NONCOMMUTATIVE HODGE-TO-DE RHAM DEGENERATION AND THE CARTIER ISOMORPHISM

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♠♠♠ Bjorn: [Unfortunately, these notes cover only the first of two lectures.]

Let K be a field of characteristic 0. Suppose that X is a smooth algebraic variety over K. Then we have the de Rham complex $\Omega^{\cdot}_X$ on X, defined in a purely algebraic way. Grothendieck proved that the hypercohomology groups $\mathbb{H}^{\cdot}(X, \Omega^{\cdot}_X)$ are "reasonable" cohomology groups in the sense that they agree with the topologically defined de Rham cohomology groups.

Hodge-to-de Rham spectral sequence:

$$H^p(X, \Omega^q_X) \implies H^{p+q}_{\mathrm{DR}}(X).$$

Theorem 0.1 (Deligne)**.** *Assume that X is proper. Then the Hodge-to-de Rham spectral sequence degenerates.*

This implies that

$$\bigoplus_{p+q=\ell} H^p(X, \Omega^q_X) \simeq H^{\ell}_{\mathrm{DR}}(X).$$

Two proofs:

(1) Analytic
(2) Method of Deligne-Illusie 1987: this is algebraic, and uses a reduction to positive characteristic (even though the degeneration theorem can fail in positive characteristic already for surfaces).

Assume $\tilde{X}$ is a smooth scheme over $W(k)$, where k is a finite field of characteristic p. Let X_k be the special fiber. Let $\mathrm{Fr}\colon X \to X$ be the absolute Frobenius. Assume that Fr lifts to $\tilde{\mathrm{Fr}}\colon \tilde{X} \to \tilde{X}$.

The morphism $\tilde{\mathrm{Fr}}\colon \Omega^{\cdot}_{\tilde{X}} \to \Omega^{\cdot}_{\tilde{X}}$ satisfies $\tilde{\mathrm{Fr}}(f) \equiv f^p \pmod{p}$ and $\tilde{\mathrm{Fr}}^*(df) \equiv d(f^p) \pmod{p} = 0 \pmod{p}$. So $\tilde{\mathrm{Fr}}$ is divisible by p^i on $\Omega^i_{\tilde{X}}$.

We can define $C^{-1}\colon \langle \Omega^{\cdot}_{\tilde{X}}, pd\rangle \to \Omega^{\cdot}_{\tilde{X}}$.

Fact: This is a quasi-isomorphism.

$$C^{-1}\colon \bigoplus_i \Omega^i_{X_k}[-i] \to \Omega^{\cdot}_{X_k}.$$

Fact: $C^{-1}\colon \Omega^i_{\tilde{X}} \to \mathcal{H}^{\cdot}_{\mathrm{DR},X_k}$ is independent of choices, and inverse to the Cartier isomorphism.

Fact: $C^{-1}\colon \bigoplus_i \Omega^i_{X_k}[-i] \to \Omega^{\cdot}_{X_k}$ depends on $\tilde{X} \otimes_{W(k)} W_2(k)$ and depends on $\tilde{\mathrm{Fr}}$ mod p^2 up to a canonical homotopy.

Conclusion: Assume that X_k is given with a lift to a smooth $\tilde{X}$ over $W_2(k)$. Then there exists $C^{-1}\colon \bigoplus_i \Omega^i_{X_k}[-i] \xrightarrow{\sim} \Omega^{\cdot}_{X_k}$.

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We have the Hodge-to-de Rham spectral sequence

$$H^i(X_k, \Omega^j_{X_k}) \implies H^{i+j}_{\mathrm{DR}}(X_k). \tag{1}$$

There is also a local-to-global spectral sequence

$$H^i(X_k, \mathcal{H}^j_{\mathrm{DR}}) \implies H^{i+j}_{\mathrm{DR}}(X_k). \tag{2}$$

If (2) degenerates, then (1) degenerates.

We need a replacement for

(1) differential forms (known in 1962)
(2) the de Rham differential (known in 1982)
(3) Cartier isomorphism (known ∼1995 by topologists, 2005 by algebraists)

Let A be an associative algebra over k. Then A-bimodules are $A^{\mathrm{op}} \otimes A$-modules. An example is A itself.

