On the topology of complex surfaces with an action of the additive group of complex numbers

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The purpose of this note is to study the topology of some regular affine surfaces over the complex numbers which carry an algebraic action of the additive group $G = (\mathbb{C}, +)$.

The generating examples of such surfaces are the varieties of DANIELEWSKI [1] which he studied in connection with the cancellation problem. The simplicity of these varieties makes it desirable to have a detailed picture of them. Here is one result about Danielewski's varieties.

Theorem A. *Let $k \geq 1$ be an integer. The following smooth manifolds are diffeomorphic preserving the orientation:*

(1) *The regular affine hypersurface*

$$U(k) = \{(x, y, z) \in \mathbb{C}^2 \mid x^2 - 4yz^k = 1\}.$$

(2) *The orbit manifold $V(k)$ of the free, proper, right $\mathbb{C}^*$–action*

$$(a, b, c, d) \cdot \lambda = (\lambda^k a, \lambda^{-k} b, \lambda c, \lambda^{-1} d)$$

on the regular affine hypersurface

$$W(k) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^4 \mid ad^k - bc^k = 1\}.$$

(3) *The quotient $D(k)$ of $\mathbb{C}^2 \amalg \mathbb{C}^2$ under the identification*

$$\mathbb{C}^* \times \mathbb{C} \ni (x, u) \sim (x, u + x^{-k}) \in \mathbb{C}^* \times \mathbb{C}$$

of the parts $\mathbb{C}^* \times \mathbb{C}$ in the two copies of $\mathbb{C}^2$.

(4) *The complex line bundle $H(-2k)$ over the projective line $\mathbb{P}^1$ with Euler class $-2k$.*

The manifolds (1), (2), *and* (3) *are actually algebraic varieties and they are isomorphic as varieties. However, the isomorphism of* (2) *with* (4) *is not complex analytic.*

From the topological view point, the equivalence with (4) is most interesting, because it implies immediately the following:

Theorem B. (1) *The manifold $D(k)$ is the interior of a smooth compact manifold $B(k)$ with boundary a lens space $L(2k,1)$.*

(2) *The generator of $H_2(D(k);\mathbb{Z}) \cong \mathbb{Z}$ has self-intersection number $-2k$.*

From Theorem B we see that $D(k)$ has fundamental group at infinity $\mathbb{Z}/2k$. It had been shown earlier by Fieseler (without the construction of the boundary, by an exhaustion method) that the homology at infinity is $\mathbb{Z}/2k$. Another method of determining the topology at infinity stems from the observation that the boundary is a Seifert manifold, compare [3]. The interest in these topological invariants comes from the fact that they distinguish the varieties for different k topologically and hence algebraically. Part (2) of Theorem B is more interesting, since it allows to distinguish the manifolds by intrisic invariants.

From part (4) in Theorem A we see moreover that the manifolds $H(-2k) \oplus \mathbb{R}$ for different k are diffeomorphic. This is due to the fact that the three-dimensional orientable vector bundle $H(-2k) \oplus \mathbb{R}$ over $\mathbb{P}^1 \cong S^2$ is trivial (since $\pi_2(BSO(3)) \cong \mathbb{Z}/2$ and we are dealing with even powers of the Hopf-bundle). The interesting fact, which was observed by Danielewski [1], is that the $D(k) \times \mathbb{C}$ are all isomorphic as affine varieties and hence give simple examples where cancellation does not hold (see also [3]).

Another feature of our construction is pointed out by the model (2). Obviously, $W(1) \cong SL(2,\mathbb{C})$. This provides the manifolds with large symmetry groups. We present a detailed study from the view point of transformation groups. From the equivalence with (4) we obtain more subtle and hidden symmetries of the underlying real manifolds.

The differential topology at infinity of the manifolds in question (in particular the fundamental group and the homology at infinity) is known once a normal crossing compactification divisor of the affine variety is determined. The boundary of a tubular neighbourhood of such divisors is the boundary of the variety as a differentiable manifold. These boundaries are given by a plumbing construction. By well known techniques, the homology and the fundamental group can be read off from the weighted dual graph of the compactification divisor.

The analysis of the varieties $D(k)$ leads to a topological description of some related varieties which were investigated in [3]. The variety $V(k)$ carries a free involution $J : (a,b,c,d) \mapsto ((-1)^{k-1}b, -a, d, -c)$. The orbit space is an affine variety $Z(k)$. For $k = 2l$ even, $Z(k)$ is the hypersurface $x^2 - 4yz^{l+1} = z$, see [3]. The Hopf-bundle $H(-2k)$ carries a free involution $(x,y;\mu) \mapsto (\bar{y}, -\bar{x}; (-)^k\bar{\mu})$. The quotient is a nonorientable real plane bundle $E(k)$ over the real projective space $\mathbb{R}P^2$ with Euler class k, see [2], p. 343.

