ITERATED MODULAR SYMBOLS

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$$\begin{tikzcd}
\boxed{\text{Classical modular symbols}} \arrow[dr] & & \boxed{\text{Multiple zeta values}} \arrow[dl] \\
& \boxed{\text{Iterated modular symbols}} &
\end{tikzcd}$$

1. Arithmetic functions and Dirichlet series

An arithmetic function is a sequence $(a_n)_{n\geq 1}$: e.g., $a_n = \sum_{d|n} d^k =: \sigma_k(n)$ or

$$a_n = \begin{cases} 1, & \text{if } n = p^k \\ 0, & \text{otherwise} \end{cases}$$

or

$$\Phi(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24} =: \sum a_n e^{2\pi i n z}.$$

Many of these have the property $(n, m) = 1 \implies a_{nm} = a_n a_m$.

Question 1.1. What is $\sum_{n\leq N} a_n$?

Question 1.2. Formulas for a_n?

The classical approach to these questions is to use generating functions: define the Dirichlet series

$$L(s) := \sum_{n\geq 1} \frac{a_n}{n^s},$$

which hopefully converges when $\operatorname{Re} s$ is sufficiently large. The property $(n, m) = 1 \implies a_{nm} = a_n a_m$ is equivalent to $L(s) = \prod_{\text{prime } p} L_p(s)$ where $L_p(s) = \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}}$.

Question 1.1 translates into analytic properties of $L(s)$. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \zeta_p(s)$. Then

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\zeta_p'(s)}{\zeta_p(s)};$$

this can be used to answer Question 1.1 for

$$a_n = \begin{cases} 1, & \text{if } n = p^k \\ 0, & \text{otherwise.} \end{cases}$$

using contour integration, etc.

Date: August 9, 2006.

2. MELLIN TRANSFORM

The Mellin transform relates $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and $L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$.

The formal Mellin transform takes $f(z)$ to

$$\int_0^{i\infty} f(z) \left(\frac{z}{i}\right)^s \frac{dz}{z}.$$

where we use the branch of $\log(z/i)$ that is real on the positive imaginary axis. This equals

$$\sum_{n=1}^{\infty} \int_0^{i\infty} a_n e^{2\pi i n z} \left(\frac{z}{i}\right)^s \frac{dz}{z}.$$

If we set $z = iT/n$, this becomes

$$\sum_{n=1}^{\infty} \int_0^{i\infty} a_n e^{-t} \left(\frac{T}{2\pi n}\right)^s \frac{dT}{T} = \sum_{n=1}^{\infty} (2\pi)^{-s} n^{-s} \Gamma(s) = (2\pi)^{-s}\Gamma(s) L_f(s).$$

The factor $(2\pi)^{-s}\Gamma(s)$ should be interpreted as Euler factor at ∞.

General remark: For many (a_n), the series $\sum a_n e^{2\pi i n z}$ has an invariance property with respect to $z \mapsto -1/z$ (or more generally $z \mapsto -1/Nz$).

Example: $\zeta(2s) = \sum \frac{1}{n^{2s}}$ which corresponds to

$$\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z} - 1\right).$$

and

$$\sum e^{2\pi i n^2 z} \sim \sum e^{-2\pi i n^2/z}.$$

Then $\int_0^{i\infty} = \int_0^{i} + \int_i^{i\infty}$ and we can change variables using $z \mapsto -1/z$. This leads to the functional equation relating $\zeta(s)$ and $\zeta(1-s)$ and also an analytic continuation. In this way we get access to the mysterious critical strip for ζ where $0 \le \operatorname{Re} s \le 1$.

The question about distribution of primes translates into questions of poles of $\zeta'(s)/\zeta(s)$.

prime numbers (Euclid) $\leftrightarrow$ zeros of ζ (Riemann) $\leftrightarrow$ Frobenius conjugacy classes in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (Galois

These are three views of the same thing.