Definition 0.2. Let $HH_\cdot(A) := \mathrm{Tor}_{A^{\mathrm{op}} \otimes A}(A, A)$.

Theorem 0.3 (H-K-R)**.** *Suppose that A is commutative and* $\mathrm{Spec}\, A$ *is smooth. Suppose that* $\mathrm{char}\, k = 0$ *or* $\mathrm{char}\, k = p > \dim \mathrm{Spec}\, A$. *Then* $HH_i(A) = \Omega^i(A)$.

The analogue of the de Rham differential was discovered by Connes, Loday-Quillen, Tsygen-Feigin (1982).

$$HC_\cdot(A) = K^+_\cdot(A).$$

Use bar-resolution to compute $HH_\cdot(A)$:

$$\begin{array}{c} A \\ \uparrow{\scriptstyle b} \\ A \otimes A \\ \uparrow{\scriptstyle b} \\ A \otimes A \otimes A \\ \uparrow \end{array}$$

One can extend this to

$$\begin{array}{ccccc} & & & & A \\ & & & & \uparrow{\scriptstyle b} \\ & & A & \xrightarrow{B} & A \otimes A \\ & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\ A & \xrightarrow{B} & A \otimes A & \xrightarrow{B} & A \otimes A \otimes A \\ & & \uparrow & & \uparrow \end{array} \tag{3}$$

where B is Connes' differential.

In the H-K-R situation,

$$A$$

$$A \xrightarrow{B} \Omega^1(A)$$

$$A \xrightarrow{B} \Omega^1(A) \xrightarrow{B} \Omega^2(A)$$

Here B is just the usual de Rham differential d.

$K.(A) \to HC.(A)$.

One can extend (3) to the right (and up) as well; this defines the periodic cyclic homology $HP.(A)$.

$HP.(A) = H^{\cdot}_{\mathrm{DR}}(\operatorname{Spec} A)[u, u^{-1}]$ where u corresponds to the shift: $\deg u = 2$.

$$\begin{aligned} HH.(A)[u^{-1}] &\implies HC.(A) \\ HH.(A)[u, u^{-1}] &\implies HP.(A) \end{aligned}$$

Theorem 0.4. *Let k be a field of characteristic* 0*. Let X be a variety over k. Let $\mathcal{A}$ be a sheaf of associative algebras over X. Assume*

1. *X has finite homological dimension*
2. *$\mathcal{A}$-bimod has finite homological dimension*
3. *$\mathcal{A}$-bimod is* Ext*-finite (i.e., for all ℓ and all $M, N \in \mathcal{A}$-bimod,* $\operatorname{Ext}^\ell(M, N)$ *is finite-dimensional over k.*
4. *Technical finite-type assumption.*

Then the Hodge-to-de Rham spectral sequence degenerates.

Theorem 0.5. *Let k be a field of characteristic $p > 0$. Let A be a finite associative algebra over k. Then*

1. *There exists a canonical isomorphism $C^{-1}\colon H_i(HH_i(A)[u, u^{-1}]) \xrightarrow{\sim} HP_i(A)$.*
2. *$HH.(A)[u, u^{-1}] \overset{\sim}{\to}{}^{C} HP.(A)$.*

$$D^b(k - \mathrm{Vect}) \xrightarrow{i^*} \text{St-Hom}^{\cdot}$$

There is also a map i_* in the opposite direction.

$$\operatorname{Ext}^{\cdot}(i_*(k), i_*(k)) = \mathrm{St}^{\cdot}(k),$$

the Steenrod algebra. The composition $i^* i_*$ equals $\mathrm{St}^{\cdot}$ going from $D^b(k - \mathrm{Vect})$ to itself.

$$\begin{aligned} \mathrm{St}^0(k) &= k \\ \mathrm{St}^1(k) &= k \\ \mathrm{St}^2(k) &= k \\ &\vdots \\ \mathrm{St}^{2p-1}(k) &= k \end{aligned}$$

Impose the following conditions: A lifts to $W_2(k)$ and p is larger than the homological dimension of (X, A).

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