Theorem C. *The variety $Z(k)$ is diffeomorphic to the total space of the bundle $E(k)$.*

From Theorem C we conclude that $Z(k)$ is the interior of a compact smooth manifold with boundary a quaternionic lens space. This lens space is the orbit space of the action of the quaternion group

$$Q(4k) = \langle A, B \mid A^k = B^2, BAB^{-1} \rangle$$

on the three-sphere S^3 given by

$$A \cdot (x, y) = (\zeta x, \zeta y), \quad B \cdot (x, y) = (\bar{y}, -\bar{x})$$

with $\zeta = \exp(2\pi i/2k)$. In particular, the fundamental group at infinity is $Q(4k)$. This was also verified in [3], by using compactification divisors or the theory of Seifert manifolds. In this case, the varieties $Z(k)$ can be distinguished by the intersection form on the two-dimensional homology with twisted coefficients $\widetilde{\mathbb{Z}}$.

1 The varieties of Danielewski

This section is devoted to the varieties (1), (2), and (3) of Theorem A. We begin with the transformation group aspect.

For each $k \in \mathbb{N}$ we have a regular affine variety $W(k)$ in $\mathbb{C}^4$ given by

(1.1) $$W(k) = \{(a, b, c, d) \in \mathbb{C}^4 \mid ad^k - bc^k = 1\}.$$

Note that $W(1) = SL(2, \mathbb{C})$. This fact motivates some of the subsequent constructions. The varieties $W(k)$ carry a smooth free right action of $\mathbb{C}^*$

(1.2) $$(a, b, c, d) \cdot \lambda = (\lambda^k a, \lambda^{-k} b, \lambda c, \lambda^{-1} d).$$

The following polynomial functions x, y, and z on $W(k)$ are $\mathbb{C}^*$–invariant:

(1.3) $$\begin{aligned} x &= ad^k + bc^k \\ y &= ab \\ z &= cd. \end{aligned}$$

They satisfy the equation

(1.4) $$x^2 - 4yz^k = 1.$$

This equation defines a regular affine variety $U(k)$ in $\mathbb{C}^3$. Let $V(k) = W(k)/\mathbb{C}^*$ denote the orbit space. The functions (1.3) induce a morphism

(1.5) $$\alpha : V(k) \to U(k), \ \ (a, b, c, d) \mapsto (ad^k + bc^k, ab, cd).$$

(1.6) Proposition. *The space $V(k)$ carries a unique structure of a differentiable manifold such that the quotient map $W(k) \to V(k)$ is a submersion. In this structure, α is a diffeomorphism.*

Proof. One verifies that α is bijective. Since $U(k)$ is a Hausdorff space this implies that the action (1.2) is proper. Now apply [2], I(5.2). By the universal property of submersions ([2], I(4.9)), α is smooth. In order to see that α is a submersion it suffices to show that the complex differental of

$$\mathbb{C}^4 \to \mathbb{C}^3, \ (a, b, c, d) \mapsto (ad^k + bc^k, ab, cd)$$

has rank three on $W(k)$. This is easy. □

(1.7) Remark. If we view $W(1) = SL(2, \mathbb{C})$, then the action (1.2) of $\lambda \in \mathbb{C}^*$ on $W(1)$ amounts to right multiplication by the diagonal matrix $Dia(\lambda, \lambda^{-1})$. Let

$D \subset SL(2,\mathbb{C})$ be the subgroup of such matrices. Then $V(1)$ is the coset variety $SL(2,\mathbb{C})/D$. The previous Proposition also gives, by the same proof, $V(k)$ as a complex manifold and α is a holomorphic isomorphism. ♡

The varieties $V(k)$ have a large symmetry group. This is obvious for $V(1) = SL(2,\mathbb{C})/D$ because $SL(2,\mathbb{C})$ acts by left translation. Certain subgroups still acts on $V(k)$ in a similar manner.

