Ramified coverings of acyclic varieties

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A cyclic ramified covering of complex affine n–space along a hyperplane is again the affine n–space. This makes is plausible that such a covering of an acyclic variety along an acyclic hypersurface is again acyclic. We study this situation from a topological viewpoint and give some applications to the construction of acyclic affine varieties. Exotic algebraic structures on Euclidean spaces have recently attracted some attention, see e. g. [8] [11] [17] [4].

One of our applications is to ramified coverings of homology planes. The homology planes of logarithmic Kodaira dimension one contain a contractible curve [10]. We investigate under which conditions the ramified covering along such a contractible curve is again a homology plane. Moreover we determine which homology planes result from this construction. The result is necessarily somewhat technical and is stated as Theorem (4.9). By this method we are able to find new homology planes which are surfaces in affine three-space. These surfaces are actually equivariantly embedded with respect to the group of covering transformations. No such examples were known before. They are given by the following polynomials, see Theorem (4.1) and section 3 for the terminology.

Theorem A *Suppose $a \geq b > 0$ and $k > 0$ are pairwise coprime integers. Consider the polynomial*

$$P_{a,b,k}(x, y, z) = z^{-k}((z^k x + 1)^a - (z^k y + 1)^b).$$

Then the affine surface

$$X(a, b, k) = \{(x, y, z) \mid P_{a,b,k}(x, y, z) = 1\}$$

is a homology plane. The map $(x, y, z) \mapsto (x, y, z^k)$ is a k–fold cyclic ramified covering

$$\Pi\colon X(a, b, k) \to X(a, b, 1).$$

The basic condition to insure the existence of an acyclic ramified covering is the possibility to fibre the complement of the hypersurface with finite monodromy. As a typical example for the affine space forms which arise from this construction we mention:

Theorem B *Let $n \geq 2$ and let $p\colon \mathbb{C}^n \to \mathbb{C}$ be a quasi-invariant polynomial of weight l such that $p^{-1}(a) = L_a$ is a regular hypersurface for $a = 0$ and connected for $a \neq 0$. Let r and s be coprime natural numbers which are prime to l. Then*

$$X = \{(x, y, u) \mid x^r + y^s + p(u) = 0\}$$

is diffeomorphic to $\mathbb{C}^{n+1}$. The general fibre

$$X_a = \{(x, y, u) \mid x^r + y^s + p(u) = a \neq 0\}$$

has the homotopy type of $\mathbb{Z}/r * \mathbb{Z}/s * L_a$, hence is not contractible if L_a is not contractible.

We remark that the polynomial in Theorem B cannot be transformed into a linear polynomial if the general fibre is not contractible. Polynomials of this type have been considered by DIMCA [8] and KALIMAN [11]. Our proof of Theorem B only uses standard results and methods.

Sections one and two present the general theory and section four gives the application to homology planes. The proof of Theorem B can be found at the of section 3. The general theory further develops some ideas of KALIMAN [11]. In fact we reprove in Theorem (3.8) one of his theorems.

1 Ramified coverings

We begin by recalling the basic definition.

(1.1) Definition. A *k–fold cyclic ramified covering* of smooth manifolds consists of a continuous map $p\colon X \to Y$ between smooth oriented manifolds of the same dimension and a smooth orientation preserving action of the cyclic group $G = \mathbb{Z}/k$ on X such that the following holds:

(1) The map p is G–invariant and induces a homeomorphism of the orbit space X/G with Y.

(2) The fixed point set $V = X^G$ has codimension 2 in X. The image $p(V) = W$ is a smooth submanifold of Y of codimension 2.

(3) The restriction $p\colon X\backslash V \to Y\backslash W$ is a k–fold smooth covering (submersion). The covering transformations are given by the G–action.

(4) The restriction $p\colon V \to W$ is a diffeomorphism. ♡

(1.2) Lemma. *Suppose given a smooth orientation preserving action of G on X with fixed point set V of codimension 2 and free action on $X \setminus V$. Then there is a smooth structure on the orbit space $Y = X/G$ such that the orbit map is a cyclic k–fold ramified covering.*

PROOF. We have the subspace $W = p(V) \subset Y$. There is a unique smooth structure on $Y \setminus W$ such that $p\colon X \setminus V \to Y \setminus W$ is a submersion. Let $\nu\colon E \to V$ denote the G–equivariant normal bundle of V in X and let $\tau\colon U \to X$ denote an equivariant tubular map for $V \subset X$ from an open equivariant disk bundle

$U \subset E$. The bundle ν carries the structure of a complex line bundle such that the G–action on U, given by scalar multiplication of $G \subset S^1 \subset \mathbb{C}^*$, makes τ equivariant. Let ν^k: $E(k) \to V$ denote the k–fold tensor-power of the complex line bundle ν. Then there is a canonical diffeomorphism α: $(E \backslash V)/G \to E(k) \backslash V$. We use the pushout

$$\begin{array}{ccc} (U \setminus V)/G & \xrightarrow{\tau/G} & (X \setminus V)/G \\ \downarrow \alpha & & \downarrow \\ UE(k) & \longrightarrow & Y \end{array}$$

to define a smooth structure of Y in a neighbourhood of $p(V) = W$. Here $UE(k)$ denotes the disk bundle in $E(k)$ which consists of the image of α together with the zero section. □

Suppose, conversely, that we are given a smooth oriented manifold Y with a smooth oriented closed submanifold W of codimension 2. A k–fold cyclic covering of $Y \setminus W$ is determined by a surjective homomorphism of the fundamental group $\pi_1(Y \setminus W) \to G$. Since G is abelian we need only specify a surjection of the first homology group ρ: $H_1(Y \setminus W) \to G$. Let $p^\#$: $X^\# \to Y \setminus W$ denote the covering associated to a given surjection ρ. We want to make $X = X^\# \amalg W$ into a smooth manifold such that the map p: $X \to Y$ which is $p^\#$ on $X^\#$ and the identity on W becomes a k–fold cyclic ramified covering in the sense of (1.1). When this is possible we say: (Y, W, ρ) admits a k–fold ramified covering.

