_n) = -q^{-2n-1}E(C_n) \\ &E(B_n)^2 = ([2n]_q + 1)E(B_n), \quad E(C_n)^2 = ([2n+1]_q - 1)E(C_n). \end{aligned}$$

Here we have used the quantum numbers $[h]_q = (q^h - q^{-h})/(q - q^{-1})$. For tangle theory and Birman-Wenzl algebras we need relations between E- and Y-matrices. It turns out that other normalizations of the Y-matrices have better properties. From now on we use the previously defined matrices multiplied by p^{-1}. In this case, F^{-1} is obtained from F by reflection in the skew diagonal and replacement of p by p^{-1}. It satisfies $(F - q^{-1})(F + 1) = 0$.

(4.7) Theorem. *The matrices Y just defined satisfy in the cases B_n and C_n the identity $YXYE = E$.*

Proof. As in previous proofs we decompose $V \otimes V$ into invariant subspaces. On the D_8-orbits the matrices E are zero. For the remaining part we use the notations of (4.2) and (4.5). The identity in question is then equivalent to the two equations

$$(AZA + qBC)E = E, \qquad (AZB + qBD)E = 0.$$

Inspection shows that with our new normalization of Y the matrix $AZA + qBC$ is actually the identity, thus the first equation holds. (This seems also a better explanation for (1.7, I).) The eigenvalue relation (1.7, II) yields by the very definition of E the second equation. □

(4.8) Remark. The decomposition of $V \otimes V$ into eigenspaces is the decomposition into irreducible representations of the relevant quantum group. Except for the one-dimensional trivial representation the second symmetric power S^2V and second alternating power $\Lambda^2 V$ are relevant. ♡

(4.9) Remark. One can also establish a four braid relation for automorphisms of the form $1 \otimes F$. This is obtained from the previously considered form by conjugation with the interchange automorphism $\tau\colon V \otimes V \to V \otimes V, v \otimes w \mapsto w \otimes v$. This is due to the fact that $\tau \circ X(q) \circ \tau = X(q^{-1})^{-1}$. ♡

(4.10) Remark. A tedious computation shows that for $X(B_2)$ and a bottom-right triangular matrix F_2 the four braid relation holds if and only if the matrix is the one used in (4.2). In this sense, the case $X(B_1)$ seems to be exceptional. I am grateful to R. Häring [15] for an independent verification of this computation.

5. The example $H^\infty B_2$

As an example for later investigations we communicate some computations for the generalized Hecke algebra $H^\infty B_2$. This is the associative algebra with 1 over $\mathfrak{K}$ with generators t, g and relations $tgtg = gtgt$ and $g^2 = (v-1)g + v$ with $v \in \mathfrak{K}^*$. The following commutation rule is basic for many computations.

(5.1) Theorem. *For $k \in \mathbb{Z}$ and $l \in \mathbb{N}_0$ the following commutation rule holds:*

$$gt^k gt^l - t^l gt^k g = (1-v) \sum_{j=1}^{l} (t^j gt^{k+l-j} - t^{k+l-j} gt^j).$$

We can write this rule more symmetrically as a commutator rule

$$[gt^k g + (v-1) \sum_{j=1}^{l} t^j gt^{k-j}, t^l] = 0.$$

For $0 \le k \le l$ the same relation holds for the sum from 0 to k; thus we have an element that commutes with t^l. By multiplication from right and left with t^{-l} one obtains similar results for $l < 0$.

Proof. By induction over k and l for $k, l \in \mathbb{N}$. For details see [1]. □

The commutation rules are used to derive a basis for the algebra (compare section 8).

(5.2) Theorem. *The algebra $H^\infty B_2$ has the following bases: Either $t^l gt^k$, $t^l gt^k g^{-1}$ for $k, l \in \mathbb{Z}$ or $t^l gt^k$, $t^l gt^k g$ for $k, l \in \mathbb{Z}$.* □

We use these results to determine all traces on the algebra.

(5.3) Theorem. *The traces T on $H^\infty B_2$ with $T(1) = 1$ correspond bijectively to pairs of families $(\tau(n) \mid n \in \mathbb{Z})$ and $(\sigma(k,l) = \sigma(l,k) \mid (k,l) \in \mathbb{Z}^2)$.*

PROOF. By (5.2), we can define a linear form T on the algebra by

$$T(1) = 1, \quad T(t^k g t^l) = \tau(k+l), \quad T(t^k g t^l g) = \sigma(k,l).$$

We have to show that any such linear form satisfies $T(xy) = T(yx)$ for all basis elements x and y. We make a series of deductions.

(1) We have that $T(t^{k+l}g) = T(t^k g t^l) = \tau(k+l)$ depends only on $k+l$.

(2) We must have $T(t^k g t^l) = T(t^l g t^k)$ for any trace T. The commutation rule (=CR) (7.1) shows that $\sigma(a,b)$ has to be symmetric in a, b.

(3) $T(t^u g t^k g t^l) = T(t^{u+l} g t^k g)$.

PROOF. We apply the CR to $g t^k g t^l$ and obtain

$$t^u g t^k g t^l = t^u t^l g t^k g + (1-v) \sum_j (t^{u+j} g t^{k+l-j} - t^{u+k+l-j} g t^j).$$

Because of (1) the sum terms cancel.

(4) $T((t^u g t^v)(t^k g t^l)) = T((t^k g t^l)(t^u g t^v))$.

PROOF. By (3), the left, right side equals $T(t^{u+l} g t^{v+k} g), T(t^{v+k} g t^{l+u} g)$, respectively, and both are equal to $\sigma(l+u, v+k)$ by (2).

(5) $T(g t^a g t^b g) = (v-1)\sigma(a,b) - v\tau(a+b)$.

PROOF. We apply the CR to $(g t^a g t^b) g$ and obtain

$$g t^a g t^b g = t^b g t^a g^2 + (1-v) \sum_{j=1}^{b} (t^j g t^{a+b-j} g - t^{a+b-j} g t^j g).$$

The sum terms cancel, by (2). Now use the quadratic relation for g^2.

(6) $T(t^m g t^k g t^l g) = T(t^l g t^m g t^k g)$.

PROOF. We apply the CR on both sides to $t^m g t^k g$. The resulting sum terms cancel by (3). Then one uses (5) to show

$$T(g t^k g t^{m+l} g) = T(t^l g^2 t^k g t^m).$$

(7) The trace of $t^a g t^b g t^c g$ is invariant under permutations of a, b, c.

PROOF. The CR is applied in $t^l g t^m g t^k g$ to $t^m g t^k g$ in order to show that the trace is invariant under permutation of k, l. The CR is applied to $t^l g t^m g$ in oder to show that the trace is invariant under permutation of m, k.

(8) $T(t^u g t^v g t^k g t^l) = T(t^k g t^l g t^u g t^v g)$.

Write gt^vgt^kg as a linear combination of the basis and use (3) in order to show that the left hand side of the claim has the trace $T(t^{u+l}gt^vt^kg)$. Then use (7).

(9) $T(t^agt^bgt^cgt^dg) = T(t^dgt^agt^bgt^cg)$.

PROOF. Write gt^bgt^cg in terms of the basis and use the invariance properties already known. □

The left hand side of (9) is invariant under all permutations of a, b, c, d. Does a similar invaraince hold for longer products of this form?

The algebra $H^\infty = H^\infty B_2$ is isomorphic to the group algebra of the corresponding infinite Weyl group. The group algebra W^∞ of this Weyl group has generators T, G and relations $G^2 = 1$ and $GTGT = TGTG$.

(5.4) Theorem. *Suppose $v + 1 \in \mathfrak{K}^*$. The following assigments define inverse isomorphisms $H^\infty \cong W^\infty$:*

$$G \mapsto \frac{1}{v+1}(2g + (1-v)), \quad T \mapsto \frac{1}{v+1}(1vt + (v-1)gt),$$

$$g \mapsto \frac{1}{2}((v+1)G + (v-1)), \quad t \mapsto ((1-v)G + (1-v))T.$$

PROOF. One has to show that the assignments yield well defined maps. The elements

$$e_1 = \frac{g+1}{v+1}, \quad e_2 = \frac{v-q}{v+1}$$

are orthogonal idempotens with sum 1 in H^∞. Therefore

$$e_1 - e_2 = \frac{1}{v+1}(2g + (1-v))$$

satisfies $T^2 = 1$. The compatibility relations are verified more easily from other presentations of the algebras involved. We state these presentations but do not translate the results to the present situation. □

The algebra H^∞ has the presentation (see section 8) H'^∞ with generators f, g, c, c^{-1} and relations $g^2 = (v-1)g + v$, $f^2 = (v-1)f + v$, $cgc^{-1} = f$, $cfc^{-1} = g$, $cc^{-1} = c^{-1}c = 1$. The correspondence is given by the following table:

H^∞	H'^∞
g	g
t	$g^{-1}c$
gt	c
$gtgt^{-1}g^{-1}$	f

The algebra W'^∞ has a similar presentation with generators G, F, C, C^{-1} and relations above for $v = 1$.