Let $G(m)$ denote the semi–direct product $\mathbb{C} \times \mathbb{C}^*$ with multiplication

(1.8)
$$(a,\alpha)\cdot(b,\beta) = (a+\alpha^m b, \alpha\beta).$$

If $B \subset SL(2,\mathbb{C})$ denotes the group of upper triangular matrices we have an isomorphism

(1.9)
$$G(2) \to B,\ (a,\alpha) \mapsto \begin{pmatrix} \alpha & a\alpha^{-1} \\ 0 & \alpha^{-1} \end{pmatrix}.$$

The group $G(m)$ acts on $\mathbb{C}$ by

$$(a,\alpha)\cdot z = \alpha^m z + a.$$

The resulting $G(m)$–space $\mathbb{C}$ will be denoted $\mathbb{C}(m)$. The group $G(2m)$ acts on $W(m)$ by

(1.10)
$$(u,\alpha)(a,b,c,d) = (\alpha^m a + u\alpha^{-m}c^m, \alpha^m b + u\alpha^{-m}d^m,\ \alpha^{-1}c, \alpha^{-1}d).$$

This action commutes with the right $\mathbb{C}^*$–action (1.2) and induces therefore an action of $G(2m)$ on $V(m)$. We also note:

(1.11) Proposition. *The projection*

$$W(k) \to \mathbb{P}^1, \quad (a,b,c,d) \mapsto [c,d]$$

is a principal $G(2k)$–bundle. □

This $G(2m)$–action can be transported by the isomorphism (1.5) into an action on $U(k)$. There exists an algebraic action of $G(2m)$ on $\mathbb{C}^3$ which induces this action on $U(k)$. One verifies that the following action has this property:

$$(u,\alpha)\cdot(x,y,z) = (x + 2u\alpha^{-2m}z^m, \alpha^{2m}z^m, \alpha^{2m}y + ux + u^2\alpha^{-2m}z^m, \alpha^{-2}z)$$

for $(u,\alpha) \in G(2m)$, $(x,y,z) \in \mathbb{C}^3$. The $G(2)$–action on $W(1)$ corresponds under (1.9) to the left translation action of B on $SL(2,\mathbb{C})$.

For the following, recall the complex line bundles $H(-2k) \to \mathbb{P}^1$, see [2], p. 251.

(1.12) Proposition. *The projection $h : V(1) \to \mathbb{P}^1$, $(a,b,c,d) \mapsto (a,c)$ is diffeomorphic over $\mathbb{P}^1$ to the line bundle $H(-2k) \to \mathbb{P}^1$.*

Proof. As we have just seen, the projection $h : V(1) \to \mathbb{P}^1$ can be identified with the canonical quotient map

$$SL(2,\mathbb{C})/D = SL(2,\mathbb{C}) \times_B B/D \to SL(2,\mathbb{C})/B.$$

We identify

$$\sigma : B/D \xrightarrow{\cong} \mathbb{C}, \quad \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \mapsto \lambda\mu.$$

The left translation by $\left(\begin{smallmatrix} \alpha & a\alpha^{-1} \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \in B$ on B/D is transformed under σ into the action $z \mapsto \alpha^2 z + a$. Thus $B/D \cong \mathbb{C}(2)$. The differential topology of the bundle above is obtained by restriction to the compact form $SL(2,\mathbb{C})/B \cong SU(2)/T$ with maximal torus $T = SU(2) \cap B$. Thus we obtain precisely the description of the bundle $H(-2)$ over $\mathbb{P}^1$, namely $SU(2) \times_T \mathbb{C}(2) \to SU(2)/T$. The isomorphism $SU(2) \times_T \mathbb{C}(2) \to V(2)$ is given explicitly as follows:

$$(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \mu) \mapsto \begin{pmatrix} a & a\mu - \bar{b} \\ b & b\mu + \bar{a} \end{pmatrix}.$$

□

We have already explained in [3] that the $G(2k)$–variety $x^2 - 4yz^k = 1$ is obtained from $\mathbb{C}^2 \amalg \mathbb{C}^2$ by the identification $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}$, $(x, u) \mapsto (x, u + x^{-k})$. A basic observation of Danielewski is that $D(k)$ is a principal $(\mathbb{C}, +)$–bundle over the affine line with two origins $\tilde{\mathbb{C}}$. This is obvious from the gluing construction (3); and in [3] we have described the corresponding action on $U(k)$. It is interesting to see this action in the model $V(k)$. If we restrict the $G(2k)$–action (1.10) on $V(k)$ to the normal subgroup $G = (\mathbb{C}, +)$, then we obtain a locally trivial G–action on $V(k)$. The map

$$W(k) \to \mathbb{C}^2 \setminus 0, \quad (a, b, c, d) \mapsto (c, d)$$

is equivariant with respect to (1.2) and the $\mathbb{C}^*$–action $(c, d) \cdot \lambda = (\lambda c, \lambda^{-1} d)$ on $\mathbb{C}^2 \setminus 0$. The orbit space $(\mathbb{C}^2 \setminus 0)/\mathbb{C}^*$ is $\tilde{\mathbb{C}}$ and the orbit map $V(k) \to \tilde{\mathbb{C}}$ is the projection of the principal G–bundle. In the $SL(2,\mathbb{C})$–case the principal bundle comes from the left U–action on $SL(2,\mathbb{C})/D$, where $U = \{\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)\}$.