Let t: $T \to W$ denote the unit sphere bundle of the normal bundle of $W \subset Y$ and let σ: $T \to Y \setminus W$ be a restriction to T of a tubular map for $W \subset Y$. The covering $p^\#$ induces via σ a covering q: $S \to T$. In order to attach W as the fixed point set, we need that $s{:} = tq$: $S \to W$ is again a sphere bundle. A necessary condition is that s has connected fibres. This condition is also sufficient. The next Proposition states the relevant topological fact without proof.

(1.3) Proposition. *Let t: $T \to W$ be a principal S^1–bundle and q: $S \to T$ a principal $\mathbb{Z}/k$–bundle. Suppose the pre-image under q of each fibre of t is connected. Then $s := tq$: $S \to W$ has the structure of a principal S^1–bundle such that q satisfies $q(xz) = q(x)z^k$ for $x \in S$ and $z \in S^1$. For the associated complex line bundles E_s and E_t of s and t the map q induces an isomorphism $E_s^{\otimes k} \cong E_t$.* □

In order to apply (1.3) in our context, we state:

(1.4) Proposition. *Let W be connected. The connectivity condition in* (1.3) *on q is equivalent to: The element in $H_1(Y \setminus W)$ which is represented by a fibre of t is mapped under ρ onto a generator of G.* □

Consider the exact homology sequence

$$H_2(Y) \to H_2(Y, Y \setminus W) \xrightarrow{\partial} H_1(Y \setminus W) \to H_1(Y).$$

By the Thom isomorphism and the connectedness of W the group $H_2(Y, Y\backslash W) \cong \mathbb{Z}$ is generated by a fibre of the normal bundle. This shows:

(1.5) Proposition. *Suppose W is connected, $H_1(Y)$ a torsion group with order prime to k and ∂ injective. Then (Y, W, ρ) admits a k–fold ramified covering.* $\square$

The main problem which we address in this note is:

(1.6) Problem. Suppose (Y, W, ρ) admits a k–fold ramified covering $p\colon X \to Y$. When is X acyclic or contractible? $\heartsuit$

The next section describes conditions under which this question can be answered.

2 Fibred manifolds

We assume given a cyclic group $L = \mathbb{Z}/l \subset S^1 \subset \mathbb{C}^*$ and a *connected* oriented S^1–manifold F with orientation preserving right action $(x, \lambda) \mapsto x\cdot\lambda$ of L. Consider the following diagram. Its details will be explained in a moment.

(2.1)
$$\begin{array}{ccccc} X_0 = F(k) \times_L \mathbb{C}^* & \xrightarrow{\Pi} & F \times_L \mathbb{C}^* = Y_0 \\ \downarrow P & & \downarrow p \\ \mathbb{C}^* & \xrightarrow{\pi} & \mathbb{C}^* \end{array}$$

Let $k > 1$ be a natural number which is prime to l. The space Y_0 is the quotient of $F \times \mathbb{C}^*$ under the relation $(x, z) \sim (x\cdot\lambda, \lambda^{-1}z)$ for $\lambda \in L$; and X_0 is the quotient under the relation $(x, z) \sim (x\cdot\lambda^k, \lambda^{-1}z)$; here $F(k)$ stands for F with the twisted action. The maps in the diagram are defined as follows:

$$\begin{aligned} \Pi(x, z) &= (x, z^k) \\ \pi(z) &= z^k \\ P(x, z) &= z^l \\ p(x, z) &= z^l. \end{aligned}$$

The maps are well-defined and the diagram is commutative. The map π is a cyclic covering of degree k and Π is the induced covering. We have an action of $G = \mathbb{Z}/k$ on X_0 given by

(2.2)
$$\mu\cdot(x, z) = (x\cdot\mu^k, \mu^{-1}z), \quad \mu^{kl} = 1.$$

The action is free. It induces a homeomorphism $X_0/K \cong Y_0$. Therefore this action is the group of covering transformations of Π.

We compare the homology of X_0 and Y_0.

(2.3) Proposition. *Let C be a finite field. If $H_i(Y_0; C) = 0$ for $i \geq 2$, then $H_i(X_0; C) = 0$ for $i \geq 2$.*

Proof. We use the fibre bundles P and p of (2.1). The homology of the total space of such bundles can be computed by using the Wang sequence (see [12], p. 67 and [16], p. 456). The Wang sequence for X_0 has the following form:

$$\to H_i(F) \xrightarrow{\alpha_* - \mathrm{id}} H_i(F) \to H_i(X_0) \to H_{i-1}(F) \to .$$

The map α: $F \to F$ is the monodromy of the fibration P. It is given by right multiplication with $\exp(2\pi ik/l) \in L$. The Wang sequence holds for homology with an arbitrary coefficient group. We use C–coefficients. Since we are then working with finite homology groups it suffices for the proof to verify that $\alpha_* - \mathrm{id}$ is injective on $H_j(F)$ for $j \geq 1$. We compare with the Wang sequence of p with monodromy β. By construction $\alpha = \beta^k$. Since $H_i(Y_0) = 0$ for $i \geq 2$, we know from the Wang sequence of p that $\beta_* - \mathrm{id}$ is injective, i. e. β_* does not have the eigenvalue one. Since β has period l and k is prime to l we have $\beta = \alpha^j$ if $jk \equiv 1 \bmod l$. Therefore α_* does not have the eigenvalue one. □

By transfer theory of transformation group theory (see [1], III.2 or [2], II.9), we obtain:

(2.4) Proposition. *Let $R = \mathbb{Z}[k^{-1}]$. Supppose the K–action on $H_*(X_0; R)$ is trivial. Then the map P induces an isomorphism P_*: $H_*(X_0; R) \to H_*(Y_0; R)$.* □

We now combine the considerations above with the situation of the previous section and assume $Y \setminus W = Y_0$, for an oriented smooth manifold Y and a closed connected oriented smooth submanifold W of codimension 2.