(5.5) Theorem. *The following assignments define inverse isomorphisms $H'^{\infty} \cong W'^{\infty}$:*

$$G \mapsto \frac{1}{v+1}(2g+1-v), \quad F \mapsto \frac{1}{v+1}(2f+1-v), \quad C \mapsto c$$

$$g \mapsto \frac{1}{2}((v+1)G+v-1), \quad f \mapsto \frac{1}{2}((v+1)F+v-1), \quad c \mapsto C.$$

□

By combining the isomorphisms, one sees that the elements

$$\frac{1}{v+1}(2vt+(v-1)gt), \quad \frac{1}{v+1}(2vgtg^{-1}+(v-1)gt)$$

commute in H^{∞}; these elements correspond to CG and GC, respectively.

Suppose $v = p^2$ and set $d = p + p^{-1}$. Let $T^{\infty}B_2$ denote the algebra over $\mathfrak{K}[D]$ with generators e, τ, τ^{-1} and relations $\tau\tau^{-1} = \tau^{-1}\tau = 1$, $e^2 = de$, $\tau^2 e = e = e\tau^2$, and $e\tau^{\pm 1}e = De$. This is a kind of Temperley-Lieb algebra. We use the presentation of $H^{\infty}B_2$ with generators g and c, c^{-1}. The assignment $\varphi(g) = pe-1$ and $\varphi(c) = \tau$ yields a homomorphism φ: $H^{\infty}B_2 \to T^{\infty}B_2$. We also have the Temperley-Lieb algebra $T\tilde{A}$ with generators e, f and relations $e^2 = de$, $f^2 = df$, $efe = D^2 e$, $fef = D^2 f$. The assignment $e \mapsto e$, $f \mapsto \tau e \tau^{-1}$ yields an embedding ψ: $T\tilde{A} \to T^{\infty}B_2$. The algebra $T\tilde{A}$ has the automorphism τ which interchanges e and f. In analogy to (8.23) we have an isomorphism

$$T\tilde{A} \otimes \mathfrak{K}[\tau, \tau^{-1}] \cong T^{\infty}B_2.$$

(The tensor product is twisted by τ.) The homomorphism φ above is related to the Kauffman calculus of B_2-braids. The algebra $T^{\infty}B_2$ has a geometrically defined trace which comes from an interpretation by $\mathbb{Z}$-equivariant bridges (section 10). The parameter D counts snakes, in the sense of section 10.

6. Some formulas

Let R be an integral domain. for use in the next section, we collect some formulas for the algebra $A = R[X]/I$, where I is the ideal generated by

$$p = (X-a_1)(X-a_2)\cdots(X-a_n), \qquad a_j \in R.$$

We consider the generic case when the a_j are pairwise different. We assume that the a_j are units in R. Then we have:

(6.1) Proposition. *The element $X \in A$ is invertible.* □

We set

$$E'_j = \prod_{i, i\neq j}(X - a_i).$$

From $p = 0$ we see that the E'_j are pairwise orthogonal

(6.2)
$$E'_j E'_k = 0 \qquad j \neq k$$

and satisfy

(6.3)
$$XE'_j = E'_j X = a_j E'_j.$$

The E'-elements are almost idempotent

(6.4)
$$E'^2_j = \alpha_j E'_j,$$

with

(6.5)
$$\alpha_j = \prod_{i, i \neq j} (a_j - a_i).$$

If we assume that

(6.6)
$$\Delta = \prod_{i<j} (a_i - a_j)$$

is invertible in R, then the elements

(6.7)
$$E_j = \alpha_j^{-1} E'_j$$

are pairwise orthogonal idempotents. The sum $\sum_i E_i$ is formally a polynomial of degree $n-1$ in $R[X]$ which assumes the value 1 at the a_j and is therefore the constant 1. Hence:

(6.8) Proposition. *If $\Delta \in R^*$, then the $E_1, \ldots, E_n$ are an R-basis of A and a decomposition of* 1 *into orthogonal idempotents.* □

The R-module A is always free of rank n with basis $1, X, \ldots, X^{n-1}$. The E'_j have with respect to this basis, up to sign, the determinant Δ.

We have

(6.9)
$$\prod (X^{-1} - a_j^{-1}) = \prod (a_j X)^{-1} \prod (a_j - X).$$

If we define A over $R = \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$, then the assignment $a_j \mapsto a_j^{-1}$, $X \mapsto X^{-1}$ defines an involution ι on A. The elements $\iota(E_j)$ are therefore also an orthogonal decomposition of 1 (in $A[\Delta^{-1}]$). From the defining equation $\iota(E'_j) = \prod_{i, i \neq j} (X^{-1} - a_j^{-1})$ we see that $\iota(E_j) E_i = 0$ for $i \neq j$. Therefore we have:

(6.10)
$$\iota(E_j) = E_j, \quad \iota(E'_j) = (-1)^{n-1} \prod_{i, i \neq j} (a_j - a_j^{-1}) E'_j.$$

What we have done so far just makes explicit the decomposition of A provided by the Chinese remainder theorem.

7. Birman-Wenzl algebras

We recall the definition of Birman-Wenzl algebras for simply laced Coxeter graphs, see [2] and [33].

Given a Coxeter graph (S, m) with values $m(s,t) \in \{1,2,3\}$. We choose parameters $q, x, \lambda \in \mathfrak{K}$ and assume $\mathfrak{K}^*$. The Birman-Wenzl algebra

$$BW(S,m) = BW(S,m;q,x,\lambda)$$

is the associative algebra with 1 with generators $(G_s, G_s^{-1}, E_s \mid s \in S)$ and the following relations ($m = m(s,t)$ always):

$$\begin{array}{rrcll}
(1) & G_sG_s^{-1} & = & G_s^{-1}G_s \,=\, 1 & \\
(2) & G_sG_t & = & G_tG_s, & m=2 \\
(3) & G_sG_tG_s & = & G_tG_sG_t, & m=3 \\
(4) & G_s - G_s^{-1} & = & (q-q^{-1})(E_s+1) & \\
(5) & E_sE_t & = & E_tE_s, & m=2 \\
(6) & E_sE_tE_s & = & E_tE_sE_t, & m=3 \\
(7) & E_s^2 & = & (x-1)E_s & \\
(8) & E_sG_t^{\pm 1}E_s & = & \lambda^{\pm 1}E_s, & m=3 \\
(9) & G_sE_s & = & E_sG_s \,=\, \lambda^{-1}E_s. & \\
(10) & G_sG_tE_s & = & E_tG_sG_t, & m=3 \\
(11) & G_sG_tE_s & = & E_tE_s, & m=3 \\
(12) & E_tG_sG_t & = & E_tE_s, & m=3 \\
(13) & G_sE_tG_s & = & G_t^{-1}E_sG_t^{-1}, & m=3 \\
(14) & G_sE_tE_s & = & G_t^{-1}E_s, & m=3 \\
(15) & E_sE_tG_s & = & E_sG_t^{-1}, & m=3
\end{array}$$

For $q \neq 1$, the parameter x is determined by the identity $(q-q^{-1})x = \lambda - \lambda^{-1}$. The original Birman-Wenzl algebra [2] belongs to the graph A_{n-1}. There are corresponding generalized Brauer centralizer algebras; they are the special cases $q = 1$, $\lambda = 1$. The Brauer centralizer algebras for the graph B_n have a different definition. They are studied in [27]. If $q - q^{-1} \in \mathfrak{K}^*$, then E_s can be computed from X_s; hence we can eliminate these generators. This yields:

(7.1) Proposition. *Suppose $q - q^{-1} \in \mathfrak{K}^*$. Then the relations* (10)-(15) *are consequences of the remaining ones. Another consequence of the relations is $(X_s - \lambda^{-1})(X_s + q^{-1})(X_s - q) = 0$.* □

This proposition can be used to redefine the algebra $BW(S,m)$, in the case that $q - q^{-1} \in \mathfrak{K}^*$, by generators $(X_s \mid s \in S)$ and relations (2), (3), (8) and the cubic relation of the last proposition. Going back and forth uses section 6.

The algebra $BW(S,m)$ has the Hecke algebra $H(S,m)$ as a quotient: Set $E_s = 0$. The $(E_s \mid s \in S)$ generate a subalgebra $T(S,m)$, a Temperley-Lieb algebra [9].

