Cylinder braiding for quantum linear groups

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1. Introduction and results

The braid group ZB_k associated to the Coxeter graph B_k has generators $t, g_1, \ldots, g_{k-1}$ and relations

$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & i > 1 \\ g_ig_jg_i & = & g_jg_ig_j & |i-j| = 1 \\ g_ig_j & = & g_jg_i & |i-j| > 1. \end{array}$$

This group is isomorphic to the group of topological braids with k strings in the cylinder $(\mathbb{C} \setminus 0) \times [0,1]$.

It is the purpose of this paper to describe a general construction of tensor representations of ZB_k, based on quantum linear groups.

A tensor representation of ZB_k is obtained from any *four braid pair* (X, F) on a module V over a commutative ring $\mathfrak{K}$. By definition, this consists of a Yang-Baxter automorphism (= R-matrix) $X\colon V \otimes V \to V \otimes V$ which satisfies the braid relation $(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X)$ on $V \otimes V \otimes V$ and an automorphism $F\colon V \to V$ such that the *four braid relation*

$$X(F \otimes 1)X(F \otimes 1) = (F \otimes 1)X(F \otimes 1)X$$

holds on $V \otimes V$. A representation of ZB_k on the k-fold tensor power $V^{\otimes k}$ is then obtained by the assignment

$$\begin{array}{rcl} t & \mapsto & F \otimes 1 \otimes \cdots \otimes 1 \\ g_i & \mapsto & 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1 = X_{(i)}, \end{array}$$

where the morphism X in $X_{(i)}$ acts on the factors i and $i+1$.

The results of this paper are based on a fundamental four braid pair (1.3) which we now describe. Let $q \in \mathfrak{K}^\times$ (= the units of $\mathfrak{K}$) and write $\delta = q - q^{-1}$. Let

$$P = (p_{ij} \mid 1 \le i,\, j \le n)$$

be a matrix with entries in $\mathfrak{K}$ such that $p_{ii} = q$ and $p_{ij}p_{ji} = 1$ for $i \neq j$. Let V be a free $\mathfrak{K}$-module with basis $v_1, \ldots, v_n$. The assignment

$$X(v_i \otimes v_j) = \begin{cases} p_{ij}v_j \otimes v_i & i = j \\ p_{ij}v_j \otimes v_i & i > j \\ p_{ij}v_j \otimes v_i + \delta v_i \otimes v_j & i < j \end{cases} \tag{1.1}$$

is a Yang-Baxter operator. (Compare [5, p. 171], devide by $p^{1/2}$ and replace q by $p^{-1/2}q$.) The *standard case* $p_{ij} = 1$ for $i \neq j$ is the most interesting. We work with the multi-parameter version, since it makes some of the computations more transparent.

Let β_j $(1 \leq j \leq n)$ and w be elements in $\mathfrak{K}$ such that

(1) $z = \beta_j \beta_{n+1-j}$ is independent of j;
(2) $\beta_l^2 = w\beta_l + z$ in case $2l = n+1$;
(3) $\prod_j \beta_j \in \mathfrak{K}^\times$.

Define an automorphism $F\colon V \to V$ by

(1.2)
$$F(v_j) = \begin{cases} \beta_j v_{n+1-j} & \text{for} \quad 2j \leq n+1 \\ \beta_j v_{n+1-j} + w v_j & \text{for} \quad 2j > n+1. \end{cases}$$

We use the notation $j' = n+1-j$. Here is the basic four braid pair; the proof will be given in the next section.

(1.3) Theorem. *Suppose* $p_{ij}p_{jk} = p_{ij'}p_{j'k}$ *whenever* $i \neq j, j'$ *and* $k \neq j, j'$. *Then* (X, F), *defined by* (1.1) *and* (1.2), *is a four braid pair.*

Henceforth we assume the hypothesis of (1.3).