For the rest of this section we suppose that P: $(X, W) \to (Y, W)$ is a cyclic ramified covering with ramification locus W such that the restriction P: $X \setminus W = X_0 \to Y_0 = Y \setminus W$ is the covering just constructed. Then we have:

(2.5) Proposition. *Supppose the G–action on $H_*(X_0; R)$ is trivial. Then the map P: $X \to Y$ induces an isomorphism P_*: $H_*(X; R) \to H_*(Y; R)$.*

Proof. By the five-lemma and the previous Proposition, it suffices to show that

$$P_*: H_*(X, X \setminus W; R) \to H_*(Y, Y \setminus W; R)$$

is an isomorphism. By excision, we can restrict to suitable tubular neighbourhoods of W in X and Y. Then we use the Thom isomorphism and notice that P_* maps the Thom class of (X, W) to k–times the Thom class of (Y, W). Since $k \in R$ is invertible, the claim follows. □

(2.6) Lemma. *Consider homology with integral coefficients. Suppose $H_1(Y) \cong 0 \cong H_2(Y)$. Then the exact sequence*

$$0 \to H_2(X, X_0) \xrightarrow{\partial} H_2(X_0) \to H_1(X) \to 0$$

splits.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \cong H_2(X, X_0) \arrow[r, "\partial"] \arrow[d, "P_*"] & H_1(X_0) \arrow[d, "P_*"] \\
\mathbb{Z} \cong H_2(Y, Y_0) \arrow[r, "\cong"] & H_1(Y_0).
\end{tikzcd}$$

The left P_* is multiplication by k (see the proof of (2.5)).

Since F is connected, the homotopy sequence of the fibrations Π and π yield surjections $\pi_1(X_0) \to \pi_1(\mathbb{C}^*)$ and $\pi_1(Y_0) \to \pi_1(\mathbb{C}^*)$. Therefore the map $\pi_*\colon H_1(Y_0) \to H_1(\mathbb{C}^*)$ is an isomorphism and the commutativity of (2.1) shows that $P_*(H_1(X_0)) \subset k\mathbb{Z}$. The diagram above now shows that ∂ is an injection as a direct summand. □

If C is a coefficient group, then a space Z is called C–acyclic if $\tilde{H}_*(Z; C) = 0$.

(2.7) Theorem. *Let C be a finite field. Suppose Y and W are C–acyclic. Then X is C–acyclic.*

Proof. We use homology with coefficients in C. The exact homology sequence of (Y, Y_0) and the Thom isomorphism $H_{i-2}(W) \cong H_i(Y, Y_0)$ show $H_i(Y_0) = 0$ for $i \geq 2$. Hence, by (2.3), $H_i(X_0) = 0$ for $i \geq 2$. The exact homology sequence of (X, X_0) now shows $H_i(X) = 0$ for $i \geq 3$. The cases $i = 1, 2$ follow if we show that the boundary map $\partial\colon H_2(X, X_0) \to H_1(X_0)$ is an isomorphism. The Wang sequence proof of (2.3) shows that $H_1(X_0) \cong C$. If we use this fact together with (2.6) we see that ∂ is an isomorphism. □

(2.8) Corollary. *Suppose Y and W are $\mathbb{Z}$–acyclic. Then X is $\mathbb{Z}$–acyclic.* □

Suppose $H_1(Y; \mathbb{Z}) \cong 0 \cong H_2(Y; \mathbb{Z})$. Then Y admits a unique k–fold cyclic ramified covering with ramification locus W; see (1.4).

(2.9) Theorem. *Suppose the G–action on $H_*(X_0; R)$ is trivial. Suppose Y is $\mathbb{Z}$–acyclic and W is $\mathbb{Z}/k$–acyclic. Then X is acyclic.*

Proof. Taking (2.5) into account, the proof is similar to the proof of (2.7). □

We now deal with the fundamental group of X.

(2.10) Proposition. *Suppose F, W, and Y are simply connected. Then X is simply connected.*

Proof. Since F is simply connected, the homotopy sequences of the fibrations P and p yield isomorphisms

$$p_*\colon \pi_1(Y_0) \cong \pi_1(\mathbb{C}^*), \qquad P_*\colon \pi_1(X_0) \cong \pi_1(\mathbb{C}^*).$$

Let U be a tubular neighbourhood of W in Y. The theorem of Seifert and van Kampen, applied to the pushout

$$\begin{tikzcd} U \setminus W \arrow[r] \arrow[d] & Y \setminus W \arrow[d] \\ U \arrow[r] & Y, \end{tikzcd}$$

and our assumptions imply that $\pi_1(U \setminus W) \to \pi_1(Y \setminus W)$ is an isomorphism. Hence $\pi_1(Y \setminus W) \cong \mathbb{Z}$ is generated by a normal 1–sphere to W. Since W has codimension 2 in X, the inclusion $X \setminus W \to X$ induces a surjection of fundamental groups (transversality theorem). The normal sphere of W in X is mapped under Π: $X_0 \to Y_0$ to k–times the normal sphere of W in Y. Therefore, if we apply the fundamental group functor to (2.1), we conclude that $\pi_1(X \setminus W)$ is generated by a normal sphere. A normal sphere is contractible in X, therefore $\pi_1(X) = 0$. $\square$

3 Affine varieties

We apply the topological results of the previous section to affine varietes. The aim is to construct acyclic or contractible varieties.

We assume given an action $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$, $(\lambda, x) \mapsto \lambda \cdot x$ of $\mathbb{C}^*$ on $\mathbb{C}^n$. For the time being, the action is assumed to be continuous, if we are just dealing with topological properties. But in an algebraic context it seems more reasonable to assume that left translations $x \mapsto \lambda \cdot x$ are polynomial automorphisms.