3. MODULAR GROUP AND MODULAR FORMS

The function $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is obviously invariant under $z \mapsto z+1$. If it is also (almost) invariant under $z \mapsto -1/z$, then it is (almost) invariant under $\mathrm{PSL}(2,\mathbb{Z}) \subseteq \mathrm{PSL}(2,\mathbb{R})$ in which $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as $z \mapsto \frac{az+b}{cz+d}$.

- Specify the behavior ("invariance") of $f(z)$ with respect to $\mathrm{PSL}(2,\mathbb{Z})$ or a finite index subgroup $G \subset \mathrm{PSL}(2,\mathbb{Z})$.
- Study the Mellin transforms $\int_0^{i\infty}$ of such functions.

The supply of arithmetic functions and their Dirichlet series obtained in this way contains a great amount of arithmetically interesting L-functions.

If we replace $\mathrm{PSL}(2, \mathbb{R})$ by reductive groups, and consider discrete subgroups, we get the Langlands program.

4. CLASSICAL MODULAR SYMBOLS

Consider $\mathrm{PSL}_2(\mathbb{Z}) \supset G$. Each $g \in G$ acts on the upper half plane $H := \{z : \operatorname{Im} z > 0\}$. Let $f(z) = \sum_{n=1}^{\infty} e^{2\pi i z/q}$. Ask for $f(z)(dz)^k$ to be invariant under $g \in G$. In this case, $f(z)$ is called a *modular form of weight* $2k$ *with respect to* G. The Mellin transform studies integrals of the form $\int_0^{i\infty} f(z) z^{s-1}\, dz$.

An element g maps the path from 0 to $i\infty$ to a path between the rational numbers $g(0)$ and $g(i\infty)$.

Systematically study $\int_\alpha^\beta f(z) z^{s-1}\, dz$ where f is a modular form.

The set of classical modular symbols is a simple thing:

$$f \overset{\{\alpha,\beta\}}{\mapsto} \int_\alpha^\beta f(z) z^{s-1}\, dz,$$

where f ranges in a space of modular forms. We will replace the simple integral with an iterated integral.

5. ARITHMETIC APPLICATIONS OF MODULAR SYMBOLS (EXAMPLES)

Recall

$$\Phi(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24} =: \sum \tau(n) e^{2\pi i n z}.$$

The $\tau(n)$ are called Ramanujan numbers. Ramanujan proved that if $(m, n) = 1$ then $\tau(m)\tau(n) = \tau(mn)$. Then

$$\tau(n) = \sum_{d|n} d^{11} + \sum_{n=\Delta\Delta'+\delta\delta'}{}^{\prime} \frac{691}{18}\left(\Delta^8\delta^2 - \Delta^2\delta^8\right) + \frac{691}{6}\left(\Delta^6\delta^4 - \Delta^4\delta^6\right)$$

where the sum is over representations of n as $\Delta\Delta' + \delta\delta'$ such that $\Delta > \delta > 0$ and either $\Delta' > \delta' > 0$ or Δ/n, $\Delta' = n/\Delta$, $\delta' = 0$, $0 < \delta/\Delta \leq 1/2$.

Corollary 5.1 (Ramanujan). $\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}$.

The 691 has something to do with Galois representations.

The values of the Mellin transform of Φ at integers in the critical strip

$$r_k(\Phi) = \int_0^{i\infty} \Phi(z) z^k\, dz$$

for $0 \leq k \leq 10$ are such that $(r_0 : r_2 : r_4) = \left(1 : -\frac{691}{2^2 3^4 5} : \frac{691}{2^3 3^2 5\cdot 7}\right)$ and $(r_1 : r_3 : r_5) = \left(1 : -\frac{5^2}{2^4 3} : -\frac{5}{2^2 3}\right)$.

6. MULTIPLE ZETA VALUES

For a finite sequence of integers $m_1, \dots, m_k \in \mathbb{Z}_{\geq 1}$ with $m_k > 1$, define

$$\zeta(m_1, \dots, m_k) := \sum_{0<n_1<\cdots<n_k} \frac{1}{n_1^{m_1} \cdots n_k^{m_k}}.$$

The condition $m_k > 1$ guarantees convergence.