We now define the restricted BW-algebras of type B and the corresponding Brauer centralizer algebras (symmetric without free points in the terminology of

Reich [27]). The algebra $BW(B_n)$ has generators $G_0, G_1, \ldots, G_{n-1}, E_1, \ldots, E_{n-1}$. The relations between the G_j, E_j for $1 \le j \le n-1$ are as above for the graph A_{n-1}. The additional relations are:

$$\begin{array}{lrcll}
(16) & G_0G_1G_0G_1 & = & G_1G_0G_1G_0 & \\
(17) & G_0G_i & = & G_iG_0, & i > 1 \\
(18) & G_0E_i & = & E_iG_0, & i > 1 \\
(19) & (G_0 - q^{-1})(G_0 + 1) & = & 0 & \\
(20) & G_0G_1G_0E_1 & = & E_1. &
\end{array}$$

Again, there is a version which does not use the E_j.

(7.2) Remark. *The algebra $BW(B_n)$ has dimension $2^n \cdot (2n-1)!!$.*

Proof. This is easily shown for the Brauer algebras, compare [27]. □

(7.3) Theorem. *The irreducible representations of $BW(B_n)$ are indexed by pairs of Young diagrams of size $n - 2k$. The Bratteli diagram is given by adding or omitting a box in the Young diagrams.*

The proof of the last theorem will be given in a subsequent paper of the series. For a further study of these algebras see also [16].

Finally, we define unrestricted BW-algebras $BW^*(B_n)$ of type B_n. They are infinite dimensional algebras. In addition to the generators for $BW(B_n)$, there is another generator E_0. The new additional relations are as follows: (23) replaces (19) above; and

$$\begin{array}{lrcll}
(21) & E_0G_j & = & G_jE_0, & j > 1 \\
(22) & E_0^2 & = & DE_0 & \\
(23) & G_0 - G_0^{-1} & = & (Q - Q^{-1})(E_0 + 1) & \\
(24) & E_1E_0E_1 & = & FE_1. &
\end{array}$$

Here D, F, and Q are new parameters. Relation (23) is analogous to (4). Occasionally, it is more convenient to use the relation $G_0 + G_0^{-1} = (Q + Q^{-1})(E_0 - 1)$. This latter relation came up in section 3, with E, F in place of E_0, G_0.

The algebra $BW(B_n)$ has the Hecke algebra $H(B_n)$, with suitable parameters, as a quotient: Again set $E_s = 0$. Similarly for the algebra $BW^*(B_n)$. The algebra $BW^*(B_n)$ contains a Temperley-Lieb algebra of type B_n: The subalgebra generated by the $E_0, \ldots, E_{n-1}$ (compare [7, 9]).

The R-matrix identities of section 4 are used to construct tensor representations of these algebras and quantum traces; and the latter give Kauffman polynomials [23] of type B. The representation of $BW(B_n)$ is on $V^{\otimes n}$. Here we use the presentation by generators $G_0, \ldots, G_{n-1}$ alone. The definition is as in (1.5). It is only necessary to check the additional cubic relation and (20). The definition of quantum traces is modelled after [29], [28]. We defer the discussion to another paper, but see [8].

(7.4) Remark. There are similar definitions for BW-algebras associated to the affine root systems $\tilde{B}_n$ and $\tilde{C}_n$. Remark (4.9) allows the construction of tensor representations for these extended algebras. ♡

We describe in detail the example $BW(B_2)$ in order to motivate the definition. Consider the algebra with generators X, X^{-1}, Y, E and relations

(1) $XX^{-1} = X^{-1}X = 1$
(2) $XYXY = YXYX$
(3) $XE = EX = \lambda E$
(4) $(Y - q^{-1})(Y + 1) = 0$
(5) $X - X^{-1} = (q - q^{-1})(E + 1)$
(6) $E^2 = dE$
(7) $YXYE = E$

Here λ and q are invertible parameters in the ground ring $\mathfrak{K}$ and $d \in \mathfrak{K}$. They have to satisfy $(d + 1)(q - q^{-1}) = \lambda - \lambda^{-1}$, as follows from (3), (5), and (6).

(7.5) Proposition. *The following assertions are implied by the relations above:*

(1) $(X - \lambda)(X - q)(X + q^{-1}) = 0$.
(2) $XYXY$ *is contained in the center.*
(3) $EYXY = E$.
(4) $EYE = \frac{q+\lambda^{-1}}{q+1}E =: aE$.

Proof. The first assertion is a consequence of (3) and (5). The second follows from (2) and (5). For the third, we compute

$$\lambda YXY = EXYXY = XYXYE = YXYXE = \lambda YXYE = \lambda E.$$

For the final identity we compute

$$\begin{aligned} EYE &= (\delta^{-1}(X - X^{-1}) - 1)YE \\ &= \delta^{-1}(XYE - X^{-1}YE) - YE \\ &= \delta^{-1}(Y^{-1}E - X^{-1}(q^{-1}Y^{-1} + (q^{-1} - 1))E) - YE \\ &= \delta^{-1}(Y^{-1}E - q^{-1}YE - (q^{-1})\lambda^{-1}E) - YE \\ &= \delta^{-1}((qY + (q - 1)E - q^{-1}YE - (q^{-1} - 1)\lambda^{-1}E) - YE \\ &= \delta^{-1}((q - 1) - (q^{-1} - 1)\lambda^{-1})E \\ &= \frac{q + \lambda^{-1}}{q + 1}E. \end{aligned}$$

We have used (4), (5), and (7) in the computation. □

The algebra is generated by

$$1, X, Y, XY, YX, YXY, XYX, XYXY, E, EY, YE, YEY.$$

It is easily checked from the relations that left and right multiplication by X, Y, E always leads to linear combinations of the displayed elements. If we take the quotient by the relation $E = 0$, then we obtain a Hecke algebra of type B_2.

The kernel of the quotient to the Hecke algebra is the twosided ideal spanned by E, YE, EY, YEY. It decomposes into the left ideals spanned by E, EY and YE, YEY. The matrices of X, Y, and E in the basis Y, EY are

$$X = \begin{pmatrix} \lambda & q-1 \\ 0 & q \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}-1 \end{pmatrix}, \quad E = \begin{pmatrix} d & a \\ 0 & 0 \end{pmatrix}.$$

One can verify directly that these matrices satisfy the relations of the algebra and thus define a two-dimensional representation. From the Hecke algebra, we obtain 4 one-dimensional representations and another two-dimensional one. Altogether, we see that the algebra has dimension 12 and the elements above form a basis.

The corresponding Brauer algebra of symmetric Brauer graphs (without free points) is obtained by using the relations with $\lambda = 1$ and $q = 1$.

Because of the inhomogeneous nature of the relation $YXYE = E$, the algebra is sensitive to the quadratic relation for Y. If one uses the relation $Y - Y^{-1} = Q - Q^{-1}$, then the algebra is at most 9-dimensional, and 8-dimensional if $\lambda \neq Q^{\pm 2}$.

8. Braid groups of type B

The braid group ZB_n associated to the Coxeter graph B_n is, by definition, the group generated by $t, g_1, \ldots, g_{n-1}$ with relations

(8.1)
$$\begin{array}{lrcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j| = 1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j| \geq 2 \\ (3) & tg_i & = & g_it, & i \geq 2 \\ (4) & tg_1tg_1 & = & g_1tg_1t. & \end{array}$$

We also need another presentations of this group.

Let $Z'B_n$ be the group with generators $c, g_1, \ldots, g_{n-1}$ and relations

(8.2)
$$\begin{array}{lrcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j| = 1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j| \geq 2 \\ (3) & cg_i & = & g_{i-1}c, & i \geq 2 \\ (4) & c^2g_1 & = & g_{n-1}c^2. & \end{array}$$

We abbreviate $g = g_{n-1}g_{n-2}\cdots g_1$.

(8.3) Proposition. *The assignment $\varphi(g_i) = g_i$, $1 \leq j \leq n-1$, and $\varphi(t) = g^{-1}c$ defines an isomorphism $\varphi\colon ZB_n \to Z'B_n$.*

Proof. The relations (1) and (2) yield in both groups

(8.4) $$g_{i-1}g = gg_i, \qquad i > 1.$$

We define in ZB_n (resp. $Z'B_n$) an element c (resp. t) by $gt = c$. From (1), (2) and (8.4) we see that the relations $cg_i = g_{i-1}c$ and $g_it = tg_i$ are equivalent for $i > 1$.