A polynomial q: $\mathbb{C}^n \to \mathbb{C}$ (or any function) is called *quasi-invariant* of *weight* $l \in \mathbb{Z}$ with respect to the given action if

$$q(\lambda \cdot x) = \lambda^l q(x), \quad x \in \mathbb{C}^n, \ \lambda \in \mathbb{C}^*$$

holds.

We assume given a quasi-invariant polynomial q of weight l. We denote the fibre $q^{-1}(c)$ by F_c. The k–fold *ramified covering* of $\mathbb{C}^n$ along $q^{-1}(0)$ is the affine variety

$$X = \{(u, x) \in \mathbb{C} \times \mathbb{C}^n \mid u^k + q(x) = 0\}, \quad k \geq 1.$$

We shall exhibit conditions on q under which X will be diffeomorphic to Euclidean space.

We begin with some elementary remarks about ramified coverings.

(3.1) Lemma. *If F_0 is regular, then X is regular.*

Proof. Compute the partial derivatives of $p(u, x) = u^k + q(x)$. $\square$

(3.2) Lemma. *The map σ: $X \to \mathbb{C}^n$, $(u, x) \mapsto x$ is a k–fold ramified covering with ramification locus F_0.*

Proof. Given $x \in \mathbb{C}^n$, there exists u such that $u^k + q(x) = 0$. Therefore σ is surjective. Consider the $\mathbb{Z}/k$–action $\mu\cdot(u,x) = (\mu u, x)$ on X. Then σ induces a homeomorphism of the orbit space $X/(\mathbb{Z}/k)$ with $\mathbb{C}^n$. $\square$

(3.3) Lemma. *$F_0 = \{(u,x) \mid u = 0,\ q(x) = 0\}$ is a regular submanifold of X. The intersection of X with $\{u = 0\}$ in $\mathbb{C}^{n+1}$ is transverse.* $\square$

(3.4) Proposition. *The map $q\colon \mathbb{C}^n \setminus F_0 \to \mathbb{C}^*$ is locally trivial with typical fibre F_1 and structure group $\mathbb{Z}/l$.*

Proof. The quasi-invariance of q implies that $(\omega, x) \mapsto \omega\cdot x$ is a $\mathbb{Z}/l$–action on F_c. We use this action to form $\mathbb{C}^* \times_L F_1$, $(\lambda, x) \sim (\lambda\omega, \omega^{-1}\cdot x)$, $\omega \in L = \mathbb{Z}/l$. The map $\varphi\colon \mathbb{C}^* \times_L F_1 \to \mathbb{C}^n \setminus F_0$, $(\lambda, x) \mapsto \lambda\cdot x$ is well-defined and satisfies $q \circ \varphi = \mathrm{pr}_1$. Let $Y = \{(\lambda, x) \mid \lambda^l = q(x)\} \subset \mathbb{C}^* \times (\mathbb{C}^n \setminus F_0)$. Then $\mathrm{pr}_2\colon Y \to \mathbb{C}^n \setminus F_0$ is a quotient map. The morphism $Y \to \mathbb{C}^* \times_L F_1$, $(\lambda, x) \mapsto (\lambda, \lambda^{-1}\cdot x)$ induces an inverse to φ. $\square$

(3.5) Proposition. *Suppose $\pi_1(\mathbb{C}^n \setminus F_0) \cong \mathbb{Z}$ and F_1 is connected. Then $\pi_1(F_c) = 0$.*

Proof. Consider the exact homotopy sequence of the fibration (3.4)

$$0 \to \pi_1(F_1) \to \pi_1(\mathbb{C}^n \setminus F_0) \xrightarrow{\ q_*\ } \pi_1(\mathbb{C}^*) \to 0.$$

The hypothesis implies that q_* is an isomorphism. $\square$

We study the topology of X via $\gamma\colon X \to \mathbb{C}$, $(u,x) \mapsto u$. Let $\Gamma_c = \gamma^{-1}(c)$.

(3.6) Proposition. *The morphism $\gamma\colon X \setminus \Gamma_0 \to \mathbb{C}^*$ is locally trivial with typical fibre F_1 and structure group $\mathbb{Z}/l$.*

Proof. The fibre $\Gamma_c = \{(u,x) \mid q(x) = -u^k, u = c\}$ is isomorphic to F_{c^k}. We have the $\mathbb{Z}/l$–action $\omega\cdot(1,x) = (1, \omega\cdot x)$ on Γ_1. The map $\varphi'\colon \mathbb{C}^* \times \Gamma_1 \to X \setminus \Gamma_0$, $(\lambda, (1,x)) \mapsto (\lambda^l, \lambda^k\cdot x)$ is well-defined. Let $jk \equiv 1 \bmod l$. Then $(\lambda,(1,x))$ and $(\omega^j\lambda, (1, \omega^{-1}\cdot x))$ have the same image under φ' for $\omega \in \mathbb{Z}/l = L$. We therefore use the $\mathbb{Z}/l$–action $(\omega, \lambda) \mapsto \omega^j\lambda$ on $\mathbb{C}^*$ in order to form $\mathbb{C}^* \times_L \Gamma_1$ and obtain from φ' a morphism $\varphi\colon \mathbb{C}^* \times_L \Gamma_1 \to X \setminus \Gamma_0$ which satisfies $\gamma \circ \varphi = p$ with $p\colon \mathbb{C}^* \times_L \Gamma_1 \to \mathbb{C}^*$, $(\lambda, x) \mapsto \lambda^l$. We obtain an inverse to φ as follows. Write $X \setminus \Gamma_0$ as quotient of

$$Z = \{(v,u,x) \mid v^l = u,\ (u,x) \in X \setminus \Gamma_0\}$$

and define

$$Z \to \mathbb{C}^* \times_L \Gamma_1, \quad (v,u,x) \mapsto (v, (1, v^{-k}\cdot x)).$$