2 The diffeomorphism type

This section is devoted to the equivalence of (2) and (4) in Theorem A and to the proof of Theorem C. We begin with the former.

For topological reasons we shall have to use a slightly different definition of $V(k)$ as a smooth manifold. Let $W_0(k)$ denote the intersection of $W(k)$ with $\{(a, b, c, d) \mid |a|^2 + |c|^{2k} = 1\}$. This intersection is transverse. Therefore $W_0(k) \subset W(k)$ is a closed smooth submanifold. It carries the S^1–action (1.2) and, as smooth manifolds, $W_0(k)/S^1 \cong V(k)$. We use the model $W_0(k)/S^1 =: V_0(k)$ of $V(k)$ for the following investigations. Let

$$p_k : V(k) \to \mathbb{P}^1, \quad (a, b, c, d) \mapsto [c, \bar{d}\,].$$

This is well defined in the new model and a smooth map. We shall see in a moment that p_k is diffeomorphic to a bundle projection. We also use a different model for the bundles. Let $S^3(k) = \{(x, y) \in \mathbb{C} \mid |x|^{2k} + |y|^{2k} = 1\}$. This manifold is diffeomorphic to the sphere S^3 and carries the standard S^1–action $\lambda \cdot (x,y) = (\lambda x, \lambda y)$. We use

$$H_0(-2k) = S^3(k) \times_{S^1} \mathbb{C}(2k)$$

as a model for the complex line bundle $h_k : H(-2k) \to \mathbb{P}^1$. The bundle projection is given by $h_k(x,y;\mu) = [x,y]$.

The following formulas define an inverse pair of diffeomorphisms $\sigma_k : V_0(k) \to H_0(-2k)$ and $\tau_k : H_0(-2k) \to V_0(k)$:

$$\sigma_k(a,b,c,d) = (N_k(c,\bar{d}), b\bar{a} + d^k \bar{c}^k),$$

$$\tau_k(x,y;\mu) = (\frac{y^k - \bar{\mu}\bar{x}^k}{N}, \frac{-\bar{x}^k(1+|\mu|^2 + y^k\mu}{N}, \frac{x}{\sqrt[k]{N}}, \frac{\bar{y}}{\sqrt[k]{N}}).$$

We used the notation

$$N_k : \mathbb{C}^2 \setminus 0 \to S^3(k), \quad (x,y) \mapsto \frac{(x,y)}{|(x,y)|_{2k}}$$

with $|(x,y)|_{2k}^{2k} = |x|^{2k} + |y|^{2k}$ and $N^2 = |y^k - \bar{\mu}\bar{x}^k|^2 + |x^k|^2$. One verifies $h_k\sigma_k = p_k$, hence σ_k is a morphism over the identity of $\mathbb{P}^1$. The verification that σ_k and τ_k are mutually inverse shows that (2) and (4) in Theorem A are diffeomorphic. □

The map $(a,b,c,d) \mapsto (a,b,c^k,d^k)$ induces smooth maps

$$\Phi_k : W(k), W_0(k), V(k) \to W(1), W_0(1), V(1).$$

Let $\varphi_k : \mathbb{P}^1 \to \mathbb{P}^1$, $[u,v] \mapsto [u^k, v^k]$.

(2.1) Proposition. *The diagram*

(2.2)
$$\begin{tikzcd} V(k) \arrow[r, "\Phi_k"] \arrow[d, "p_k"] & V(1) \arrow[d, "p_1"] \\ \mathbb{P}^1 \arrow[r, "\varphi_k"] & \mathbb{P}^1. \end{tikzcd}$$

is a pullback.