Associated to the R-matrix X above is, by the FRT-construction, a cobraided bialgebra $A = A(X)$ with *braid form* $r = r_X\colon A \otimes A \to \mathfrak{K}$. (See [5, VIII 5, VIII 6] or [6, 9.1] for back ground; r is called universal r-form in these references.) In our case, the algebra A is generated by T_i^j, $1 \leq i, j \leq n$, with relations

$$\begin{aligned} qT_i^m T_i^n &= p_{mn} T_i^n T_i^m \\ qT_i^m T_j^m &= p_{ji} T_j^m T_i^m \\ p_{ji} T_j^m T_i^n &= p_{mn} T_i^n T_j^m \\ \delta T_j^m T_i^n &= p_{mn} T_j^n T_i^m - p_{ij} T_i^m T_j^n \end{aligned}$$

for $m > n$ and $i > j$. The comultiplication μ of A is determined by $\mu(T_i^j) = \Sigma_k T_i^k \otimes T_k^j$ and the counit ε by $\varepsilon(T_i^j) = \delta_i^j$.

It was shown in [2] that any four braid pair (X, F) induces a so-called cylinder form $f = f_{(X,F)}\colon A \to \mathfrak{K}$. We recall the notion of a cylinder form on a cobraided bialgebra (A, r). We denote the multiplication in the dual algebra C^* of a coalgebra C as convolution $*$. A *cylinder form* on (A, r) is a $\mathfrak{K}$-linear map $f\colon A \to \mathfrak{K}$ such that f is convolution invertible and satisfies in the convolution algebra $(A \otimes A)^*$ the identity

(1.4)
$$fm = (f \hat{\otimes} \varepsilon) * r\tau * (\varepsilon \hat{\otimes} f) * r,$$

where $m\colon A \otimes A \to A$ is the multiplication, $\varepsilon\colon A \to \mathfrak{K}$ the counit, $f \hat{\otimes} g\colon a \otimes b \mapsto f(a)g(b)$, and $\tau(x \otimes y) = y \otimes x$. In terms of formal notation $\mu(a) = \sum_{(a)} a_1 \otimes a_2$ and $(\mu \otimes 1)\mu(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3$ for the (iterated) comultiplication, (1.4) assumes the following form

$$f(ab) = \sum_{(a),(b)} f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3)$$

for $a, b \in A$.

The quantum linear group $GL_X(n)$ is a localization of $A(X)$, and $SL_X(n)$ is a quotient of $A(X)$. In order to define them, one has to use the quantum determinant D (see [6, 9.2], [8], [9], [12] for this topic). In our case it is given as a sum over the symmetric group S_n (compare [3, p. 157])

(1.5)
$$D = \sum_{\sigma \in S_n} \alpha(\sigma) T_1^{\sigma(1)} \cdots T_n^{\sigma(n)} = \sum_{\sigma \in S_n} \alpha(\sigma)^{-1} T_n^{\sigma(n)} \cdots T_1^{\sigma(1)}$$

with
$$\alpha(\sigma) = \prod_{\substack{\mu < \nu \\ \sigma(\mu) > \sigma(\nu)}} (-q p_{\sigma(\nu)\sigma(\mu)}).$$

It is not totally obvious that the two expressions of D define the same element of A. Since this fact is important for our applications, we indicate a proof as follows. We think of $A(X)$ as being defined formally over the ring $\mathbb{Z}[p_{ij}, p_{ij}^{-1}]$. The algebra has an anti-involution ι defined by $\iota(T_i^j) = T_i^j$, $\iota(q) = q^{-1}$, $\iota(p_{ij}) = p_{ij}^{-1}$. It sends the first expression for D into the second one. We have the exterior algebra as a comodule over A. The exterior algebra is the algebra generated by $v_1, \dots, v_n$ with relations $v_i v_i = 0$, $v_i v_j + q^{-1} p_{ij} v_j v_i = 0$ for $i < j$. The element $v_1 v_2 \dots v_n = v$ generates a one-dimensional subcomodule with coaction μ: $v \mapsto D \otimes v$. We also have an anti-involution ι on the exterior algebra which is the identity on the v_i and defined on the coefficient ring as for A. The coaction μ is compatible with the anti-involutions $\mu \circ \iota = (\iota \otimes \iota) \circ \mu$. This compatibility and the fact that ι sends v to a multiple of v implies $\iota(D) = D$.