We set $h = g_{n-1}\cdots g_2$, $k = g_{n-2}\cdots g_1$ and infer from (8.4)

(8.5)
$$gh = kg.$$

We use this to show that the relations $c^2 g_1 = g_{n-1} c^2$ and $tg_1tg_1 = g_1tg_1t$ are equivalent, provided (1), (2), and (3) hold. We compute

$$g_{n-1}^{-1} c^2 g_1 = g_{n-1}^{-1} g_{n-1} kthg_1tg_1 = khtg_1tg_1$$

$$c^2 = gthg_1t = ghtg_1t = kgtg_1t = khg_1tg_1t$$

and see the equivalence. □

The braid group $Z\tilde{A}_{n-1}$ of the Coxeter graph with n vertices $\tilde{A}_{n-1}$ has, by definition, generators $g_1, \dots, g_n$ and relations

(8.6)
$$\begin{array}{rclr} g_i g_j g_i & = & g_j g_i g_j, & m(i,j) = 3 \\ g_i g_j & = & g_j g_i, & m(i,j) = 2. \end{array}$$

Indices will be considered $\bmod n$ in this case. We have $m(i,j) = 3$ if and only if $i \equiv j \pm 1 \bmod n$. All this holds for $n \geq 3$. For $n = 2$, the group is the free group generated by g_1 and g_2.

The graph $\tilde{A}_{n-1}$ has an automorphism which permutes the vertices cyclically. We have an induced automorphism s of $Z\tilde{A}_{n-1}$ given by

$$s(g_i) = g_{i-1}, \qquad i \bmod n.$$

The n-th power of s is the identity.

We use s to form the semi-direct product

(8.7)
$$Z\tilde{A}_{n-1} \to G_n \to \mathbb{Z};$$

the generator $1 \in \mathbb{Z}$ acts through s on $Z\tilde{A}_{n-1}$. There is a similar semi-direct product where $\mathbb{Z}$ is replaced by $\mathbb{Z}/nk$. The semi-direct product is the group structure on the set $Z\tilde{A}_{n-1} \times \mathbb{Z}$ defined by $(x,m) \cdot (y,n) = (x \cdot s^m(y), m+n)$. The group G_n has the following description by generators and relations. Let G'_n denote the group with generators $s, g_1, \dots, g_n$ and relations (8.6) for the g_j together with

(8.8)
$$sg_i = g_{i-1}s, \qquad i \bmod n.$$

(8.9) Proposition. *The assignment $\psi(g_i) = (g_i, 0)$ and $\psi(s) = (e,1)$ yields an isomorphism $\psi\colon G'_n \to G_n$ (neutral element e).*

Proof. One verifies that ψ is compatible with relations (8.6) and (8.8). This is obvious for (8.6). The relation $(e,1)(x,0)(e,1)^{-1} = (s(x),0)$ is used to show compatibility with (8.8).

An element $x \in Z\tilde{A}_{n-1}$ has an image $x' \in G'_n$, induced by $g_i \mapsto g_i$. This assignment has the property $(s(x))' = sx's^{-1}$. We have the homomorphism $G_n \to G'_n$, $(x,m) \mapsto x's^m$, by (8.4). It is inverse to ψ. □

(8.10) Proposition. *The assignment $\alpha(g_i) = g_i$, $1 \le i \le n-1$, and $\alpha(c) = s$ defines an isomorphism $\alpha\colon Z'B_n \to G'_n$.*

Proof. The assignment is compatible with the relations of $Z'B_n$, since

$$\alpha(c^2 g_1 c^{-2}) = s^2 g_1 s^{-2} = s g_n s^{-1} = g_{n-1}.$$

An inverse to α is induced by the assignment $\beta(g_i) = g_i$, $\beta(g_n) = c g_1 g^{-1}$, and $\beta(s) = c$. In order to see that β is well defined, one has to check, in particular, the relations

$$g_{n-1} g_n g_{n-1} = g_n g_{n-1} g_n, \qquad g_1 g_n g_1 = g_n g_1 g_n.$$

In the first case, this amounts to the equalitiy of

$$g_{n-1} c g_1 c^{-1} g_{n-1} = c^2 g_1 c^{-1} g_1 c g_1 c^{-2}$$

and

$$c g_1 c^{-1} g_{n-1} c g_1 c^{-1} = c g_1 c g_1 c^{-1} g_1 c^{-1}.$$

We compute

$$c g_1 g_2 g_1 c^{-1} = c g_2 g_1 g_2 c^{-1} = c g_2 c^{-1} c g_1 c^{-1} c g_2 c^{-1} = g_1 c g_1 c^{-1} g_1$$

and hence

$$c(g_1 c g_1 c^{-1} g_1) c^{-1} = c^2 g_1 g_2 g_1 c^{-2}.$$

On the other hand, $g_1 c^{-1} g_1 c g_1 = g_1 g_2 g_1$. This yields the desired equality.

The second relation above leads to the same situation. $\square$

If we combine the foregoing, we obtain a semi-direct product

(8.11) $$Z\tilde{A}_{n-1} \to ZB_n \to \mathbb{Z}.$$

In terms of the original generators, the inclusion $Z\tilde{A}_{n-1} \subset ZB_n$ is given by

(8.12) $$g_n \mapsto g t g_1 t^{-1} g^{-1}; \quad g_i \mapsto g_i, \quad 1 \le i \le n-1.$$

The homomorphism $ZB_n \to \mathbb{Z}$ in (8.11) is given by $g_i \mapsto 0$ and $t \mapsto 1$.

Different types of Weyl groups (= Coxeter groups) are related to these braid groups. We have the Coxeter groups $W\tilde{A}_{n-1}$ and WB_n associated to the graphs $\tilde{A}_{n-1}$ and B_n. In addition, we will also use a group $W^\infty B_n$. It is obtained from ZB_n by adding the relations $g_j^2 = 1$, but no relation for t. The reason for introducing this group is a semi-direct product in analogy to (8.11). The arguments which lead to (8.11) also give a semi-direct product

$$W\tilde{A}_{n-1} \to W^\infty B_n \to \mathbb{Z}.$$

We give another interpretation and describe these groups as groups of permutations.

Let $t_n\colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x+n$ be the translation by n. Let P_n denote the group of t_n-equivariant permutations $\sigma\colon \mathbb{Z} \to \mathbb{Z}$. Equivariance means $\sigma(i+n) = \sigma(i)+n$.

Hence σ induces $\overline{\sigma}$: $\mathbb{Z}/n \to \mathbb{Z}/n$, and $\sigma \mapsto \overline{\sigma}$ is a homomorphism π: $P_n \to S_n$ onto the symmetric group S_n.

(8.13) Proposition. *The kernel of π is isomorphic to $\mathbb{Z}^n$. The group P_n is isomorphic to the semi-direct product $\mathbb{Z}^n \to P'_n \to S_n$ in which S_n acts on $\mathbb{Z}^n$ by permutations.*

PROOF. Let $\sigma_1 \in P_n$. Then there exists a permutation α of $\{1, \dots, n\}$ and an n-tuple $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $\sigma(i+tn) = \alpha(i)+(k_i+t)n$. We denote this map by $\sigma_1 = \sigma(\alpha; k_1, \dots, k_n)$. Suppose $\sigma_2 = \sigma(\beta; l_1, \dots, l_n)$ is another permutation written in this form. Then

$$\sigma_2 \circ \sigma_1 = \sigma(\beta\alpha; l_{\alpha(1)} + k_1, \dots, l_{\alpha(n)} + k_n).$$

If we think of $P'_n = S_n \times \mathbb{Z}^n$ as sets, then the desired isomorphism is given by $(\alpha; k_1, \dots, k_n) \mapsto \sigma(\alpha; k_1, \dots, k_n)$. □

The semi-direct product P'_n has a normal subgroup Q'_n which is given as a semi-direct product

(8.14) $$N \to Q'_n \to S_n$$

with $N = \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \subset \mathbb{Z}^n$. The homomorphism

$$\varepsilon\colon P'_n \to \mathbb{Z}, \quad (\alpha; k_1, \dots, k_n) \mapsto \sum k_i$$

is a surjection with kernel Q'_n. The canonical sequence

(8.15) $$Q'_n \to P'_n \to \mathbb{Z}$$

is itself a semi-direct product; the assignment $1 \mapsto (\mathrm{id}; 1, 0, \dots, 0)$ gives a splitting of ε. Under the isomorphism (8.13) the subgroup Q'_n corresponds to the subgroup

$$Q_n = \{\sigma \in P_n \mid 1 + 2 + \dots + n = \sigma(1) + \dots + \sigma(n)\}.$$

(8.16) Proposition. *The groups $W^\infty B_n$ and P_n are isomorphic. The groups $W\tilde{A}_{n-1}$ and Q_n are isomorphic. The element g_i is mapped to the transposition $(i, i+1)$, $i \in n\mathbb{Z}$. The element t is mapped to $\sigma(i) = i + n$ for $i \equiv 1 \bmod n$ and $\sigma(j) = j$ otherwise.*

The proof is given after the proof of (8.21). In the proof of (8.16) we use the following:

(8.17) Lemma. *The elements*

$$t_0 = t, \quad t_1 = g_1 t g_1, \quad \dots, \quad t_{n-1} = g_{n-1} \dots g_2 g_1 t g_1 g_2 \dots g_{n-1}$$

of the braid group ZB_n pairwise commute.

