THE ANDRÉ-OORT CONJECTURE AND MANIN-MUMFORD

EMMANUEL ULLMO

1. Introduction

Abelian varieties	Shimura varieties ($\mathcal{A}_{g,1}$)
$A = \mathbb{C}^n/\Gamma$ + polarization	$S = \Gamma\backslash X$ + complex structure X hermitian symmetric space $X \simeq G(\mathbb{R})/Z_G(\mathbb{R})K_\infty$, $G_\mathbb{Q}$ a $\mathbb{Q}$-reductive group
torsion points	CM points
torsion subvarieties $X = B + P$, $P \in A_{\text{tors}}$, B abelian subvariety	component of translate by Hecke operator of a Shimura subvariety $S_H = \Gamma \cap H(\mathbb{R})\backslash X_H$, $H_\mathbb{Q} = T.H_\mathbb{Q}^{\text{der}}$
$[n]$	Hecke operator
Manin-Mumford conjecture	André-Oort conjecture

The conjecture on each side says: A component of the Zariski closure of a set of special points is a special subvariety. Equivalently: Let $Y \subseteq S$ be a subvariety; then there exist special subvarieties $Z_1, \dots, Z_r$ such that every special subvariety $Z \subseteq Y$ is contained in $\bigcup_{i=1}^r Z_i$.

Raynaud proved the Manin-Mumford conjecture. The GRH implies the André-Oort conjecture (Klinger-Yafaev and Ullmo-Yafaev, using some results of Clozel-Ullmo and the strategy of Edixhoven-Yafaev).

Equidistribution plus Galois implies André-Oort and Manin-Mumford.

Define strongly special subvarieties X as follows:

$X = B$ abelian subvariety	special subvariety associated with a semisimple Mumford-Tate group

Equidistribution of strongly special subvarieties:

Fourier series	theorems of Ratner and Margulis-Dani

Classification (and equidistribution) of special subvarieties such that $\#\{Z^\sigma : \sigma \in \mathrm{Gal}(\overline{K}/K)\}$ is bounded, where K is a number field over which S is defined (Ullmo-Yafaev).

$Z = P + B$ with $\mathrm{ord}(P)$ bounded	"T-special subvarieties" with finitely many T's, where T is the center of the Mumford-Tate group

2. Hindry's proof of Manin-Mumford revisited

2.1. **Equidistribution.** Let $A = \mathbb{C}^n/\Gamma$. Let μ_A be the associated normalized Haar measure. Let $Z = B + P$ and let μ_Z be the associated probability measure.

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Proposition 2.1. *Let $Z_n = B_n + P_n$ be a sequence of special subvarieties with* $\mathrm{ord}(P_n)$ *bounded. Then there exists a special subvariety $Z = P + B$ and a subsequence Z_{n_k} such that $\mu_{Z_{n_k}}$ converges weakly to μ_Z and $Z_{n_k} \subseteq Z$ for all sufficiently large k.*

Sketch of proof. Let $G_{\mathbb{Q}} = \mathbb{Q}^n$. Let $\Gamma = \mathbb{Z}^n$. Let $X = \mathbb{Z}^n \backslash \mathbb{R}^n$. Let $\pi \colon \mathbb{R}^n \to X$ be the projection. A (weakly) special subvariety of X is by definition of the form $Z = \pi(H \otimes \mathbb{R})$ for a $\mathbb{Q}$-vector subspace of $H_{\mathbb{Q}} \subseteq G_{\mathbb{Q}}$. Such a Z is endowed with a canonical probability measure. ☐

Lemma 2.2. *Let Z_n be a sequence of (weakly) special subvarieties. Then there exists a (weakly) special subvariety $Z \subset X$ such that $\mu_{Z_{n_k}}$ converges weakly to μ_Z and $Z_{n_k} \subset Z$ for all $k \gg 1$.*

Proof. Weyl's criterion say that we need to check only that for all complex characters χ of X,

$$\int_X \chi \mu_{Z_{n_k}} \to \int_X \chi \mu_Z.$$

☐

We have $A = \Gamma \backslash \mathbb{C}^m \xrightarrow{\sim} \mathbb{Z}^{2m} \backslash \mathbb{R}^{2m}$. The abelian subvarieties of A are a subset of the set of weakly special subvarieties of $\mathbb{Z}^{2m} \backslash \mathbb{R}^{2m}$. We need only check that if B_n is a sequence of special subvarieties of A such that $\mu_{B_{n_k}} \to \mu_Z$, then Z is in fact an abelian subvariety.

2.2. **Galois orbits.** Let K be a number field. Suppose A is defined over K. There exists a finite extension K' of K such that any abelian subvariety B of A is defined over K'. Without loss of generality, assume that $K' = K$.

Proposition 2.3 (Serre)**.** *For all $\epsilon > 0$, there exists a constant $C = C(A, K, \epsilon) > 0$ such that for all $x \in A_{\mathrm{tors}}$ with* $\mathrm{ord}(x) = n$,

$$\#\{x^\sigma : \sigma \in \mathrm{Gal}(\overline{K}/K)\} \geq C n^{1-\epsilon}.$$

Let B be an abelian subvariety. Then there exists an abelian subvariety B' such that $A = B + B'$ with $\#(B \cap B') \ll 1$ (Bertrand); i.e. bounded in terms of A only. We say that the description of Z as $B + P$ is normalized if $P \in B'$ (given by Bertrand). We assume this from now on.

Remark 2.4. Let $Z_n = B_n + P_n$ (in normalized form). Then $\#\{Z_n^\sigma : \sigma \in \mathrm{Gal}(\overline{K}/K)\} \to \infty$ if and only if $\mathrm{ord}(P_n) \to \infty$.