In the standard case and some other cases, the quantum determinant D is a group-like central element of A (see [3, 6.3]). If this is the case, then the algebra $GL(n) = GL_X(n)$ is defined as the localization of $A(X)$ with respect to D, i. e., $GL_X(n) = A(X)[y]/(Dy - 1)$. This is now a Hopf algebra, and the braid form r_X extends to a braid form on $GL_X(n)$; see [3, 3.1] for a more general result. (In the standard case, these algebras are denoted $\mathcal{O}(GL_q(n))$ and $\mathcal{O}(SL_q(n))$ in [6, 9.2].)

We assume $q^{1/n} \in \mathfrak{K}$ and renormalize our R-matrix $X' = q^{-1/n} X$. We assume that D is group-like and central. The quotient of $A(X')$ by the twosided ideal generated by $D - 1$ is then a Hopf algebra. We denote it by $SL_X(n) = SL(n)$. The braid form $r = r_{X'}$ passes to $SL(n)$ and yields a braid form.

The braid form $r = r_{X'}$ on $A = A(X') = A(X)$ satisfies

(1.6)
$$r(x \otimes D) = r(D \otimes x) = \varepsilon(x)$$

for $x \in A$. By the axioms of a braid form, we have $r(xy \otimes D = r(x \otimes D) r(y \otimes D)$, since D is group-like. Therefore it suffices to prove (1.6) for $x = T_i^j$. But then it is a restatement of the equality $\beta = \tau$ in [3, 4.2].

The next proposition collects the formal results about cylinder forms.

(1.7) Proposition. *Suppose that the quantum determinant is a group-like central element. Then the cylinder form $f = f_{(X,F)}$ satisfies $f(xD) = f(x)f(D)$ for*

$x \in A$. If $f(D) \in \mathfrak{K}^\times$, then the assignment $f(xD^{-l}) = f(x)f(D)^{-l}$ yields a cylinder form on $GL(n)$. If $f' = f_{(X',F)}$ satisfies $f'(D) = 1$, then f' factors through the quotient map $A(X') \to SL(n)$ and induces a cylinder form on $SL(n)$.

PROOF. The identity (1.4) together with (1.6) and the fact that D is group-like yields (in formal notation)

$$\begin{aligned} f(xD) &= \sum_{(x)} f(x_1) r(D \otimes x_2) f(D) r(x_3 \otimes D) \\ &= \sum_{(x)} f(x_1)\varepsilon(x_2) f(D)\varepsilon(x_3) \\ &= f(x)f(D); \end{aligned}$$

the last equality follows from the counit axiom. From $f(xD) = f(x)f(D)$ we see that $xD^{-s} \mapsto f(x)f(D)^{-s}$ is a well-defined map on $GL(n)$. By using the same property, one verifies that (1.4) still holds for the extended form on $GL(n)$. Finally, it is clear that f vanishes on the ideal generated by $D-1$ if $f(D) = 1$.□

In order to make use of (1.7) we have to know the value $f(D)$. Theorem (1.8) will be obtained from a general analysis of tensor representations in section 3.

(1.8) Theorem. *The cylinder form $f_{(X,F)}$ assumes the following value on the quantum determinant*

$$f(D) = (-q)^{-n(n-1)/2} \prod_{i>j} p_{i'j} \prod_j \beta_j.$$

In the standard case $p_{ij} = 1$ for $i \neq j$, we have

$$\prod_{i>j} p_{i'j} = q^l \quad \text{for} \quad n = 2l, n = 2l+1.$$

If one works with X' in place of X, then this value has to be multiplied by q^{1-n}, since the definition of f via t_n (see section 3) involves $n(n-1)$ R-matrices X.