If we replace v by $\omega^j v$, $\omega \in L$, the resulting element has the same image under this map. Thus we obtain an induced map $\psi\colon X \setminus \Gamma_0 \to \mathbb{C}^* \times_L \Gamma_1$ which is the inverse of φ. $\square$

(3.7) Proposition. *Under the hypothesis of* (3.5) *we have $\pi_1(X) = 0$.*

Proof. This is an application of (2.10). We remark that the diagram (2.1) corresponds in the present context to the diagram

$$\begin{array}{ccc} X \setminus \Gamma_0 & \xrightarrow{\sigma} & \mathbb{C}^n \setminus F_0 \\ \downarrow \gamma & & \downarrow q \\ \mathbb{C}^* & \xrightarrow{\tau} & \mathbb{C}^* \end{array}$$

with $\tau(z) = -z^k$. □

We can now verify the hypothesis of (2.4) in the present situation. We have the orbit map $\sigma\colon X \to \mathbb{C}^n$ of the $\mathbb{Z}/k$–action $\mu\cdot(u,x) = (\mu u, x)$, see (3.2). This action can be exended to an S^1–action, up to an automorphism of $\mathbb{Z}/k$. The S^1–action $\lambda(u,x) := (\lambda^l u, \lambda^k \cdot x)$ restricts to a $\mathbb{Z}/k$–action, which coincides with the original action up to $\lambda \mapsto \lambda^l$. This uses that k and l are coprime. As a consequence of this S^1–action we see that $\mathbb{Z}/k = G$ acts trivially on $H_i(X;\mathbb{Z})$.

The next result is a direct consequence of (2.9), (3.4), (3.5), and (3.7). It is due to Kaliman [11].

(3.8) Theorem. *We assume the following:*

(1) *$q\colon \mathbb{C}^n \to \mathbb{C}$ is quasi-invariant of weight l.*

(2) *$F_0 = q^{-1}(0)$ is regular and $\mathbb{Z}/k$–acyclic. The integers k and l are coprime.*

(3) *$F_c = q^{-1}(c)$ for $c \neq 0$ is connected.*

Then

$$X = \{(x,u) \in \mathbb{C}^n \times \mathbb{C} \mid u^k + q(x) = 0\}$$

is acyclic. If, moreover, $\pi_1(\mathbb{C}^n \setminus F_0) \cong \mathbb{Z}$, then X is contractible. □

Proof of Theorem B.
(The proof in this case is actually somewhat simpler and more transparent than the one for (3.8).) The hypersurface $\{(y,u) \mid y^s + p(u) = 0\} = Y$ carries a $\mathbb{Z}/s$–action $\lambda\cdot(y,u) = (\lambda y, u)$ which can be extended to an S^1–action since p is quasi-invariant and s prime to l. The orbit space of this $\mathbb{Z}/s$–action is $\mathbb{C}^n$. By transfer theory of transformation group theory as before, Y is $\mathbb{Z}/s$–acyclic. Applying the same reasoning once more, we see that X is acyclic. There is a Milnor fibration $\mu\colon \mathbb{C}^{n+1} \setminus Y \to \mathbb{C}^*$. The fibre has the homotopy type of the join $\mathbb{Z}/s * L_a$ for $a \neq 0$; this follows along the lines of the join-theorem of Sebastiani and Thom [15] or Oka [14]. Since L_a is connected, the space $\mathbb{Z}/s * L_a$ is simply connected. The homotopy sequence of μ now yields $\pi_1(\mathbb{C}^{n+1} \setminus Y) \cong \mathbb{Z}$. From (3.7) we conclude that X is contractible. The statement about the homotopy type of X_a follows again from the join-theorem. □

(3.9) Remark. In connection with (3.8) we mention without proof the following useful fact. Let $p\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then there exists a finite set $S \subset \mathbb{C}$ such that $p\colon \mathbb{C}^n \setminus p^{-1}(S) \to \mathbb{C} \setminus S$ is a locally trivial fibration with general fibre $F_a = p^{-1}(a)$, $a \notin S$. If F_a is connected, then $\pi_1(\mathbb{C}^n \setminus F_a) \cong \mathbb{Z}$. ♡

4 Homology planes

The general results can be applied to construct homology planes which are ramified coverings of other homology planes. In particular, we find new homology planes which are surfaces in $\mathbb{C}^3$ and carry, moreover, a group action which extends to $\mathbb{C}^3$. No such examples were known before. The construction is based on the surfaces which were found in [5]. For details about homology planes see [6], [7].

(4.1) Theorem. *Suppose $a \geq b > 0$ and $k > 0$ are pairwise coprime integers. Consider the polynomial*

$$P_{a,b,k}(x,y,z) = z^{-k}((z^k x+1)^a - (z^k y+1)^b).$$

Then the affine surface

$$X(a,b,k) = \{(x,y,z) \mid P_{a,b,k}(x,y,z) = 1\}$$

is a homology plane. The map $(x,y,z) \mapsto (x,y,z^k)$ is a k–fold cyclic ramified covering

$$\Pi\colon X(a,b,k) \to X(a,b,1).$$

Proof. Let $M(a,b)$ denote the complement in $\mathbb{C}^2$ of the curve $C(a,b) = \{(x,y) \mid x^a = y^b\}$. The variety $M(a,b)$ carries the $\mathbb{C}^*$–action $\lambda\cdot(x,y) = (\lambda^b x, \lambda^a y)$. The map

$$p\colon M \to \mathbb{C}^*, \quad (x,y) \mapsto x^a - y^b$$

is quasi-invariant of weight ab.

A homology plane $X(a,b)$ is obtained as follows: Blow up $\mathbb{C}^2$ in a regular point of $C(a,b)$ and take the complement of the proper transform of $C(a,b)$. The contractible curve W is the part of the exceptional divisor in $X(a,b)$. Moreover,

$$X(a,b) \setminus W \cong M(a,b)$$

carries a $\mathbb{C}^*$–action and a quasi-invariant map p of weight ab. As explained in section 3, we can write $M(a,b)$ as a fibration over $\mathbb{C}^*$ with monodromy of finite order ab. By Theorem (3.8), the k–fold ramified covering of $(X(a,b),W)$ is acyclic. We verify that the map Π of (4.1) is this covering.