PROOF. The diagram is commutative by construction. The map p_1 is a submersion. Therefore the pullback of (φ_k, p_1)

$$V'(k) = \{[u,v], (a,b,c,d) \mid [u^k, v^k] = [c, \bar{d}]\}$$

is a smooth closed submanifold of $\mathbb{P}^1 \times V(1)$. By the universal property, (p_k, Φ_k) induces a smooth map $\omega : V(k) \to V'(k)$. An inverse to this map is given as follows:

Let $x = ([u,v],(a,b,c,d)) \in V'(k)$. There exists a unique $\lambda \in \mathbb{C}^*$ such that $u^k = \lambda c$ and $v^k = \lambda \bar{c}$. This λ depends smoothly on x. Map x to the point $y = (\lambda a, \lambda^{-1} b, u, \bar{v}) \in V(k)$. One checks that $x \mapsto y$ is a well defined smooth inverse of ω. □

Proof of Theorem C. The involution $J : (a, b, c, d) \mapsto ((-1)^{k-1}b, -a, d, -c)$ on $V(k)$ lives, after normalization, on $V_0(k)$. The induced involution $J' = \tau_k J \sigma_k$ on $H_0(-2k)$ is computed to be

$$(x, y; \mu) \mapsto (N_k(\bar{y}M(k)^{-1}, -\bar{x}M(k)); (-1)^k \bar{\mu}).$$

The number $M(k) > 0$ is defined by $M(k)^k = 1 + |\mu|^2$. Let $a(k)$ be the square root of $M(k)$. The diffeomorphism

$$H_0(-2k) \to H_0(-2k), \quad (x, y; \mu) \mapsto (N_k(a(k)x, a(k)^{-1}y); \mu)$$

transforms J' into the involution

$$(x, y; \mu) \mapsto (\bar{y}, -\bar{x}; (-1)^k \bar{\mu}).$$

This finishes the proof. □

There are other simple varieties which have the diffeomorphism type of vector bundles. For the sake of completeness we mention some.

Consider the divisor in $\mathbb{P}^1 \times \mathbb{P}^1$ which is given by

$$C(k) = \{[x, y], [x^k, y^k] \mid [x, y] \in \mathbb{P}^1\}.$$

Let V_k be the complement of $C(k)$.

(2.3) Proposition. *The manifold V_k is diffeomorphic to $H(-2k)$.*

PROOF. The following formula yields a diffeomorphism $V_k \to S^3 \times_{S^1} \mathbb{C}(2k)$

$$[x, y], [u, v] \mapsto (x, y), \frac{u\bar{x}^k + v\bar{y}^k}{-uy^k + vx^k}.$$

We assume in this formula that $|x|^2 + |y|^2 = 1$. □

(2.4) Remark. There are higher dimensional varieties of the same type. The variety $V = SL(n, \mathbb{C})/D$ with D the diagonal subgroup has a quotient map to the flag variety $SL(n, \mathbb{C})/B$ with B the Borel subgroup of upper triangular matrices. This projection is diffeomorphic to a vector bundle. The unipotent radical $U \subset B$ acts by left translations on V and makes this variety into a principal U–bundle. Varieties of this type will be studied elsewhere. ♡

3 The local model

This section serves as a preparation for the determination of compactification divisors.

We denote by $G = G_a$ the additive group $(\mathbb{C}, +)$ of complex numbers. Let $k \in \mathbb{N}_0$ and denote by $\mathbb{C}^2(k)$ the affine space $\mathbb{C}^2$ with action

(3.1) $$G \times \mathbb{C}^2 \to \mathbb{C}^2, \quad t \cdot (x, y) = (x, y + tx^k).$$

The set $F = 0 \times \mathbb{C} \subset \mathbb{C}^2(k)$ is the fixed point set $(k > 0)$ and the other orbits are free. The action on $\mathbb{C}^2(0)$ is free. The action (3.1) has a canonical compactification. Let $\Sigma(k)$ be the Hirzebruch manifold

$$\Sigma(k) = (\mathbb{C}^2 \setminus 0) \times_{\mathbb{C}^*} \mathbb{P}^1; \tag{3.2}$$

it is defined as the $\mathbb{C}^*$–orbit space

$$(x_0, x_1), [y_0, y_1] \sim (\lambda x_0, \lambda x_1),\ [\lambda^k y_0, y_1]$$

for $\lambda \in \mathbb{C}^*$, where $[y_0, y_1]$ are homogeneous coordinates of points in the projective line $\mathbb{P}^1$. The compactified action is

$$t \cdot ((x_0, x_1), [y_0, y_1]) = ((x_0, x_1), [y_0 + t x_0^k y_1, y_1]). \tag{3.3}$$

The action (3.1) is contained as the chart $x_1 = 1 = y_1$. The complement of $\mathbb{C}^2(k)$ in $\Sigma(k)$ consists of the divisors $L = \{x_1 = 0\}$ and $D = \{y_1 = 0\}$. The divisor D is fixed under G and L is G–stable. For $k > 0$ we have $E_0 = \{x_0 = 0\}$ as a further G–fixed divisor.