Finally, we recall from [2] that a cylinder form f on a cobraided bialgebra (A, r) induces a cylinder braiding on the category of left A-comodules. Let $\mu_M\colon M \to A \otimes M$, $x \mapsto \sum x^1 \otimes x^2$ be a comodule structure on M in formal notation. Define $t_M\colon M \to M$ by $x \mapsto \sum f(x^1)x^2$. Let $z_{M,N}\colon M \otimes N \to N \otimes M$ denote the braiding morphisms induced by the braid form r. Then for any two left A-comodules M and N the following identities hold

$$t_{M\otimes N} = z_{N,M}(t_N \otimes 1) z_{M,N}(t_M \otimes 1) = (t_M \otimes 1) z_{N,M}(t_N \otimes 1) z_{M,N}.$$

Isomorphisms $t_M\colon M \to M$ which satisfy these identities are called a *cylinder braiding* on the category of left A-comudules. In particular, we obtain a four braid pair $(z_{M,M}, t_M)$ for each comodule M.

We remark that the algebra of cylinder braidings can be used to construct invariant of coloured ribbon tangles in the cylinder along the lines of Turaev [13].

2. The four braid pair

This section is devoted to the proof of (1.3). Consider the involutions $\sigma(i,j) = (n+1-i,j)$ and $\tau(i,j) = (j,i)$ on the set of indices $J = \{(i,j) \mid 1 \le i,\, j \le n\}$. Since $\sigma\tau\sigma\tau = \tau\sigma\tau\sigma$, they formally generate the dihedral group D_8 of order 8. We decompose J into the orbits under the D_8-action. Recall the notation $n+1-i = i'$.

Let $n = 2k$. Then we have the orbit $(i,i),(i',i),(i,i'),(i,i')$ of length 4. This is the orbit of (i,j) if $i = j$ or $i = j'$. If $i \ne j, j'$, then the orbit of (i,j) has length 8.

Let $n = 2\ell - 1$. Then we have the fixed point (ℓ,ℓ). There is another orbit type of length 4, namely $(\ell,j),(j,\ell),(j',\ell),(\ell,j')$ for $j \ne \ell$. In addition, we have the orbits of (i,i), $i \ne \ell$, of length 4; and for $i \ne j, j'$ and $i \ne \ell \ne j$ the orbit of (i,j) of length 8.

For each D_8-orbit $x \in J/D_8$ the subspace of $V \otimes V$ spanned by $\{v_i \otimes v_j \mid (i,j) \in x\}$ is invariant under X and $F \otimes 1 = Y$.

Therefore it suffices to verify the four braid relation on these subspaces. The structure of the matrices involved depends only on the D_8-isomorphism type of the orbit. When we display matrices we use the antilex(icographical) order: $(i,j) < (a,b)$ if and only if $j < b$ or $j = b$ and $i > a$. This ordering is applied to the basis $\{v_i \otimes v_j \mid (i,j) \in J\}$ of $V \otimes V$.

The fixed point orbit is trivial. Therefore we have to consider three cases.

Case (I): The orbit $(i,i),(j,i),(i,j),(j,j)$ with $i < j = i'$. The matrices are

$$Y = \begin{pmatrix} 0 & \beta_j & & \\ \beta_i & w & & \\ & & 0 & \beta_j \\ & & \beta_i & w \end{pmatrix} \quad X = \begin{pmatrix} q & & & \\ & 0 & p_{ij} & \\ & p_{ji} & \delta & \\ & & & q \end{pmatrix}.$$

One computes $YXYX$ to be the matrix of 2×2-blocks

$$\begin{pmatrix} 0 & qp_{ij}\beta_j F_{ij} \\ q\beta_i p_{ji} F_{ij} & (q^2-1)F_{ij}^2 + wF_{ij} \end{pmatrix} \quad \text{with} \quad F_{ij} = \begin{pmatrix} 0 & \beta_j \\ \beta_i & w \end{pmatrix}.$$

One obtains $XYXY$ by transposition of $YXYX$ and the changes $p_{ab} \mapsto p_{ba}$, $\beta_a \mapsto \beta_{a'}$. This shows the four braid relation this case.