Problem: Understand the arithmetic of these values. For instance, Euler proved that $\zeta(2m) = b_m \pi^{2m}$ where b_m is a rational number related to a Bernoulli number. But $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, ... are mysterious, apparently unrelated to π and to each other.

More specific problem: Describe the polynomial relations over $\mathbb{Q}$ between values $\zeta(m_1, \dots, m_k)$. We can describe some relations. Where do all these numbers appear? In Euler, and in work of V. Drinfeld in the 1980s and 1990s on quantum groups, in work of M. Kontsevich on quantization, in work of Connes and Marcolli, and in work of D. Zagier.

Relations: Shuffles with repetitions.

Simplest case: For $p, q \geq 2$,

$$\zeta(p)\zeta(q) = \sum_{n_1 \geq 1} \frac{1}{n_1^p} \sum_{n_2 \geq 1} \frac{1}{n_2^q} \qquad = \sum_{n_1<n_2} + \sum_{n_2<n_1} + \sum_{n_1=n_2} = \zeta(p,q) + \zeta(q,p) + \zeta(p+q).$$

Thus for instance, $\zeta(2,2) = (\zeta(2)^2 - \zeta(4))/2$.

More sophisticated: combinatorics of shuffles.

$$\begin{aligned}\zeta(p_1, \dots, p_k)\zeta(q_1, \dots, q_\ell) &= \sum \frac{1}{n_1^{p_1} \cdots n_k^{p_k} {n'_1}^{q_1} \cdots {n'_\ell}^{q_\ell}} \\ &= \sum_{s\colon [1,\dots,k] \coprod [1',\dots,\ell'] \to [1,\dots,k+ell]} \zeta(\cdots),\end{aligned}$$

where $1', \dots, \ell'$ denote a copy of the integers from 1 and ℓ marked so that we can take the disjoint union above.

Result: We get relations of degree 2, homogeneous with respect to the grading $\deg \zeta(m_1, \dots, m_k) := \sum_{i=1}^k m_i$.

Question: Let $Z(m_1, \dots, m_k)$ be formal symbols (indeterminates). Form the graded ring $\mathbb{Q}[Z(m_1, \dots, m_k)]/(\text{shuffle relations})$. What is the dimension of the D-th graded part? Is the map from this ring to $\mathbb{Q}[\zeta(m_1, \dots, m_k)]$ injective? (It turns out that the answer is no: the $\zeta(m_1, \dots, m_k)$ satisfy additional relations.)

7. INTEGRAL REPRESENTATIONS OF MULTIPLE ZETA VALUES

7.1. **Shuffle relations.** Simplest case:

$$\zeta(m) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{m-2}} \frac{dt_{m-1}}{t_{m-1}} \int_0^{t_{m-1}} \frac{dt_m}{1-t_m}.$$

The inner integral is

$$\int_0^{t_{m-1}} \sum_{k\geq 0} t_m^k dt_m = \sum_{k \geq 0} \frac{t_{m-1}^{k+1}}{k+1}.$$

The second inner integral is

$$\sum_{k\geq 0}\frac{t_{m-2}^{k+1}}{(k+1)^2},$$

and continuing, we find that the whole iterated integral gives

$$\sum_{k\geq 0}\frac{1}{(k+1)^m}=\zeta(m).$$

In general,

$$\zeta(m_1,\ldots,m_k)=\underbrace{\int_0^1\frac{dt_1}{t_1}\cdots\int\frac{dt_{m_k-1}}{t_{m_k-1}}\int\frac{dt_{m_k}}{1-t_{m_k}}}\underbrace{\int\cdots\int}\cdots\underbrace{\int\cdots\int},$$

where each block of integrals involves m_k variables, m_{k-1} variables, $\ldots$, m_1 variables, always ending with $\frac{dt}{1-t}$. This is the integral of $\omega[m_k,\ldots,m_1]$ over the region $\Delta_{m_1+\cdots+m_k}$ where

$$\omega[m_k,\ldots,m_1]:=\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots$$

and $\Delta_{m_1+\cdots+m_k}$ is defined by $1\geq t_1\geq t_2\geq\cdots\geq t_{m_1+\cdots+m_k}\geq 0$. (There is a sign issue here that I will not discuss.)