By computing partial derivatives one verifies that $X(a,b,k)$ is a regular surface in $\mathbb{C}^3$. It was shown in [5] that $X(a,b)$ equals $X(a,b,1)$. The curve W corresponds to the divisor W' given by $z=0$ in $X(a,b,1)$. Thus Π is a ramified covering of $(X(a,b,1),W')$ with the correct properties. $\square$

In order to place Theorem (4.1) into a wider context, we recall that a homology plane Y of logarithmic Kodaira dimension $\bar\kappa(Y) = 1$ contains a regular contractible curve W ($\mathbb{C}$–curve for short) and this curve is actually unique; see [9], [10]. Thus we ask for ramified coverings of (Y,W). We can apply the results of section 2 if we have a fibration $p\colon Y \setminus W \to \mathbb{C}^*$ with monodromy of finite order. In view of section 3, it is natural to look for $\mathbb{C}^*$–actions on $Y \setminus W$ and quasi-invariant polynomials.

(4.2) Theorem. *Let Y be a homology plane of logarithmic Kodaira dimension one with regular contractible curve $W \subset Y$. Then $Y \setminus W$ carries a $\mathbb{C}^*$–action and there exists a fibration $p\colon Y \setminus W \to \mathbb{C}^*$ with monodromy of finite order.*

In the course of proving this theorem we determine the order of the monodromy. We recall the construction of homology planes, following [9]. We also use the terminology of [7] and add further details to these sources. In accordance with [7] the letter V will later denote a homology plane.

Consider the projective bundle

$$\sigma\colon \Sigma(b) := (\mathbb{C}^2 \setminus 0) \times_{\mathbb{C}^*} \mathbb{P}^1 \to \mathbb{P}^1.$$

Here the Hirzebruch variety $\Sigma(b)$ is defined by the equivalence relation

$$(x, y; u, v) \sim (\lambda x, \lambda y; \lambda^b u, v)$$

for $(x, y) \in \mathbb{C}^2$, $[u, v] \in \mathbb{P}^1$, and $\lambda \in \mathbb{C}^*$. Moreover, $\sigma(x, y; u, v)) = [x, y]$. We have the standard sections of σ

$$E_0 = \{v = 0\}, \qquad E_\infty = \{u = 0\}.$$

We single out a finite set of points

$$z_0 = [1, 0],\ z_1 = [x_1, 1],\ \ldots,\ z_r = [x_r, 1]$$

in $\mathbb{P}^1$. Let F_j denote the fibre of σ over z_j. Consider the divisor

$$D = E_0 + E_\infty + \sum_{j=0}^{r} F_j. \tag{4.3}$$

Let $\zeta_t = E_\infty \cap F_t$. The homology planes in question are constructed from the map $\sigma\colon \Sigma(b) \setminus D \to \mathbb{P}^1$ by adding singular fibres over z_j. The fibre over ζ_0 is isomorphic to $\mathbb{C}$ and the remaining singular fibres are isomorphic to $\mathbb{C}^*$. For our purpose it is useful to add the singular fibres by a patching procedure.

Let the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\chi_j"] \arrow[d, "\mathrm{pr}_1"] & \Sigma(b) \setminus E_\infty \arrow[d, "\sigma"] \\ \mathbb{C} \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

be an affine chart such that $\chi_j(\mathbb{C} \times 0) \subset E_0$ and $\chi_j(0 \times \mathbb{C}) = F_j \setminus \zeta_j$. Choose coprime natural numbers (m_j, n_j) with $m_j > n_j$ and consider the matrix

$$M_j = \begin{pmatrix} u_j & m_j \\ v_j & n_j \end{pmatrix} \in SL(2, \mathbb{Z})$$

with $0 < u_j < m_j$ and $v_j \geq 0$. This matrix is uniquely determined by (m_j, n_j) and called a multiplicity matrix. For each matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ let

$$A\colon \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}, \quad (u, v) \mapsto (u^a v^b, u^c v^d).$$

Consider the Zariski open sets

$$U_j = \chi_j^{-1}(\Sigma(a) \setminus D) \subset \mathbb{C}^* \times \mathbb{C}^*$$

and

$$V_j = M_j^{-1}(U_j) \cup \mathbb{C}^* \times 0 \subset \mathbb{C}^* \times \mathbb{C}.$$

Define a variety $V(M)$ from $\Sigma(b) \setminus D$ by simultaneous patching with the diagrams

$$V_j \xleftarrow{M_j^{-1}} U_j \xrightarrow{\chi_j} \Sigma(b) \setminus D.$$

Here M refers to the family $\{M_j \mid 0 \leq j \leq r\}$. We have $\Sigma(b) \setminus D \subset V(M)$ and σ can be extended canonically to $V(M)$. The effect of passing from $\Sigma(b) \setminus D$ to $V(M)$ is the addition of a singular fibre E_j over z_j which corresponds to $\mathbb{C}^* \times 0 \subset V_j$. It can be shown that, in the terminology of [7], the variety $V(M)$ is obtained from $\Sigma(b)$ by cutting cycles of the dual graph of D at the points $E_s \cap F_j$ with multiplicities (m_j, n_j). The patching with the matrix M_j is also called the standard expansion M_j.