PROOF. We set

$$g(i,j) = g_i g_{i+1} \dots g_j, \qquad i \le j$$

$$g(i,j) = g_i g_{i-1} \dots g_j, \qquad i \geq j.$$

The braid relations imply immediately

$$g(1,j)g_{j+1}g(j,1) = g(j+1,2)g_1 g(2,j+1)$$

and (8.5) yields

$$g(2,j+1)g(1,j+1) = g(1,j+1)g(1,j).$$

By commutation of g_j-elements, it suffices to show $t_i t_{i+1} = t_{i+1} t_i$. We compute

$$\begin{aligned} t_j t_{j+1} &= g(j,1)tg(1,j)g_{j+1}g(j,1)tg(1,j+1) \\ &= g(j,1)tg(j+1,2)g_1 g(2,j+1)tg(1,j+1) \\ &= g(j,1)g(j+1,2)tg_1 tg(2,j+1)g(1,j+1) \\ &= g(j,1)g(j+1,2)[tg_1 tg_1]g(2,j+1)g(1,j). \end{aligned}$$

A similar computation works for $t_{j+1}t_j$. $\square$

The semi-direct product relation (8.13), (8.16) between $W^\infty B_n$ and WA_{n-1} has a counterpart for the braid groups. The homomorphism

$$\lambda\colon ZB_n \to ZA_{n-1}, \quad g_j \mapsto g_j, \quad t \mapsto 1$$

splits by $g_j \mapsto g_j$. Therefore we have a semi-direct product

(8.18) $$K_n \to ZB_n \to ZA_{n-1}.$$

The elements

$$y_0 = t, \quad , y_1 = g_1 t g_1^{-1}, \quad \dots, \quad y_{n-1} = g_{n-1} \dots g_1 t g_1^{-1} \dots g_{n-1}^{-1}$$

are contained in the kernel K_n of λ.

(8.19) Lemma. *The elements y_j have the following conjugation properties with respect to ZA_{n-1}:*

(1) $g_k^{-1} y_j g_k = y_j, \qquad k > j,\ k < j-1$
(2) $g_k^{-1} y_k g_k = y_{k-1},$
(3) $g_k^{-1} y_{k-1} g_k = y_{k-1} y_k y_{k-1}^{-1}.$

Proof. (2) follows directly from the definitions.
(1) If $k > j$, then g_k commutes with every generator in the definition of y_j. In the case $k < j-1$ one uses the commutation relation between generators and $g_{k+1}g_k g_{k+1}^{-1} = g_k^{-1} g_{k+1} g_k$ (and the inverse) to cancel g_k^{-1} and g_k.
(3) is proved by induction on k. The verification for $k=0$ is easy. We calculate with (1) and (2)

$$g_k^{-1} y_k y_{k+1} y_k^{-1} g_k = y_{k-1} y_{k+1} y_{k-1}^{-1} = g_{k+1} y_{k-1} y_k y_{k-1}^{-1} g_{k+1}^{-1}.$$

On the other hand, by (1) and (2)

$$\begin{aligned} g_{k+1}^{-1}g_k^{-1}g_{k+1}^{-1}y_k g_{k+1}g_k g_{k+1} &= g_k^{-1}g_{k+1}^{-1}g_k^{-1}y_k g_k g_{k+1}g_k \\ &= g_k^{-1}g_{k+1}^{-1}y_{k-1}g_{k+1}g_k \\ &= g_k^{-1}y_{k-1}g_k. \end{aligned}$$

This yields the induction step. □

(8.20) Proposition. *The group K_n is the free group generated by $y_0, \dots, y_{n-1}$.*

Proof. By the previous Lemma, the group K_n^0 generated by the $y_0, \dots, y_{n-1}$ is invariant under conjugation by elements of ZA_{n-1}. Since $t \in K_n^0$ and t together with ZA_{n-1} generates ZB_n, we must have equality $K_n^0 = K_n$.

Let F_n denote the free group generated by $y_0, \dots, y_{n-1}$. We define homomorphisms $\gamma_1, \dots, \gamma_{n-1}$: $F_n \to F_n$ by imitating (8.20):
(1) $\gamma_k(y_j) = y_j$, $\quad k > j,\ k < j-1$
(2) $\gamma_k(y_k) = y_{k-1}$,
(3) $\gamma_k(y_{k-1}) = y_{k-1}y_k y_{k-1}^{-1}$.
We claim:

(8.21) Lemma. *The γ_j are automorphisms and satisfy the braid relations*

$$\gamma_i\gamma_j\gamma_i = \gamma_j\gamma_i\gamma_j, \qquad |i-j| = 1, \quad \textit{and} \quad \gamma_i\gamma_j = \gamma_j\gamma_i, \qquad |i-j| \geq 2.$$

Proof. First we check that the homomorphism δ_k: $F_n \to F_n$
(1) $\delta_k(y_j) = y_j$, $\quad k > j,\ k < j-1$
(2) $\delta_k(y_{k-1}) = y_k$,
(3) $\delta_k(y_k) = y_k^{-1}y_{k-1}y_k$
is inverse to γ_k. Hence γ_k is an isomorphism. Since γ_k fixes y_j for $j \notin \{k-1, k\}$, the second braid relation is obviously satisfied. For the first relation, the reader may check the following values of $\gamma_1\gamma_2\gamma_1$ and $\gamma_2\gamma_1\gamma_2$ on y_0, y_1, y_2:

$$y_0 \mapsto y_0y_1y_2y_1^{-1}y_0^{-1}, \quad y_1 \mapsto y_0y_1y_1^{-1}, \quad y_2 \mapsto y_0.$$

We use this Lemma to define a semi-direct product

$$F_n \to \Gamma_n \to ZA_{n-1}, \tag{8.22}$$

in which $g_j \in ZA_{n-1}$ acts on F_n through δ_j. By (8.19) and $K_n^0 = K_n$, we have a canonical epimorphism μ: $\Gamma_n \to ZB_n$. We show that μ is an isomorphism. As a set, $\Gamma_n = F_n \times ZA_{n-1}$. An inverse to μ has to send $g_j \mapsto (1, g_j)$ and $t \mapsto (y_0, 1)$. We have to check that this assignment is compatible with the relations of ZB_n. This is obvious for the g_j. Moreover:

$$\begin{aligned} tg_1tg_1 &\mapsto (y_0,1)(1,g_1)(y_0,1)(1,g_1) \\ &= (y_0,g_1)(y_0,g_1) \\ &= (y_0\delta_1(y_0), g_1^2) \\ &= (y_0y_1, g_1^2) \end{aligned}$$

$$\begin{aligned} g_1tg_1t &\mapsto (1,g_1)(y_0,1)(1,g_1)(y_0,1) \\ &= (y_1,g_1)(y_1,g_1) \\ &= (y_1\delta(y_1),g_1^2) \\ &= (y_0y_1,g_1^2). \end{aligned}$$

This finishes the proof of Proposition (8.20). □

Proof of (8.16). The elements t_j of (8.17) and the elements y_j coincide in $W^\infty B_n$, since $g_j = g_j^{-1}$ in this group. Lemma (8.19) shows that conjugation $y \mapsto g_k^{-1}yg_k$ acts on the set $(y_0,\dots,y_{n-1})$ by interchanging y_{k-1} and y_k. The proof of (8.20) is now easily adapted to show the isomorphism $W^\infty B_n \cong P_n'$. This isomorphism restricts to an isomorphism $W\tilde{A}_{n-1} \cong Q_n'$. □

We now apply the previous results to Hecke algebras. We have the Hecke algebras HA_{n-1}, $H\tilde{A}_{n-1}$, and HB_n associated to the corresponding Coxeter graphs. We consider algebras over the ground ring $\mathfrak{K}$. The first one is given by generators $g_1,\dots,g_{n-1}$, the braid relations between them and the quadratic relations $g_j^2 = (q-1)g_j + q$ with a parameter $q \in \mathfrak{K}$. The second one has generators $g_1,\dots,g_n$, the braid relations (8.6) and the same quadratic relations. The algebra HB_n has generators $t, g_1,\dots,g_{n-1}$, the braid relations (8.1), the quadratic relations above for the g_j and $t^2 = (Q-1)t + Q$ with another parameter $Q \in \mathfrak{K}$. If we omit the quadratic relation for t, then we obtain the definition of $H^\infty B_n$. This is not a Hecke algebra in the formal sense, i. e. associated to a Coxeter graph. It is a deformation of the group algebra of $W^\infty B_n$.