7.2. **Relations involving "shuffles without repetitions".**

$$\zeta(p)\zeta(q)=\pm\int_{\Delta_p}\omega[p]\int_{\Delta_q}\omega[q].$$

We can write $\Delta_p\times\Delta_q$ as a union of copies of Δ_{p+q} indexed by shuffles s without repetitions. Hence (exercise) we obtain

$$\begin{aligned}\zeta(p)\zeta(q)&=\sum_{s\colon\,[1,\ldots,p]\coprod[1',\ldots,q']\overset{\sim}{\to}[1,\ldots,p+q]}\zeta(\cdots)\\&=\sum_s\zeta(a,b).\end{aligned}$$

Funny thing: For $\zeta(2)\zeta(2)$, we obtain a different relation than we obtained earlier. For instance, $\zeta(1,3)$ appears in this new relation: this corresponds to the reshuffling

$$\frac{d\tau}{\tau}\frac{d\tau}{\tau}\frac{d\tau}{1-\tau}\frac{d\tau}{1-\tau}$$

of

$$\frac{d\tau}{\tau}\frac{d\tau}{1-\tau}\frac{d\tau}{\tau}\frac{d\tau}{1-\tau}.$$

7.3. **Third way of getting relations.**

- Choose a regularization of divergent series so that we can allow $m_k=1$.
- Initiate shuffling for convergent and regularized series.
- Eliminate the new values.

Conjecture 7.1. The relations obtained in these three ways are all the relations.

8. CONNECTION WITH MODULAR SYMBOLS

The differential forms $\frac{dt}{t}$ and $\frac{dt}{1-t}$ have poles at $0, 1, \infty$ on the $\mathbb{P}^1(\mathbb{C})$ with coordinate t. The iterated integrals we have been considering have been along the path from 0 to 1 in $\mathbb{P}^1(\mathbb{C})$.

The extended upper half plane $H \coprod \mathbb{P}^1(\mathbb{Q})$ uniformizes $\mathbb{P}^1(\mathbb{C})$ in various ways: namely one can take the quotient of $H \coprod \mathbb{P}^1(\mathbb{Q})$ by

$$\Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : c \equiv 0 \pmod 4 \right\}$$

or

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : b, c \equiv 0 \pmod 2 \right\}$$

and $0, 1, \infty$ are the cusps $\phi(\mathbb{P}^(\mathbb{Q}))$, where $\phi \colon H \coprod \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{C})$ is the uniformization. Now $\phi^* \left(\frac{dt}{t}\right)$ is an Eisenstein series of weight 2 times dz, and so is $\phi^* \left(\frac{dt}{1-t}\right)$.

9. ITERATED MODULAR SYMBOLS

9.1. **Iterated integrals of holomorphic 1-forms on a Riemann surface.** This is a subset of Chen's theory of iterated integrals on C^∞-manifolds.

Let X be a connected Riemann surface; e.g., the upper half plane. Let V be a finite set (not ordered), and let $\{\omega_v : v \in V\}$ be a finite family of holomorphic 1-forms. Let $\{A_v : v \in V\}$ be free associative non-commuting variables. Let $\mathbb{C}\langle\langle A_v : v \in V\rangle\rangle$ be the noncommutative power series ring in these variables. Let $\Omega := \sum_{v \in V} A_v \omega_v$. Let $\gamma \colon [0, 1] \to X$ be a path. The *total iterated integral of* $\{\omega_v\}$ *along* γ is

$$\mathcal{J}_\gamma(\Omega) = 1 + \sum_{n \geq 1} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{m-1}} \gamma^*(\Omega)(t_m).$$