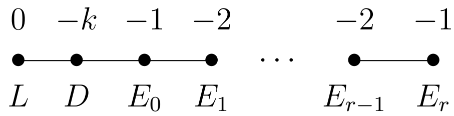
We define a standard equivariant expansion of $\Sigma(k)$. An expansion of $\Sigma(k) = X_0$ is a sequence

$$\pi : X_r \xrightarrow{\pi_r} X_{r-1} \to \ldots \to X_1 \xrightarrow{\pi_1} X_0 \tag{3.4}$$

of σ–processes π_j where each π_j blows up a point $z_{j-1} \in E_{j-1} \subset X_{j-1}$ to the exceptional divisor $E_j = \pi_j^{-1}(z_{j-1})$. We assume that the exceptional set of π, i. e. the set

$$\Sigma_\pi = \{z \mid \pi^{-1}(z) \quad \text{is not a point}\},$$

consists of a single point in F. Moreover, we assume that z_{j-1} is not contained in the proper transforms of the previous E_i, $i < j-1$. The pre-image of Σ_π in X_r is a divisor $E_1 \cup E_2 \cup \ldots \cup E_r$ where E_j also denotes the proper transform of $E_j \subset X_j$. The total transform of $L \cup D \cup E_0$ in X_r has the following linear weighted dual graph.



The dotted part consists of (-2)–curves. This finishes the description of the standard expansion. The next lemma states its equivariance properties.

(3.5) Lemma. *Suppose $r \le k$. The expansions are G–equivariant with respect to an algebraic action on the X_j. The divisors E_j, $0 \le j < k$ consist of fixed points. The divisor E_k carries a free action on $E_k \setminus (E_k \cap E_{k-1})$.*

Proof. The proof is by induction on r. We show that a single σ–process applied to a point of F in $\mathbb{C}^2(k)$ produces a situation which is isomorphic to $\mathbb{C}^2(k-1)$ in a neighbourhood of the exceptional divisor $(k > 0)$.

Let $X = (\mathbb{C}^2 \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}$ with $(x, y; u) \sim (\lambda x, \lambda^{-1} u)$ for $\lambda \in \mathbb{C}^*$. We have the G–action

$$\textbf{(3.6)} \qquad t \cdot (x, y, u) = (x, y + t x^k u^{k-1}; u)$$

on X. The map

$$\textbf{(3.7)} \qquad \pi : X \to \mathbb{C}^2(k),\ (x, y; u) \mapsto (xu, c + yu)$$

is equivariant und blows up the point $(0, c) \in F$. The complement of the proper transform of F is isomorphic to $\mathbb{C}^2(k-1)$ under the map

$$\textbf{(3.8)} \qquad (u, y) \mapsto (1, y; u). \qquad \square$$

Let W_k denote the G–variety which is the complement of the divisor

$$L \cup D \cup E_0 \cup \ldots \cup E_{k-1}$$

in X_k. By the previous lemma, W_k carries a free G–action. The restriction of π to W_k is a G–morphism $\pi : W_k \to \mathbb{C}^2(k)$.

The next proposition gives complete insight into the model situation.

(3.9) Proposition. *There exists an isomorphism $\varphi : \mathbb{C} \times G = \mathbb{C}^2(0) \to W_k$ of G–varieties such that $\psi = \pi\varphi : \mathbb{C}^2(0) \to \mathbb{C}^2(k)$ has the form*

$$\psi(x, u) = (x, p_k(x) + u x^k)$$

with a polynomial p_k of degree less than k. The polynomials p_k are in bijctive correspondence with the standard expansions.

Proof. Consider the morphism

$$\tau_0 : \mathbb{C} \to \mathbb{C}^2(k),\ x \mapsto (x, a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}).$$

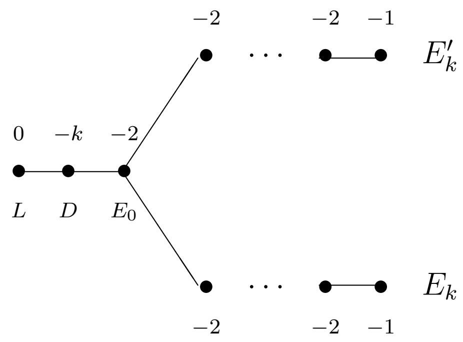
The coefficient a_0 is determined by the fact that π_1 blows up $(0, a_0)$. The proper transform of τ_0 can be written in terms of the chart (3.8) as

$$\tau_1 : \mathbb{C} \to \mathbb{C}^2(k-1),\ x \mapsto (x, a_1 + \ldots + a_{k-1} x^{k-2}).$$