Next we apply an h–fold sprouting expansion to $V(M)$ in a point of E_0. Again we describe a sprouting expansion in terms of a patching procedure. This is based on the morphism

$$\pi\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad (x, u) \mapsto (xu^h + p(u), u),$$

where $p(u)$ is a polynomial of degree at most $h - 1$ with $p(0) \neq 0$. We set $P(x, u) = xu^h + p(u)$ and $A(P) = \{(x, u) \mid P(x, u) = 0\}$. Then π induces an isomorphism

$$\pi_0\colon \mathbb{C} \times \mathbb{C}^* \setminus A(P) \to \mathbb{C}^* \times \mathbb{C}^*.$$

We use this isomorphism for the patching

$$\mathbb{C} \times 0 \cup \pi_0^{-1}(V_0') \supset \pi_0^{-1}(V_0') \xrightarrow{\pi_0} V_0' \subset V(M) \setminus E_0$$

with $V_0' = V_0 \setminus \mathbb{C}^* \times 0$. Let V denote the result. The effect of passing from $V(M)$ to V is to remove E_0 and replace it by the curve W which corresponds to $\mathbb{C} \times 0$ under the last patching. We still have a canonical morphism $\sigma\colon V \to \mathbb{P}^1$ and W is now the fibre over z_0.

The discriminant $\Delta(V)$ of V (in the sense of [7]) is the integer

(4.4)
$$\Delta(V) := bm - \sum_{j=0}^{r} \frac{n_j}{m_j} m, \quad \text{with } m = \prod_{j=0}^{r} m_j.$$

The order of the first homology group $H_1(V; \mathbb{Z})$ is $|\Delta(V)|$, provided $\Delta(V) \neq 0$.

The variety V is called a homology plane if $|\Delta(V)| = 1$ and a $\mathbb{Q}$–homology plane if $\Delta(V) \neq 0$.

We recall from the construction above, that V is specified by the following data, called invariants.

(4.5) Invariants of V.

(1) The twisting integer b.

(2) The singular locus $S = \{z_0, \ldots, z_r\} \subset \mathbb{P}^1$.

(3) The multiplicities $M = \{(m_0, n_0), \ldots, (m_r, n_r)\}$.

(4) The sprouting parameter P.

These invariants are slightly redundant; this will not be discussed here.

The next Theorem determines the k–fold cyclic ramified covering of (V, W). In order to state it, we need more notation. We set:

(4.6) $$kq_j - n_j = d_j m_j, \quad 0 \leq d_j, \quad \text{for } j \geq 1.$$

(4.7) $$ak = \sum_{j=1}^{r} d_j + s + b, \qquad 0 \leq s < k.$$

We assume that k is prime to $m_1 \ldots m_r$. Then the equalities above determine q_j, d_j and s, a.

(4.8) Lemma. *Let $p(u)$ be a polynomial of degree at most $h-1$ with $p(0) \neq 0$ and let $l \geq 0$ be an integer. Then there exists a unique polynomial $q(u)$ of degree at most $h-1$ with $q(0) \neq 0$ such that $q(u) \equiv p(q^l(u)u) \bmod (u^h)$.*

Proof. Write $q(u) = \alpha_{h-1}u^h + \ldots + \alpha_0$ and $p(u) = \beta_{h-1}u^h + \ldots + \beta_0$. The required congruence leads to relations of the type

$$\alpha_j = \beta_j \beta_0^l + r_j$$

where r_j is a polynomial which only involves $\alpha_0, \ldots \alpha_{j-1}$. □

We write $q_{p,l}$ for the polynomial q in (4.8) to show the dependence on the initial data.

(4.9) Theorem. *Suppose k is prime to $\Delta(V)m_1 \ldots m_r$. Then a k–fold cyclic ramified covering of (V, W) exists and has the following invariants:*

(1) *The twisting integer a.*

(2) *The singular locus S.*

(3) *The multiplicities $(km_0, u_0 + sm_0)$, $(m_1, q_1), \ldots, (m_r, q_r)$.*

(4) *The sprouting parameter $xu^{kh} + q_{h,l}(x^k)$.*

Here we have to use the integer l in (4.13) *which is determined by m_0, n_0, k, s, and* (4.11).

In order to prove this Theorem we construct a variety X with the data of the Theorem together with an action of $\mathbb{Z}/k$ and a morphism $\kappa\colon X \to V$ such that the action realizes the deck transformations. The proof will be finished at □□.

The morphism κ arises from a rational morphism

$$\varphi\colon \Sigma(a) \to \Sigma(b), \quad (x, y; u, v) \mapsto (x, y; \frac{u^k}{R(x, y)}, v^k)$$

with the polynomial

$$R(x, y) = y^s \prod_{j=1}^{r} (x - c_j y)^{d_j}.$$

This definition is compatible with the defining equivalence relations when (4.7) is satisfied. Note that φ is defined on $\Sigma \setminus D$ and is a fibrewise map of degree k. We look for conditions on multiplicities and sprouting parameters such that a lifting

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d] & V \arrow[d] \\ \Sigma(a) \arrow[r, "\varphi"] & \Sigma(b) \end{tikzcd}$$

to the singular fibres exists. The next Lemma deals with the local situation.

(4.10) Lemma. *Suppose*

$$\varphi = (\varphi_1, \varphi_2)\colon \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$$

is a rational morphism of the form

$$\varphi_j(x, y) = x^{a(j)} y^{b(j)} p_j(x, y)$$

with $p_j(0, 0) \neq 0$. Then φ extends over the standard expansions

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2\mathbb{Z}), \qquad \tau = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in SL(2, \mathbb{Z})$$

to a rational map $\Phi\colon \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}$ which is defined in a neighbourhood of $(0, 0)$ if and only if

$$\beta \equiv (b, d) \begin{pmatrix} a(1) & b(1) \\ a(2) & b(2) \end{pmatrix} \begin{pmatrix} z \\ -v \end{pmatrix} = 0.$$

Proof. One computes $\tau^{-1}\varphi\sigma$. It has a first component of the form $x^\alpha y^\beta r_1(x, y)$ with $r_1(0, 0) \neq 0$. Therefore $\beta = 0$ is a necessary and sufficient condition for Φ to exist. □

We apply this Lemma in the case

$$\begin{pmatrix} a(1) & b(1) \\ a(2) & b(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -d_j & k \end{pmatrix}.$$

Then the condition of this Lemma is satisfied for

$$\sigma = \begin{pmatrix} ku_j - l_j m_j & m_j \\ v_j + d_j u_j - l_j q_j & q_j \end{pmatrix}, \qquad \tau = \begin{pmatrix} u_j & m_j \\ v_j & n_j \end{pmatrix} =: M_j$$

if l_j is suitably chosen so that σ is a multiplicity matrix. The resulting morphism Φ has the form

$$(x,y) \mapsto (x^k r_1(x,y), x^{-l_j} y r_2(x,y)).$$

This is an ordinary k–fold covering of $\mathbb{C}^* \times \mathbb{C}$ and maps the singular fibre to the singular fibre. This consideration yields already the desired morphism κ over the points z_j for $j \geq 1$.