We know from Hecke algebra theory that an additive basis of the Hecke algebra is in bijective correspondence with the elements of the Coxeter group. There is a similar relation between $W^\infty B_n$ and $H^\infty B_n$. In order to derive it, we relate $H\tilde{A}_{n-1}$ and $H^\infty B_n$.

The algebra $H\tilde{A}_{n-1}$ has an automorphism τ given by $\tau(g_i) = g_{i-1}$ (indices $\operatorname{mod} n$). We define the twisted tensor product over the ground ring $\mathfrak{K}$

(8.23) $$H\tilde{A}_{n-1} \otimes \mathfrak{K}[\tau,\tau^{-1}] =: H_n^\infty$$

by the multiplication rule $(x\otimes\tau^k)\cdot(y\otimes\tau^l) = (x\cdot\tau^k(y),\tau^{k+l})$ for $k,l \in \mathbb{Z}$ and $x,y \in H\tilde{A}_{n-1}$.

(8.24) Proposition. *The algebra* (8.23) *is canonically isomorphic to* $H^\infty B_n$.

Proof. We use the isomorphism (8.3) to redefine the algebra $H^\infty B_n$ by generators $c, g_1,\dots,g_{n-1}$ relations (8.2) and the quadratic relations for the g_j. The assignment $g_j \mapsto g_j\otimes 1$, $c \mapsto 1\otimes\tau$ induces a homomorphism $H^\infty B_n \to H\tilde{A}_{n-1}\otimes H_n^\infty$. We have a homomorphism $H\tilde{A}_{n-1} \to H^\infty B_n$, $x \mapsto x'$ induced by $g_j \mapsto g_j$ with $g_n = gtg_1t^{-1}g^{-1}$ in $H^\infty B_n$ (see (8.12)). This extends to a homomorphism $H_n^\infty \to H^\infty B_n$ by $x\otimes\tau^k \mapsto x'\cdot c^k$, since $\tau(y)' = cy'c^{-1}$. These homomorphisms are inverse to each other. □

(8.25) Corollary. *Suppose* $(b_j \mid j \in J)$ *is a* $\mathfrak{K}$*-basis of* $H\tilde{A}_{n-1}$*. Then* $(b_j'c^k \mid j \in J, k \in \mathbb{Z})$ *is a* $\mathfrak{K}$*-basis of* $H^\infty B_n$. □

9. Braids of type B

We use a theorem of Brieskorn [4] to derive some geometric interpretations of the braid group ZB_n. The starting point is the reflection representation of the Weyl group WB_n. This group is a semi-direct product

(9.1) $$(\mathbb{Z}/2)^n \to WB_n \to S_n.$$

It acts on complex n-space $\mathbb{C}^n$ as follows:
(1) S_n acts by permuting the coordinates.
(2) $(\mathbb{Z}/2)^n$ act by sign changes $(z_1, \ldots, z_n) \mapsto (\varepsilon_1 z_1, \ldots, \varepsilon_n z_n)$, $\varepsilon_i \in \{\pm 1\}$.
This group contains the reflections in the hyperplanes

$$z_i = \pm z_j, \quad i \neq j; \quad \text{and} \quad z_j = 0.$$

Let X denote the complement of these hyperplanes. From the theory of finite reflection groups it is known, that $W = WB_n$ acts freely on X. Brieskorn [4] shows:

(9.2) Theorem. *The braid group ZB_n is isomorphic to the fundamental group $\pi_1(X/W)$ of the orbit space X/W.* □

If we think of WB_n as the Coxeter group with generators $t, g_1, \ldots, g_{n-1}$, then g_j acts as the transposition $(j, j+1)$ and t as $z_1 \mapsto -z_1$.

We use (9.2) to give several interpretations of ZB_n by braids.

We remove the hyperplanes $z_j = 0$ from $\mathbb{C}^n$. It remains the n-fold product $\mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{C}^{*n}$. Removal of the remaining reflection hyperplanes yields the space X of n-tuples $(z_j) \in \mathbb{C}^{*n}$ with pairwise different squares z_j^2.

The *configuration space* $C^n(\mathbb{C}^*)$ is the space of subsets of $\mathbb{C}^*$ with cardinality n. As topological space it is defined as Y/S_n where $Y \subset \mathbb{C}^{*n}$ is the set of n-tuples (y_j) with pairwise different components.

(9.3) Proposition. *X/W is homeomorphic to $C^n(\mathbb{C}^*)$.*

Proof. We arrive at X/W in two steps: First we form $Y' = X/(\mathbb{Z}/2)^n$ and then we divide out the S_n-action. The map $(z_j) \mapsto (z_j^2)$ yields an S_n-equivariant homeomorphism $Y' \to Y$. □

By (9.2) and (9.3), $ZB_n \cong \pi_1(C^n(\mathbb{C}^*))$. The elements of $\pi_1(C^n(\mathbb{C}^*))$ will be interpreted as braids in the cylinder (cylindrical braids). We take $(1, \omega, \cdots, \omega^{n-1})$, $\omega = \exp(2\pi i/n)$, as base point in $C^n(\mathbb{C}^*)$. A loop in $C^n(\mathbb{C}^*)$ lifts to a path

$$w\colon I \to Y, \quad t \mapsto (w_1(t), \ldots, w_n(t))$$

with this initial point. Thus we have
(1) $w(0) = (1, \omega, \ldots, \omega^{n-1})$.
(2) $w(1) = (\sigma(1), \ldots, \sigma(\omega^{n-1}))$, with a permutation σ of the set $\mathbb{Z}/n = \{1, \omega, \ldots, \omega^{n-1}\}$.
(3) For $j \neq k$ we have $w_j(t) \neq w_k(t)$.

These data yield a braid z_w with n strings in $\mathbb{C}^* \times [0,1]$ from $\mathbb{Z}/n \times 0$ to $\mathbb{Z}/n \times 1$

$$z_w(t) = \{w_1(t), \dots, w_n(t)\} \times t.$$

Homotopy classes of loops correspond to isotopy classes of such braids. Multiplication of loops corresponds to concatenation of braids, as usual. Thus we have:

(9.4) Theorem. *The braid group ZB_n is the group of n-string braids in the cylinder $\mathbb{C}^* \times [0,1]$.* □

A second interpretation is by symmetric braids in $\mathbb{C} \times [0,1]$. This was already used in [7]. We take the base point $(1,2,\dots,n) \in X$. We lift a loop in X/W to a path

$$w\colon I \to X, \quad t \mapsto (w_1(t), \dots, w_n(t)).$$

Then we have:
(1) $w(0) = (1,2,\dots,n)$.
(2) $w(1) = (\pm\sigma(1), \dots, \pm\sigma(n))$ with a permutation σ of $\{1,\dots,n\}$.
(3) For $j \neq k$ we have $w_j(t) \neq \pm w_k(t)$.
(4) $w_j(t) \neq 0$.

Let $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$. The data yield a braid with $2n$ strings in $\mathbb{C} \times [0,1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$, namely

$$t \mapsto \{-w_n(t), \dots, -w_1(t), w_1(t), \dots, w_n(t)\} \times t.$$

These braids are $\mathbb{Z}/2$-equivariant with respect to $(z,t) \mapsto (-z,t)$ and are therefore called *symmetric*. The theorem of Brieskorn thus gives:

(9.5) Theorem. *The group ZB_n is isomorphic to the group of symmetric braids with $2n$ strings.* □

Symmmetric braids are drawn as ordinary braids but with additional symmetry with respect to the axis $0 \times [0,1]$.

The symmetry is not the reflection in the axis, but corresponds to a spacial rotation about this axis. The relation $tg_1tg_1 = g_1tg_1t$ appears in this context as a generalized Reidemeister move.

Braids in the cylinder with n strings can be visualized as ordinary braids with $n+1$ strings — the axis of the cylinder is the additional string. This method has been used by Lambropoulou [24]. It allows the reduction of B_n-type braids to ordinary Artin braids, also with respect to proofs. The theorem of Brieskorn is then not used.

The twofold covering, ramified along the axis, of the cylinder produces a symmetric braid from a cylindrical one — and vice versa.

The cylinder $\mathbb{C}^* \times [0,1]$ has the universal covering $\mathbb{C} \times [0,1]$. Lifting cylindrical braids with n strings produces n-periodic infinite braids in $\mathbb{C} \times [0,1]$ from $\mathbb{Z} \times 0$ to $\mathbb{Z} \times 1$. They are invariant with respect to the translation $(z,t) \mapsto (z+n,t)$. This gives yet another interpretation of ZB_n by n-periodic braid pictures.