Case (II). The orbit $(\ell,i),(i,\ell),(i',\ell),(\ell,i')$ for $i < \ell$ if $n = 2\ell - 1$. The matrices are $(j = i')$

$$Y = \begin{pmatrix} \beta_\ell & & & \\ & 0 & \beta_j & \\ & \beta_i & w & \\ & & & \beta_\ell \end{pmatrix} \quad X = \begin{pmatrix} 0 & p_{i\ell} & & \\ p_{\ell i} & \delta & & \\ & & 0 & p_{\ell j} \\ & & p_{j\ell} & \delta \end{pmatrix}.$$

Recall that $\beta_\ell^2 = \beta_\ell w + \beta_i\beta_j$. (Strangely enough, Y is now an R-matrix and X has almost the form of the Y in case (I).) The necessity of the condition

$p_{i\ell}p_{\ell i} = p_{j\ell}p_{\ell j}$ results from the comparison of the 2×2-blocks on the codiagonal of $XYXY = YXYX$. If we invoke this condition and use $\beta_\ell^2 = \beta_\ell w + z$, we compute $XYXY$ to be the matrix of 2×2-blocks

$$\begin{pmatrix} 0 & \beta_j\beta_\ell p_{\ell j}P^{i\ell}_{j\ell} \\ \beta_i\beta_\ell p_{j\ell}P^{\ell j}_{\ell i} & \beta_\ell w(P^{\ell j}_{j\ell})^2 + \delta\beta_i\beta_j P^{\ell j}_{j\ell} \end{pmatrix} \qquad \text{with} \qquad P^{cd}_{ab} = \begin{pmatrix} 0 & p_{cd} \\ p_{ab} & \delta \end{pmatrix}.$$

Again, this matrix does not change under transposition and the changes as in case (I).

Case (III). The orbit of length eight for $a < b \le [\frac{n}{2}]$ is (b,a), (b',a), (a,b), (a',b), (a,b'), (a',b'), (b,a'), (b',a'). For simplicity of notation we use $n = 4$, $(a,b) = (1,2)$. The matrices are in 2×2-block form

$$Y = \begin{pmatrix} Y_{23} & & & \\ & Y_{14} & & \\ & & Y_{14} & \\ & & & Y_{23} \end{pmatrix} \qquad Y_{ab} = \begin{pmatrix} 0 & \beta_b \\ \beta_a & w \end{pmatrix}$$

$$X = \left(\begin{array}{cc|cc|cc|cc} & & p_{12} & & & & & \\ & & & & p_{13} & & & \\ \hline p_{21} & & \delta & & & & & \\ & & & & & & p_{24} & \\ \hline & p_{31} & & & \delta & & & \\ & & & & & & & p_{34} \\ \hline & & & p_{42} & & & \delta & \\ & & & & & p_{43} & & \delta \end{array}\right).$$

The necessity of the condition $p_{ai}p_{ib} = p_{ai'}p_{i'b}$ results from the comparison of the 2×2-blocks on the codiagonal of $YXYX$ and $XYXY$. If we invoke this condition, then $YXYX$ turns out to be the following matrix of 2×2-blocks.

			$p_{13}p_{34}\beta_4 Y_{23}$
		$p_{24}p_{43}Y_{14}$	$\delta p_{24}\beta_3 Y_{24}$
	$p_{34}p_{42}\beta_2 Y_{14}$	wY_{14}	$\delta p_{34}(zI + wY_{24})$
$p_{43}p_{31}\beta_1 Y_{23}$	$\delta p_{42}\beta_2 Y_{13}$	$\delta p_{43}(zI + wY_{13})$	$\delta^2 zI + (w + \delta^2 w)Y_{23}$

Here I is a unit matrix. Again, transposition and the changes of case (I) do not change the matrix.

3. The cylinder twist in the tensor representation

Let $E_{ij}\colon V \to V$ be the linear map defined by $E_{ij}(v_r) = \delta_{ir} v_j$. We use the same symbol E_{ij} for its matrix, it has 1 in position (j,i) and 0 in the other positions. If $M\colon V \to V$ is any linear map with matrix $M = (m_{ab})$, then

$$\textbf{(3.1)} \qquad E_{ji} M E_{\ell k} = m_{jk} E_{\ell i}.$$

We also use

$$E(r) = \sum_{j=1}^{r-1} E_{jj}, \quad 1 \le r \le n.$$

This is the zero matrix for $r = 1$.