We continue in this manner. The coefficient a_j is determined by π_{j+1}. The proper transform $\tau_k : \mathbb{C} \to W_k$ of τ_0 is a section of the action, i. e. meets each orbit exactly once. We obtain an isomorphism φ if we set $\varphi(x, u) = u \cdot \tau_k(x)$. $\square$

We are now going to describe a $G(2k)$–equivariant compactification of $U(k)$. We start with the G–action on $\mathbb{C}^2(k)$ and $\Sigma(k)$ and apply two standard expansions at the same time. In the generic case, when we start by blowing up two different points in F, we obtain a compactification divisor B with dual graph

(3.10)



Divisor B

The dotted parts contain $k-1$ curves with self-intersection -2. The complement U_k of $B - E_k$ is isomorphic to a variety W_k as in (3.9). Similarly, the complement U'_k of $B - E'_k$ is a variety W'_k. The variety $D(k)$ which is defined to be the complement of $B - E_k - E'_k$ carries a locally trivial G–action since it is the union $U_k \cup U'_k$. More precisely: $D(k)$ is obtained by gluing U_k and U'_k along an isomorphism $\gamma : U_k \setminus E_k \to U'_k \setminus E'_k$. From (3.9) we can derive an explicit formula for γ. Suppose

$$\varphi : \mathbb{C}^2(0) \to U_k, \quad \varphi' : \mathbb{C}^2(0) \to U'_k$$

have been chosen such that

$$\pi\varphi(x,u) = (x, p(x) + ux^k)$$

and similarly with a polynomial $q(x)$ for φ'. Then we have isomorphisms

$$(\mathbb{C} \setminus 0) \times G \xrightarrow{\varphi} U_k \setminus E_k \xrightarrow{\pi} (\mathbb{C} \setminus 0) \times G$$

and similarly for $U'_k \setminus E'_k$. The transition function is given by

(3.11) $$(x,u) \mapsto (x, u + \frac{p(x) - q(x)}{x^k}).$$

Note that $x^{-k}(p(x) - q(x))$ has the form

(3.12) $$g(x) = c_{-k}x^{-k} + \ldots + c_{-1}x^{-1}$$

with $c_{-k} \neq 0$. The gluing (3.12) is exactly the one used by Danielewski. Thus we see that the affine variety $U(k)$ is one of the varieties $D(k)$. Therefore we have shown:

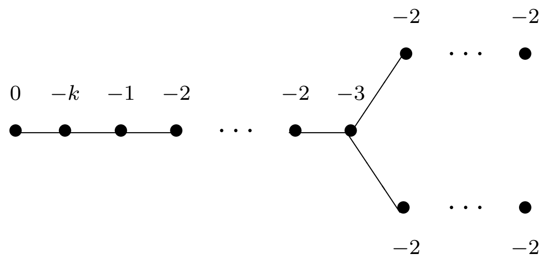
(3.13) Proposition. *$U(k)$ has the normal crossing compactification divisor $B - E_k - E'_k$.* □

By tracing the definitions we see that the compactification of $U(k)$ is $G(2k)$–equivariant.

If the two standard expansions separate at a higher infinitesimal level, then we obtain a similar picture. The only difference is that the polynomials p and q have coefficients of $x^0, \ldots, x^l$ equal so that $x^{-k}(p(x))$ has a pole of order $k - l$.

The resulting compactification divisor has a weighted dual graph of following the shape:

(3.14)



The components of the divisor are, of course, smooth rational curves.

4 The general case

The method of the previous section is easily generalized and yields equivariant compactifications for affine varieties with locally trivial G–action. We begin with the constructive side. The method also allows for an algebraic classification of the resulting varieties. This will be carried out elsewhere.