The relation between ZB_n and $Z\tilde{A}_{n-1}$ has the following geometric origin or counterpart. The map

$$\mathbb{C}^{*n} \to \mathbb{C}^*, \quad (z_1, \dots, z_n) \mapsto z_1 \cdot \ldots \cdot z_n$$

is S_n-equivariant and induces therefore a map from the configuration space

$$\alpha\colon C^n(\mathbb{C}^*) \to \mathbb{C}^*.$$

(9.6) Lemma. *The map α is a fibre bundle.*

Proof. Let

$$H = \{(z_1, \dots, z_n) \in \mathbb{C}^{*n} \mid \prod z_j = 1\}.$$

This is an S_n-invariant subset. The map

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} H \to \mathbb{C}^{*n}, \quad (z, z_1, \dots, z_n) \mapsto (zz_1, \dots, zz_n)$$

is an S_n-equivariant homeomorphism. Thus γ is the fibre bundle with fibre H assoziated to the $\mathbb{Z}/n$-principal bundle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. In $\mathbb{C}^{*n}$ we have to remove the subset

$$C = \{(z_1, \dots, z_n) \mid \text{there exists } i \neq j \text{ such that } z_i = z_j\}.$$

Let $D = H \cap C$. Then γ induces an S_n-equivariant homeomorphism

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus D) \to \mathbb{C}^{*n} \setminus S.$$

This yields the fibre bundle description

$$\mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus T)/S_n \to \mathbb{C}^*$$

for the configuration space. □

We apply the fundamental group to this fibration and obtain the exact sequence

$$1 \to \operatorname{kernel} \alpha_* \to ZB_n \to \mathbb{Z} \to 0.$$

It can be shown that this is the sequence (8.11), i. e. $Z\tilde{A}_{n-1}$ is the fundamental group of the fibre of α.

Our next aim is to describe an additive basis of the Hecke algebra $H^\infty B_n$ by geometric means, i. e. by specifying a certain canonical set of basic braids.

A cylindrical braid with n strings is called *descending*, if for $i < j$ the i-th string is always overcrossing the j-th string. The i-th string is the one starting at ω^i, $0 \leq i \leq n-1$. Overcrossing means the following: We look radially and orthogonally from infinity onto the axis. The braid is in general position if we only see transverse double points. The first string we meet, coming from infinity, is the overcrossing one.

(9.7) Theorem. *The descending braids form a $\mathfrak{K}$-basis of the algebra $H^\infty B_n$. The descending braids with winding number zero form a $\mathfrak{K}$-basis of the algebra $H\tilde{A}_{n-1}$.*

We use (8.11) to reduce the first statement to the second. For the latter Hecke algebra we have the canonical basis related to the elements of reduced form in the Weyl group, and elements of the Weyl group will be shown to correspond to descending braids. We use the description of the Weyl group elements as n-periodic permutations of $\mathbb{Z}$. We represent such a permutation by n straight lines $c_1, \ldots, c_n$ in the strip $\mathbb{R} \times [0,1]$ starting at $\{1, \ldots, n\} \times 0$ such that c_i and c_j have at most one crossing, and then repeat with period n. By slightly moving the endpoints of the c_j we can assume that the curves are in general position. The resulting crossings are used to write the permutation as a product of reflections. This product is reduced, in the sense of Coxeter group theory. It is geometrically obvious that the same configuration of crossings can be realized by a descending braid.

(9.8) Proposition. *The set*

$$\mathfrak{C} = \{y_{n-1}^k g_{n-1} g_{n-2} \ldots g_j \mid k \in \mathbb{Z}, 1 \le j \le n\}$$

is a system of representatives for the left cosets of the inclusion $W^\infty B_{n-1} \subset W^\infty B_n$.

Proof. This is an immediate consequence of the semi-direct product description. The powers of y_{n-1} are representatives for cosets of $V_{n-1} \subset V_n$, and the products $g_{n-1} \ldots g_j$ are representatives for the cosets of $S_{n-1} \subset S_n$. □

We use this Proposition to derive the following result of Lambropoulou and Przytycki which was proved by them in a purely algebraic manner.

(9.9) Theorem. *Let $\mathfrak{B}$ be the canonical basis of $H^\infty B_{n-1}$. Then $\{bc \mid b \in \mathfrak{B}, c \in \mathfrak{C}\}$ is a basis of $H^\infty B_n$.*

Proof. Represent a basis element of $H^\infty B_n$ by a descending braid. □

10. Categories of bridges

This section introduces some general terminology for certain graphical categories and algebras.

A free involution $\sigma\colon P \to P$ of a set P is calles a *P-bridge.* A free involution of P is a partition of P into 2-element subsets $\{i, \sigma(i)\}$, called the *arcs* or *strings* of the bridge. A bridge is called *oriented* if its arcs are ordered sets $\{a_1, a_2\}$.

We study bridges with a geometric terminology. Suppose $\sigma\colon P \to P$ is a bridge. The *geometric realization* $|\sigma|$ of σ is the one-dimensional simplicial complex with P as set of 0-simplices and a 1-simplex for each arc $\{i, \sigma(i)\}$ with i and $\sigma(i)$ as boundary points. We say that the arc *connects* its boundary points. The arcs are the components of $|\sigma|$.

A (P,Q)-bridge is a bridge on the disjoint union $P \coprod Q$. An arc of a (P,Q)-bridge σ is called *horizontal* if its boundary points are either contained in P or in Q. The other arcs are called *vertical*.

We use a graphical notation for (P,Q)-bridges σ. We think of $P \subset \mathbb{R} \times 0$, $Q \subset \mathbb{R} \times 1$ and we draw an arc in $\mathbb{R} \times [0,1]$ from i to $\sigma(i)$. The notation horizontal and vertical is evident in this context. The horizontal arcs with endpoints in P are called the *lower* part of the bridge, the horizontal arcs with endpoints in Q the *upper* part.

(10.1) Remark. *Suppose P has $2n$ elements. The number of P-bridges is*

$$(2n-1) \cdot (2n-3) \cdots 3 \cdot 1.$$

Proof. There are $2n-1$ possibilities to connect a fixed element of P. Having fixed this connection, a set with $2n-2$ elements remains. Now use induction. □

We will use bridges with further properties.

Let G be a group and suppose P and Q are G-sets. A G-equivariant (P,Q)-bridge is a G-equivariant free involution σ of $P \amalg Q$. Equivariant means: $\sigma(gi) = g\sigma(i)$ for $g \in G$ and $i \in P \amalg Q$.

Suppose the bridge $\sigma \colon P \to P$ is G-equivariant. We have an induced G-action on $|\sigma|$. The action on the 0-simplices is given. If $\{i, \sigma(i)\}$ is a 1-simplex, then, by equivariance, $\{gi, g\sigma(i)\}$ is a 1-simplex. It can happen that these simplices coincide. This is the case if g is in the isotropy group G_i of i. If $g \in G$ acts non-trivially on $\{i, \sigma(i)\}$, then

$$gi = \sigma(i), \quad g\sigma(i) = i, \quad g^2 i = i$$

and hence $g^2 \in G_i$. Geometrically, g acts as reflection in the barycentre of the 1-simplex $\{i, \sigma(i)\}$ in this case.

In the sequel we only consider G-sets P with the following additional properties:

(1) The isotropy groups are finite.
(2) The orbit set is finite.
(3) G acts effectively on each orbit.

Under these hypotheses we have:

(10.2) Proposition. *Let σ be a G-equivariant (P,Q)-bridge. Then the following holds:*

(1) *The G-action respects lower, upper and horizontal arcs.*
(2) *The G-action on $|\sigma|$ is proper.*
(3) *The orbit space $|\sigma|/G$ is a compact one-dimensional* CW*-complex.*

Proof. (1) is clear from the definitions.

(2) The G-action on the barycentric subdivision $|\sigma|'$ of $|\sigma|$ is a cellular action with finite isotropy groups.

(3) This follows since G acts cellularly on $|\sigma|'$ and the orbit space has a finite number of cells.

Suppose σ is a G-equivariant (P,Q)-bridge and τ a G-equivariant (Q,R)-bridge. Consider the G-space $|\tau| \cup_Q |\sigma|$. The components of this space can be of different type. Consider the G-orbit $B = Gx$ of a component x. Let H denote the isotropy group of the component x. Then B/G is homeomorphic to x/H. The orbit space of $|\tau| \cup |\sigma$ is compact. Hence x/H is compact. We use:

(10.3) Lemma. *There is no proper action of a discrete group on* $[0,1[$ *with compact orbit space.* □

This Lemma tells us that the components of $|\tau| \cup |\sigma|$ are not homeomorphic to $[0,1[$. Since the components are one-dimensional manifolds (with or without boundary), there are three cases:

(10.4) *A componente of* $|\tau| \cup |\sigma|$ *is homeomorphic to* $[0,1]$, S^1, *or* $]0,1[$. □

(10.5) Proposition. *The components of* $|\tau| \cup |\sigma|$ *which are homeomorphic to* $[0,1]$ *define a* G*-equivariant* (P,R)*-bridge.*

Proof. If the component is homeomorphic to $[0,1]$, then the boundary points are contained in $P \coprod R$.