In order to organize the following computations, we need a suitable notation for matrices in tensor products. Suppose U and W are $\mathfrak{K}$-modules with basis $u_1, \dots, u_r$ and $w_1, \dots, w_s$. We call

$$u_1 \otimes w_1, \dots, u_r \otimes w_1;\ u_1 \otimes w_2, \dots, u_r \otimes w_2; \dots u_r \otimes w_s$$

the *antilex*(icographical) basis of $U \otimes W$. This ordering of a basis is an associative process. In particular, $V(k) := V^{\otimes k}$ has for any $k \in \mathbb{N}$ a well-defined antilex basis. In the sequel, we always use the antilex basis, unless the contrary is explicitly stated.

Suppose $f\colon U \to U$ has matrix A. Then $f \otimes 1_W\colon U \otimes W \to U \otimes W$ has the block diagonal matrix $\mathrm{Dia}(A, \dots, A)$ with rank (W) diagonal terms A.

Suppose $g\colon W \to W$ has matrix $B = (b_{ij})$. Then $1_U \otimes f\colon U \otimes W \to U \otimes W$ has the block matrix B, where b_{ij} now stands for the diagonal block $b_{ij} I(u)$, with $I(u)$ the unit matrix of size $u = \mathrm{rank}\,(U)$.

The matrix X in (1.1) is in this notation the $n \times n$-block matrix with blocks of size n

$$\textbf{(3.2)} \qquad X = \begin{pmatrix} qE_{11} + \delta E(1) & p_{12}E_{12} & p_{13}E_{13} & \dots & p_{1n}E_{1n} \\ p_{21}E_{21} & qE_{22} + \delta E(2) & p_{23}E_{23} & \dots & p_{2n}E_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ p_{n1}E_{n1} & p_{n2}E_{n2} & p_{n3}E_{n3} & \dots & qE_{nn} + \delta E(n) \end{pmatrix}.$$

By the remarks above, the matrix

$$X_k = 1_{V(k-2)} \otimes X$$

on $V^{\otimes k}$ has a similar structure, but each entry is now a diagonal block of size n^{k-2}.

We still use the notation as in (3.2). Then E_{ij} is a block-matrix of size n, and its entries are blocks of size n^{k-2}. In this block notation (3.1) still holds.

Basis objects for our inductive computations are bottom-right triangular block-matrices with blocks of size m, or BRTB(m)-matrices for short. These are, by definition, $n \times n$-matrices $T = (T_{ij})$ with:

(1) $T_{ij} = 0 \quad$ for $i + j \le n$.

(2) T_{ij} is a block of size m.
(3) The blocks $T_{i,n+1-i}$ on the codiagonal are diagonal matrices.

The next proposition provides us with the basic inductive structure of BRTB-matrices with regard to the matrix X.

(3.3) Proposition. *Let $T = (T_{ij})$ be a* BRTB(n^{k-2})*-matrix. Then*

$$\tilde{T} = (\tilde{T}_{ab}) = X_k(T \otimes 1)X_k$$

is a BRTB(n^{k-1})*-matrix.*

Proof. We use the remarks above about the structure of matrices in tensor products, the structure of X_k, and the relation (3.1). Denote the $n \times n$-matrix X in (3.2) by (e_{ab}). Then, by matrix multiplication,

$$\textbf{(3.4)} \qquad \tilde{T}_{ab} = \sum_{j=1}^{n} e_{aj} T e_{jb}.$$

We use (3.1) with $e_{aj} = p_{aj}E_{aj}$, for $a \neq j$, and $e_{aa} = qE_{aa} + \delta \sum_{j=1}^{a-1} E_{jj}$. Suppose $a + b \leq n$. Then, by assumption, $T_{ab} = 0$, and (3.1) gives $\tilde{T}_{ab} = 0$.