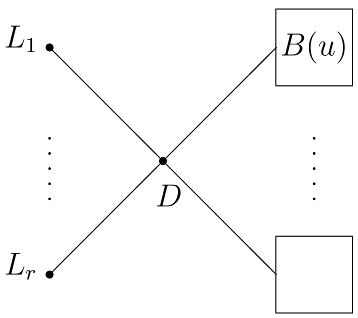
We assume given the following data:

(4.1) Data

1. Z compact Riemann surface.
2. $p : P \to Z$ holomorphic or algebraic $\mathbb{P}^1$–bundle.
3. $s : Z \to P$ a section of p and $L_1, \dots, L_r$ fibres of p over $z_1, \dots, z_r$ such that $U = Z \setminus \{z_1, \dots, z_r\}$ is affine.
4. Set $W = P \setminus (L_1 \cup \dots \cup L_r \cup s(Z))$ and assume that $p : W \to U$ is isomorphic to the projection $\mathrm{pr}_1 : U \times \mathbb{C} \to U$.
5. $\sigma : U \to \mathbb{C}$ a holomorphic function with isolated zeros.
6. A G–action on P which corresponds to $t \cdot (u, z) = (u, z + \sigma(u)t)$ on $U \times \mathbb{C} \cong W$.

Suppose σ has a zero of order k in $u \in U$. We apply a finite number of k–fold standard expansions (in the sense of section 3) to points in the fibre over u. Any such process results in a locally trivial free G–action, as explained in section 3. This action has a projection onto U. If $\sigma(u) \neq 0$, then there is a single orbit over u. If $\sigma(u) = 0$, then there are b orbits over u if we have applied b standard expansions to the fibre over u. The compactification divisor of this action has the following structure of the dual graph:

(4.2)



The box $B(u)$ contains the tree which results from the expansion of the fibre over u. The generic situation for $B(u)$ is a star-shaped tree with center-weight $-n$ and n strings of length k consisting of -2–curves if σ has a zero of order k in u. A typical non-generic case for $k = 2$ is (3.15).

By standard topology the compactification divisor determines the topology at infinity. For the fundamental group see e. g. [4]. For H_1 one has to use the intersection matrix of the weighted tree. Each box $B(u)$ contributes a certain torsion group; the projection onto the surface U is split surjective.

Some of the G–varieties appear as equivariant surfaces in $\mathbb{C}^3$. We communicate some explicit formulas.

Let $p(x) \in \mathbb{C}[x]$, $q(x, y) \in \mathbb{C}[x, y]$. We consider the affine variety $V(p, q)$ in $\mathbb{C}^3$ given by

(4.3) $$P(x,y,z) \equiv p(x)z - q(x,y) = 0.$$

For each $a(x) \in \mathbb{C}[x]$ we have a factorization

(4.4) $$q(x,y+a(x)) - q(x,y) = a(x)r_a(x,y).$$

This implies that for $a(x) = p(x)h(x) \in p(x)\mathbb{C}[x]$ the automorphism of $\mathbb{C}^3$

(4.5) $$(x,y,z) \mapsto (x, y+a(x), z+b(x,y))$$

with $b(x,y) = h(x)r_a(x,y)$ leaves the polynomial $P(x,y,z)$ and hence $V(p,q)$ invariant. Thus $V(p,q)$ has a rather large algebraic automorphism group. The minimal case of this construction gives the G–action

(4.6) $$c \cdot (x,y,z) = (x, y+cp(x), z+r(x,y,c))$$

where $q(x,y+cp(x)) - q(x,y) = p(x)r(x,y,c)$ and $c \in \mathbb{C}$.

When is the action (3.7) free? The orbit of (x,y,z) is obviously free when $p(x) \neq 0$. Suppose $p(x_0) = 0$ and $q(x,y) = \sum_r q_r(x)y^r$. Then $r(x_0,y,c) = c\sum rq_r(x_0)y^{r-1}$. Therefore the orbit of (x_0,y,z) is free if and only if $\frac{\partial q(x_0,y)}{\partial y} \neq 0$. Since $p(x_0) = 0$ implies $q(x_0,y) = 0$, the conditions $(x_0,y,z) \in V(p,q)$ and $\frac{\partial q(x_0,y)}{\partial y} = 0$ imply that y is a multiple root of $q(x_0,y)$. Thus if $q(x_0,y)$ has no multiple roots whenever $p(x_0) = 0$, then $V(p,q)$ carries a free G–action and, moreover, is a regular surface in $\mathbb{C}^3$. Suppose these conditions hold. The projection

$$\rho : V(p,q) \to \mathbb{C}, \quad (x,y,z) \mapsto x$$

is G–invariant. If $p(x_0) \neq 0$, then the fibre is isomorphic to $\mathbb{C}$ and a G–orbit. If $p(x_0) = 0$, then the fibre of ρ over x_0 consists of the orbits $(x_0,y_j) \times \mathbb{C}$, where y_j runs through the roots of $q(x_0,y)$.

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