It remains to deal with z_0. We apply again the previous Lemma, this time to

$$\sigma = \begin{pmatrix} a(1) & b(1) \\ a(2) & b(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s & k \end{pmatrix}, \qquad \tau = \begin{pmatrix} u_0 & m_0 \\ v_0 & n_0 \end{pmatrix}.$$

The conditions of the Lemma call for the relation $-b(n_0 + sm_0) + (km_0)d = 0$. We can solve this with

$$\textbf{(4.11)} \qquad \sigma = \begin{pmatrix} \mu_0 & km_0 \\ \nu_0 & n_0 + sm_0 \end{pmatrix},$$

provided the integers km_0 and $n_0 + sm_0$ are coprime.

(4.12) Lemma. *Under the hypotheses of Theorem* (4.9) *the integers km_0 and $n_0 + sm_0$ are coprime.*

Proof. We substitute $kq_j - n_j = d_j m_j$ for $j \geq 1$ into

$$\Delta(V) = bm - \sum_{j=1}^{r} \frac{n_j}{m_j} m$$

and obtain

$$m(b + \sum_{j=1}^{r} d_j) - n_0 m_1 \dots m_r \equiv \Delta \bmod k.$$

Since, by (4.7), $b + \sum d_j \equiv -s \bmod k$ we obtain

$$(sm_0 + n_0) m_1 \dots m_r \equiv -\Delta \bmod k.$$

Therefore, if k is prime to Δ and $m_1 \dots m_r$, then $sm_0 + n_0$ is prime to k. It is prime to m_0 because m_0 and n_0 are coprime. □

Thus, by the last two Lemmas, we obtain a lifting Φ of φ over the standard expansions σ and τ, and Φ has the form

$$(x,y) \mapsto (xr_1(x,y), x^l y^k r_2(x,y))$$

with

$$l = u_0\nu_0 k - \mu_0 v_0 - \mu_0 s u_0. \tag{4.13}$$

Finally, we have to lift Φ over the sprouting expansion. We disregard r_1 and r_2 in the following notations. The lifting of Φ requires to write $(x, y) \mapsto (Q, Q^l u^k)$ with $Q = xu^{kh} + q_{h,l}(x^k)$ in the form $(z, w) \mapsto (zw^h + p(w), w)$ with a suitable morphism $(x, u) \mapsto (z, w)$. This leads exactly to the conditions of Lemma (4.8) for the definition of $q_{h,l}$. For the lifting of Φ note that we are working on a set where $Q \neq 0$. This finishes the construction of the morphism κ.

It remains to verify that κ is the desired covering. We use the fibrewise $\mathbb{Z}/k$–action $\lambda{\cdot}(x, y; u, v) = (x, y; \lambda u, v)$ on $\Sigma(a)$. It can be verified that this action lifts to an action on X, free on $X \setminus W$, and with fixed point set W. Moreover, κ is the orbit map with respect to this action. □□

We remark that group actions of the type above have also been considered by Miyanishi and Sugie [13].

Proof of Theorem (4.2).
We use the construction of the homology plane (V, W) above. The variety $V \setminus W$ is isomorphic to the variety which is obtained from $\Sigma \setminus D$ by applying the standard expansion over the points $z_1, \ldots, z_r$. We consider the morphism

$$\psi\colon \Sigma(b) \setminus D \to \mathbb{C}^*, \quad (x, y; u, v) \mapsto \frac{(u/v)^\mu}{S(x, y)}$$

with $\mu = m_1 \ldots m_r$ and

$$S(x, y) = y^t \prod_{j=1}^{r} (x - c_j y)^{\mu n_j / m_j}.$$

This is well-defined if we have

$$b\mu - t - \sum_{j=1}^{r} \mu n_j / m_j = 0.$$

The morphism ψ is quasi-invariant with weight μ if we use the fibrewise $\mathbb{C}^*$–action $\lambda(x, y; u, v) = (x, y; \lambda u, v)$. It remains to extend ψ equivariantly over $V \setminus W \supset \Sigma(b) \setminus D$. This uses the following computation: The standard expansion with multiplicities (m, n) transforms

$$\mathbb{C}^2 \to \mathbb{C}, \quad (x, y) \mapsto x^{mc}/y^{nc} f(x, y)$$

with $f(0, 0) \neq 0$ into

$$\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}, \quad (x, y) \mapsto x^c g(x, y)$$

with $g(0, 0) \neq 0$. Thus the latter map is still defined for $y = 0$, i. e. for the singular fibre. Since the multiplicity matrix is

$$\begin{pmatrix} u & m \\ v & n \end{pmatrix} \in SL(2, \mathbb{Z})$$

it is easy to lift the $\mathbb{C}^*$–action. The singular fibre obtains the action $(\lambda, z) \mapsto \lambda^m z$ and the slice representation of the singular fibre is given by λ^{-u}. $\square$

(4.14) Remark. From Theorem (4.9) it is easy to determine a normal crossing divisor for the hypersurfaces of Theorem (4.1). One just has to recall the effect of the sprouting expansion and the cutting of cycles. See [7] for details. ♡

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