It remains to show that $\tilde{T}_{ab}$ for $a+b = n+1$ is a diagonal matrix. From (3.1) and (3.4) we obtain $\tilde{T}_{ab} = \sum_j p_{aj}p_{jb}T_{ab}E_{jj}$, since $\delta E(a)Te_{ab} = 0$ and $e_{ab}T\delta E(b) = 0$ by the BRTB(n^{k-2})-property of T. This is the diagonal block matrix

$$\textbf{(3.5)} \qquad \mathrm{Dia}(p_{a1}p_{1b}T_{ab}, p_{a2}p_{2b}T_{ab}, \ldots, p_{an}p_{nb}T_{ab}).$$

We apply this result to the tensor representation obtained from the four braid pair (1.3). We set $t(1) = F$ and define inductively

$$t(k) = X_k(t(k-1) \otimes 1)X_k.$$

We can apply (3.3) to the matrices $t(k)$. We determine the diagonal blocks on the codiagonal of $t(k)$. Since we are working in $V^{\otimes k}$, we use a multi-index notation for the basis and other terms. A multi-index is a function $i\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$. Let $A(k, n)$ denote the set of these functions. We write

$$v_i = v_{i(1)} \otimes \cdots \otimes v_{i(k)} \quad \text{for} \quad i \in A(k, n).$$

The diagonal positions of the diagonal block $t(k)_{a,n+1-a}$ of $t(k)$ are indexed by multi-indices i with $i(k) = a$; here i denotes the columns of $t(k)$. Let $d(i)$ denote this diagonal term. From (3.5) we obtain by induction on k:

$$\textbf{(3.6)} \qquad d(i) = \beta_{i(k)} \prod_{j=1}^{k-1} p_{i(k)i(j)}p_{i(j)i(k)'}.$$

The *cylinder twist* $t_k\colon V^{\otimes k} \to V^{\otimes k}$ is, by definition, the automorphism defined inductively by

$$t_1 = t(1) \quad \text{and} \quad t_k = (t_{k-1} \otimes 1)t(k).$$

(3.7) Theorem. *The matrix of t_k is a* BRTB(1)*-matrix. The codiagonal element $\ell(i)$ in column $i \in A(k,n)$ is*

$$\ell(i) = \prod_{a<b} p_{i(b)i(a)} p_{i(a)i(b)'} \cdot \prod_{a=1}^{k} \beta_{i(a)}.$$

Proof. The blocks of t_k are $t_{k-1}t(k)_{ab}$. This immediately gives that t_k is a BRTB(1)-matrix. The value of the codiagonal element is obtained inductively from (3.6). □

We mention the standard special case $p_{ij} = 1$ for $i \neq j$. Let $\mathrm{Or}(n)$ be the orbit set of $\{1,\dots,n\}$ under the involution $i \mapsto n+1-i$. Let $s(x)$ be the order of the isotropy group of $x \in \mathrm{Or}(n)$. In that case we have

(3.8)
$$\ell(i) = q^{\lambda(i)} \prod_{a=1}^{k} \beta_{i(a)} \quad \text{with} \quad \lambda(i) = \sum_{x\in \mathrm{Or(n)}} s(x) \binom{|i^{-1}(x)|}{2}.$$

It may be helpful to see a specific example for these computations. We consider the case $n = 3$ and $p_{ij} = 1$ for $i \neq j$. Then we obtain the matrix

$$t(2) = \left[\begin{array}{ccc|ccc|ccc}
 & & & & & & q\beta_3 & & \\
 & & & & & & & \beta_3 & \\
 & & & & & & & & q\beta_3 \\
\hline
 & & & \beta_2 & & & & \delta\beta_3 & \\
 & & & & q^2\beta_2 & & & & \\
 & & & & & \beta_2 & & \delta\beta_2 & \\
\hline
q\beta_1 & & & & & & w & & q\delta\beta_3 \\
 & \beta_1 & & \delta\beta_1 & & \delta\beta_2 & & w+\delta^2\beta_2 & \\
 & & q\beta_1 & & & & q\delta\beta_1 & & q^2 w
\end{array}\right]$$

and finally the matrix for $t_2 = (t_1 \otimes 1)t(2)$ in the antilex basis.