For each point in $P \coprod R$ there exists a component of $|\tau| \cup |\sigma|$ with this point as boundary point. Since components of type $[0,1[$ do not exist, the component has a second boundary point in $P \coprod R$. □

We denote the bridge in (10.5) by $\tau \wedge \sigma$. The components of $|\tau| \cup |\sigma|$ which are homeomorphic to S^1 are called *cycles*, the components which are homeomorphic to $]0,1[$ are called *snakes*.

(10.6) Remark. Let $H = G_x$ be the subgroup of elements which map the component x into itself. Then H acts effectively and properly on the one-dimensional manifold x. Therefore we have, up to H-homeomorphism, the following possibilities:

(1) Suppose $x \cong S^1$. Then $H \cong \mathbb{Z}/m$ or $H \cong D_{2m}, m \geq 1$, and the action is by the usual action of a subgroup of $O(2)$.

(2) Suppose $x \cong \mathbb{R}$. Then $H \cong \mathbb{Z}$ or $H \cong D_\infty$, and the action is by the usual action as a subgroup of the group of affine transformations. ♡

Let $Z(\tau,\sigma)$ denote the orbit set of the components of $|\tau| \cup |\sigma|$ which are cycles or snakes. The G-orbits of components in $Z(\tau,\sigma)$ are counted according to types. The *type* of a component x consists of the conjugacy class of G_x together with the G_x-homeomorphism type of the G_x-action. The group $\mathbb{Z}/2$ has two different actions on S^1, by rotation or by reflection. (In the latter case it is the group D_2.) It is an observation of H. Reich [27] that these two actions should be distinguished.

Let C denote the set of possible types. We denote by $k(c,\tau,\sigma)$ the number of elements in $Z(\tau,\sigma)$ of type c.

After these preparations we define the category $F(G)$ of G-bridges. The *objects* of $F(G)$ are the G-sets as above, i. e. with finite isotropy groups, finite orbit set and effective action on orbits.

We fix a ground ring $\mathfrak{K}$. The *morphism set* $\mathrm{Mor}(P,Q)$ is the free $\mathfrak{K}$-module on the set of G-equivariant (P,Q)-bridges.

In order to define the *composition* of morphisms we fix a map d: $C \to \mathfrak{K}$, called the *parameter function*. The composition of morphisms $\mathrm{Mor}(Q,R) \times \mathrm{Mor}(P,Q) \to \mathrm{Mor}(P,R)$ is assumed to be $\mathfrak{K}$-bilinear. The composition of bridges is defined to be

$$\tau \circ \sigma \prod_{c \in C} d(c)^{k(c,\tau,\sigma}\tau \wedge \sigma.$$

The *identity* $P \to P$ is represented by the bridge ι: $P \amalg P \to P \amalg P$ which connects $i \in P$ vertically with $i \in P$. We have $|\sigma| \cup |\iota| \cong |\sigma|$ and $|\iota| \cup |\sigma| \cong |\sigma|$, if defined.

Associativity of composition follows from a geometrical consideration: The cycles and snakes of $|\tau| \cup |\sigma| \cup |\rho|$ are those of $|\tau| \cup |\sigma|$, plus those of $|\sigma| \cup |\rho|$, plus those of $|\tau \wedge \sigma| \cup |\rho|$ (equal to those of $|\tau| \cup |\sigma \wedge \rho|$).

We shall mostly work with suitable subcategories of $F(G)$. For instance, we could use only free G-sets. Or we restrict the morphisms; this will be the case in the Temperley-Lieb categories.

The composition of bridges with only vertical strings is again a bridge of this form. No cycles or snakes appear. The vertical (P,P)-bridges under composition can be identified with the group of G-equivariant permutations of P.

11. References

(1) Ariki, S., and K. Koike: A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations. Adv. in Math. 106, 216 – 243 (1994).

(2) Birman, S. J., and H. Wenzl: Braids, link polynomials and a new algebra. Trans. Amer. Math. Soc. 313, 249 – 273 (1989).

(3) Brauer, R.: On algebras which are connected with the semisimple continuous groups. Ann. of Math. 38, 854 – 887.

(4) Brieskorn, E.: Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Invent. Math. 12, 57 – 61 (1971).

(5) Curtis, Ch. W., and I. Reiner: Methods of representation theory. Vol II. New York, J. Wiley & Sons 1987.

(6) tom Dieck, T.: Knotentheorien und Wurzelsysteme. Teil I. Mathematica Gottingensis 21 (1993).

(7) tom Dieck, T.: Symmetrische Brücken und Knotentheorie zu den Dynkin-Diagrammen vom Typ B. J. reine angew. Math. 451, 71 – 88 (1994).

(8) tom Dieck, T.: Knotentheorien und Wurzelsysteme. Teil II. Mathematica Gottingensis 44 (1993).

(9) tom Dieck, T.: Knot theories and root systems. Part III. Mathematica Gottingensis 22 (1994).

(10) Dipper, R., and G. James: Representations of Hecke algebras of type B_n. J. of Algebra 146, 454 – 481 (1992).

(11) Fishel, S., and I. Grononowski: Canonical bases for the Brauer centralizer algebra. Math. Res. Letters 2, 1 – 16 (1995).

(12) Geck, M., and S. S. F. Lambropoulou: Markov traces and knot invariants related to Hecke algebras of type B. To appear.

(13) Goodman, F. M., de la Harpe, P., and V. F. R. Jones: Coxeter graphs and towers of algebras. New York – Berlin, Springer 1989.

(14) Goodmam, F. M., and H. Wenzl: The Temperley-Lieb algebra at roots of unity. Pac. J. Math. 161, 307 – 334 (1993).

(15) Häring, R.: Private communication.

(16) Häring, R.: Über Birman-Wenzl-Algebren vom Typ B. Dissertation, Göttingen 1996.

(17) Hanlon, P., and D. Wales: On the decomposition of Brauer's centralizer algebras. J. of Algebra 121, 409 – 445 (1989).

(18) Hanlon, P., and D. Wales: Eigenvalues connected with Brauer's centralizer algebras. J. of Algebra 121, 446 – 476 (1989).

(19) Hanlon, P., and D. Wales: A tower construction for the radical in Brauer's centralizer algebras. J. of Algebra 164, 773 – 830 (1994).

(20) Jones, V. F. R.: An affine Hecke algebra quotient in the Brauer algebra. Enseignement Math.

(21) Kassel, Chr.: Quantum groups. New York, Springer-Verlag 1995.

(22) Kauffman, L. H.: State models and the Jones polynomial. Topology 26, 395 – 407 (1987).

(23) Kauffman, L. H.: An invariant of regular isotopy. Trans. Amer. Math. Soc. 318, 417 – 471 (1990).

(24) Lambropoulou, S. S. F.: A study of braids in 3-manifolds. Thesis, Warwick 1993.

(25) Lambropoulou, S. S. F.: Solid torus links and Hecke algebras of B-type. Proc. of quantum topology, (D. Yetter ed.), 225 – 245. World Scientific, Singapore 1994.

(26) Murphy, G. E.: On the representation theory of the symmetric group and associated Hecke algebras. J. of Algebra 152, 492 – 513 (1992).

(27) Reich, H.: Symmetrische Brauer-Algebren. Diplomarbeit, Göttingen 1994. Auch: Mathematica Gottingensis 15 (1995).

(28) Reshetikhin, N., and V. G. Turaev: Ribbon graphs and their invariants derived from quantum groups. Commun. Math. Phys. 127, 1 – 26 (1990).

(29) Turaev, V. G.: Operator invariants of tangles and R-matrices. USSR Izvestia 35, 411 – 444 (1990).

(30) Wenzl, H.: On the structure of Brauer's centralizer algebras. Ann. of Math. 128, 173 – 193 (1988).

(31) Wenzl, H.: Hecke algebras of type A_n and subfactors. Invent. Math. 92, 349 – 383 (1988).

(32) Wenzl, H.: Representations of braid groups and the quantum Yang-Baxter equation. Pacific J. Math. 145, 153 – 180 (1990).

(33) Wenzl, H.: Quantum groups and subfactors of type B, C, and D. Commun. Math. Phys. 133, 383 – 432 (1990).