If $\gamma(0) = a$ and $\gamma(1) = z$, we may write this iterated integral also as

$$\begin{aligned} \mathcal{J}_\gamma(\Omega) = \mathcal{J}_a^z(\Omega) &= 1 + \sum_{n \geq 1} \int_a^z \Omega(z_1) \int_a^{z_1} \Omega(z_2) \cdots \int_a^{z_{n-1}} \Omega(z_n) \\ &= 1 + \sum_{n \geq 1} \sum_{(v_1, \ldots, v_n) \in V^n} A_{v_1} \cdots A_{v_n} \int_a^z \omega_{v_1}(z_1) \int_a^z \omega_{v_2}(z_2) \cdots \int_a^{z_{n-1}} \omega_{v_n}(z_n). \end{aligned}$$

(Implicit in the notation is that we have fixed the homotopy class of the path.)

Proposition 9.1.

1. $d\mathcal{J}_a^z(\Omega) = \Omega(z)\mathcal{J}_a^z(\Omega)$, *where d is differentiation with respect to z. Equivalently, "$\mathcal{J}_a^z(\Omega)$ is a horizontal section of the flat connection on X given by $\nabla_\Omega := d - \ell_\Omega$", where ℓ_Ω denotes left multiplication by Ω.*
2. *If U is a simply connected neighborhood of a, then $\mathcal{J}_a^z$ is the* unique *flat section such that $\mathcal{J}_a^a(\Omega) = 1$.*
3. *Any other flat section is uniquely $\mathcal{J}_a^z(\Omega) \cdot C$ for some $C \in \mathbb{C}\langle\langle A_v : v \in V\rangle\rangle$.*
4. $\mathcal{J}_b^z(\Omega) = \mathcal{J}_a^z(\Omega)\mathcal{J}_b^a(\Omega)$. *This is the noncommutative iterated analogue of $\int_b^z = \int_a^z + \int_b^a$.*

The last part of the proposition follows from the earlier parts. Note that the order of the factors in the last part cannot be reversed.

What is the noncommutative iterated analogue of the formula

$$\int_a^z (*_1 + *_2) = \int_a^z *_1 + \int_b^z *_2 \quad ?$$

Proposition 9.2. *Define*

$$\Delta\colon \mathbb{C}\langle\langle A_v\rangle\rangle \to \mathbb{C}\langle\langle A_v\rangle\rangle \hat{\otimes} \mathbb{C}\langle\langle A_v\rangle\rangle$$

by $\Delta(A_v) = A_v \otimes 1 + 1 \otimes A_v$. *Then*

$$\Delta(\mathcal{J}_a^z(\Omega)) = \Delta(\mathcal{J}_a^z(\Omega)) \hat{\otimes} \Delta(\mathcal{J}_a^z(\Omega)).$$

This can be motivated by thinking of $\Delta(\Omega)$ as $\Omega_A + \Omega_B$ where $\Omega_A = \sum A_v\omega_v$ and $\Omega_B = \sum B_v\omega_v$.

Claim 1: $\Delta(\mathcal{J}) = \mathcal{J}\hat{\otimes}\mathcal{J}$ encodes all integral shuffle [i.e., shuffles without repetitions] relations.

Claim 2: $\log \mathcal{J}_a^z(\Omega)$ is a formal series in $\mathbb{C}[[\text{commutators}]] \hookrightarrow \mathbb{C}\langle\langle A_v\rangle\rangle$. This inclusion is isomorphic to the inclusion of the completed free Lie algebra generated by the A_v in the enveloping algebra $\mathcal{U}$.

Functoriality: If $g\colon X \xrightarrow{\sim} X$ is such that $g^*(\omega_v) = \sum_{u\in V} g_{vu}\omega_u$ where $g_{vu} \in \mathbb{C}$, define $g_*(A_u) = \sum_{v\in V} A_v g_{vu}$. Then $\mathcal{J}_{ga}^{gz}(\Omega) = g_*(\mathcal{J}_a^z(\Omega))$.

Remark 9.3. Chen's iterated integrals generalize the theory we have been discussing. One can consider C^∞ manifolds, any (ω_v), De Rham(Loop space).