$$\left[\begin{array}{ccc|ccc|ccc}
 & & & & & & & & q\beta_3^2 \\
 & & & & & & & \beta_2\beta_3 & \\
 & & & & & & q\beta_1\beta_3 & & q\beta_3 w \\
\hline
 & & & & & \beta_2\beta_3 & & \delta\beta_2\beta_3 & \\
 & & & & q^2\beta_2^2 & & & & \\
 & & & \beta_1\beta_2 & & w\beta_2 & & \delta\beta_2^2 & \\
\hline
 & & q\beta_1\beta_3 & & & & q\delta\beta_1\beta_3 & & q^2 w\beta_3 \\
 & \beta_1\beta_2 & & \delta\beta_1\beta_2 & & \delta\beta_2^2 & & w\beta_2+\delta^2\beta_2^2 & \\
q\beta_1^2 & & q\beta_1 w & & & & q^2 w\beta_1 & & q\delta\beta_1\beta_3+q^2w^2
\end{array}\right]$$

The space $V \otimes V$ decomposes into the second symmetric and exterior power $V \otimes V = S^2V \oplus \Lambda^2 V$. A basis of S^2V is $v_{22}; u_{12}, u_{23}; v_{11}, u_{13}, v_{33}$ with $v_{ij} = v_i \otimes v_j$

and $u_{ij} = v_{ij} + q^{-1}v_{ji}$. The subspaces S^2V and Λ^2V are t_2-invariant. The matrix of t_2 on S^2V in the basis above is a block diagonal matrix with blocks

$$(q^2\beta_2^2), \quad \begin{pmatrix} & q\beta_2\beta_3 \\ q\beta_1\beta_2 & q\delta\beta_2^2 + w\beta_2 \end{pmatrix}, \quad \begin{pmatrix} & & q\beta_3^2 \\ & q^2\beta_1\beta_3 & q^2w\beta_3 \\ q\beta_1^2 & (1+q^2)w\beta_1 & q^2w^2 + q\delta\beta_1\beta_3 \end{pmatrix}.$$

The roots of the characteristic polynomial are $q^2\beta_2^2$, $-\beta_1\beta_3$ and $q^2(\beta_2 - w)^2$, with multiplicities 3, 2, and 1, respectively. Thus the matrix satisfies a relation of degree three.

The space Λ^2V has the basis $w_{ij} = v_{ij} - qv_{ji}$, $i < j$. The restriction of t_2 to Λ^2V has with respect to the basis w_{12}, w_{13}, w_{23} the matrix

$$\begin{pmatrix} & & -q^{-1}\beta_2\beta_3 \\ & -\beta_1\beta_3 & \\ -q^{-1}\beta_1\beta_2 & & -\beta_1\beta_3 + q^{-2}\beta_2^2 \end{pmatrix}.$$

The characteristic polynomial has the roots $-\beta_1\beta_3, q^{-2}\beta_2^2$ with multiplicities 2, 1.

Proof of (1.8). We apply the preceding results in order to compute the value of the cylinder form on the quantum determinant D. For this purpose we have to recall from [2] the definition of the cylinder form $f\colon A \to \mathfrak{K}$. Suppose $t_k(v_i) = \sum_j F_i^j v_j$, where i and j are multi-indices of length k. The cylinder form is then defined by $f(T_i^j) = F_i^j$.

We see from Theorem (3.7) that $f(T_n^{\sigma(n)} \cdots T_1^{\sigma(1)}) = 0$ if $(\sigma(n), \dots, \sigma(1)) \neq (1, \dots, n)$. This claim is proved as follows: We have to consider the column $(n, \dots, 1)$ of the matrix (3.7) for t_n. Among the $(\sigma(n), \dots, \sigma(1))$ the element $(1, \dots, n)$ is the largest one in the antilex order. The claim thus follows from the BRTB-structure of the matrix. Thus only the term $f(T_n^1 T_{n-1}^2 \cdots T_1^n)$ contributes to $f(D)$. The value for $f(D)$, as stated in (1.8), is now a direct consequence of (3.7). Note that there is some cancellation of p_{ij}-factors due to the hypothesis of (1.3). $\square$

By dualization and linear algebra (see e. g. [10], [12]) one can obtain from the results of this paper cylinder braidings on suitable categories of modules over quantum enveloping algebras (say integrable modules over $U_q(sl_n)$ in the sense of [7]).

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