Let $X = H$ or $X = H \cup \mathbb{P}^1(\mathbb{Q})$. Let (ω_v) be forms of (cusp) modular type.

Reminder:

1. The "action of weight k" of $\gamma \in \mathrm{GL}^+(2,\mathbb{Q})$ on $f\colon H \to \mathbb{C}$ is defined by
$$(f|_{[\gamma]_k})(z) = (\det\gamma)^{k/2} f([\gamma]z)(cz+d)^{-k}$$
2. f is a modular form of weight k for $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ if and only if $f|_{[\gamma]_k}(z) = f(z)$ for all $\gamma \in \Gamma$.
3. f is a cusp form if the Fourier series at ∞ has the form $\sum_{n\geq 1} a_n e^{2\pi i n z/q}$ and $f|_{[\gamma]_k}$ also has this property.

Definition 9.4. $\omega_v = f_v(z) z^{s_v - 1}\, dz$ such that $f_v(z)$ is a modular (cusp) form of weight k_v with respect to a Γ. Call s_v the Mellin argument of ω_v.

Definition 9.5.

1. Suppose that $f_1, \ldots, f_k$ is a sequence of cusp forms with respect to Γ, and $\omega_j(z) := f_j(z) z^{s_j - 1}\, dz$. Consider the iterated Mellin transform
$$\int_{i\infty}^0 \omega_1(z_1) \int_{i\infty}^{z_1} \omega_2(z_2) \cdots \int_{i\infty}^{z_{n-1}} \omega_n(z_n)$$
as a function of $s_1, \ldots, s_n$.
2. The *total Mellin transform* of $(f_v : v \in V)$ and $s = (s_v : v \in V)$ and $\omega_v(z) = f_v(z) z^{s_v - 1}\, dz$ is $\mathrm{TM} = \mathcal{J}_{i\infty}^0(\Omega)$ where $\Omega = \sum \omega_v A_v$. Convergence follows from the f_v being cusp forms.

Assume that Γ is normalized by $g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, $z \mapsto -1/Nz$, and that $f|_{[N^{-1/2}g_N]_k} = \epsilon_f f$ with $\epsilon_f = \pm 1$.

Theorem 9.6 (Functional equation for TM)**.** *If f_v consists of eigenvectors for g_N in the sense above, then*

$$\mathrm{TM}((f_v),(s_v)) = g_{N*}(\mathrm{TM}((f_v),(k_v - s_v)))^{-1}.$$

In the many-dimensional case, we would need more than one symmetry in order to get a full analytic continuation.

10. Multiple Dirichlet series

Suppose that

$$\omega_v(z) = \left(\sum_{n \geq 1} c_{v,n} e^{2\pi i n z} \right) z^{m_v - 1}\, dz$$

with $m_v \geq 1$ and $c_{v,n} = O(n^C)$. Let

$$\begin{aligned} &L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1) \\ &:= (2\pi i z)^{j_k} \sum_{n_1,\dots,n_k \geq 1} \frac{c_{v_1,n_1} c_{v_2,n_2} \cdots c_{v_k,n_k} e^{2\pi i (n_1 + \cdots + n_k) z}}{n_1^{m_{v_1}+j_0-j_1} (n_1+n_2)^{m_{v_2}+j_1-j_2} \cdots (n_1+n_2+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}} \end{aligned}$$

Note that the denominator looks like the denominator in the sum defining a multiple zeta value, but the n_i in the numerator are the differences of the increasing sequence of integers in parentheses in the denominator.

Theorem 10.1.

$$(2\pi i)^{m_{v_1}+\ldots+m_{v_k}} I_{i\infty}^{z}(\omega_{v_k},\dots,\omega_{v_1})$$

is a linear combination with rational coefficients of $L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1)$ with $j_1 \in (0, m_{v_1} - 1)$, $j_2 \in (0, m_{v_2} - 1 + j_1)$, and so on. Here I denotes one coefficient in the total iterated integral.

We must generalize to allow the differential forms to have logarithmic poles at the endpoints of the path of integration.

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