Sketch of proof of Manin-Mumford using Equidistribution + Galois. Let $Y \subset A$ be a subvariety and let $Z_n = B_n + P_n \subset X$ be a Zariski dense (in Y) sequence of special subvarieties. Up to passing to a subsequence, at least one of the following occurs:

1. $\mathrm{ord}(P_n)$ is bounded (μ_{Z_n} is equidistributed to μ_Z, $Z_n \subset Z \subset Y$, so $Z = Y$, so Y is special)
2. $\#\{Z_n^\sigma : \sigma \in \mathrm{Gal}(\overline{K}/K)\} \to \infty$; then using Galois plus a characterization of special subvarieties, we have for all $n \gg 1$ that there exists a special subvariety Z_n' with $Z_n \subsetneq Z_n' \subset Y$.

☐

Characterization of special subvarieties: Let $X \subset A$ be a subvariety, and let $d \geq 2$ be an integer; if $[d]X \subset X$, then X is a torsion subvariety (main technical moral tool in Klinger-Yafaev).

Theorem 2.5 (Serre)**.** *There exists $c = c(A, K)$ such that for all $x \in A_{\text{tors}}$ with $p \nmid \operatorname{ord}(x)$, there exists $\sigma \in \operatorname{Gal}(\overline{K}/K)$ such that*

$$[p^c]x = x^\sigma.$$

(Main use of GRH in the Shimura case.)

Suppose $Z_n \subset Y$ where Z_n is special and $\#\{Z_n^\sigma : \sigma \in \operatorname{Gal}(\overline{K}/K)\} \to \infty$. Choose $p \nmid \operatorname{ord}(P_n)$ with $Z_n = P_n + B_n$, $p \ll \log^A(n)$, $Y \supset Y \cap [p^c]Y \supset Z_n$. Let $Y' = Y \cap [p^c]Y$. So $\deg Y' \ll \log^B(n)$. If $Y \cap [p^c]Y$ is not proper, then Y' is special, and $Z_n \subset Y' = Y$. If $Y \cap [p^c]Y$ is proper, repeat with Y' in place of Y, and iterate. Note that $Z_n = Y^{(i)} \cap [p^c]Y^{(i)}$ is not possible since $\deg(Z_n) \gg$.

We want to explain how, under GRH, Equidistribution + Galois gives the André-Oort conjecture.

Definition 2.6. A *Shimura datum* $(G_{\mathbb{Q}}, X)$ where $G_{\mathbb{Q}}$ is a $\mathbb{Q}$-reductive group and X is a $G(\mathbb{R})$-conjugacy class of morphisms

$$h_0 \colon \mathbb{S} \to G_{\mathbb{R}}$$

where $\mathbb{S} = \mathbb{C}^\times$. Let X^+ be a connected component of X. Then X^+ is a hermitian symmetric space. If $K \subset G(\mathbf{A}_f)$ is a compact open subgroup, then

$$\operatorname{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_f)/K = \coprod_{i=1}^{r} \Gamma_i \backslash X^+$$

where $G(\mathbf{A}_f) = \coprod_{i=1}^r G(\mathbb{Q})\alpha_i K$ and $\Gamma_i = G(\mathbb{Q}) \cap \alpha_i K \alpha_i^{-1}$ is an arithmetic lattice. A component $S = \Gamma \backslash X^+ \subsetneq \operatorname{Sh}_K(G, X)$ is called a *Shimura variety.* Then S has the structure of a quasi-projective scheme. Baily-Borel: If Γ is torsion-free, then S is smooth.

Definition 2.7. Call $(H_{\mathbb{Q}}, X_H) \subset (G_{\mathbb{Q}}, X)$ a *sub-Shimura datum* if $H_{\mathbb{Q}} \subsetneq G_{\mathbb{Q}}$ induces $X_H \subsetneq X$; i.e.,

$$h_0 \colon \mathbb{S} \to G_{\mathbb{R}}$$

factors through $H_{\mathbb{R}}$, and X_H is the $H(\mathbb{R})$-conjugacy class of this $h_0 \in X$. A *sub-Shimura variety* Z of S is a component of the image of $\operatorname{Sh}_{K \cap H(\mathbf{A}_f)}(H, X_H) \hookrightarrow \operatorname{Sh}_K(G, X)$.

Remarks 2.8.

1. Such a Z is equipped with a canonical probability measure μ_Z.
2. Z does not determine (H, X_H): one can change H to $\gamma H \gamma^{-1}$ for any $\gamma \in \Gamma$. We can choose for $H_{\mathbb{Q}}$ the "Mumford-Tate" group on X_H; i.e., the smallest $\mathbb{Q}$-algebraic subgroup $H'_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ such that for all $h \in X_H$, $h \colon \mathbb{S} \to H_{\mathbb{R}}$ factors through $H'_{\mathbb{Q}}$. Then $(H'_{\mathbb{Q}}, X_H) \subset (H_{\mathbb{Q}}, X_H)$ also is a sub-Shimura datum. We fix a Shimura variety $S = \Gamma \backslash X^+$ associated to $(G_{\mathbb{Q}}, X)$. We suppose $G_{\mathbb{Q}} = G_{\mathbb{Q}}^{\text{ad}}$.

Definition 2.9. *Special points* are special subvarieties of dimension 0, i.e, defined by a sub-Shimura datum (H, X_H) with $H_{\mathbb{Q}}$ a $\mathbb{Q}$-torus.

Definition 2.10. A *strongly special subvariety* is defined by $(H_{\mathbb{Q}}, X_H)$ with $H_{\mathbb{Q}}$ semisimple.

Definition 2.11. Fix a $\mathbb{Q}$-torus $T \subset G_{\mathbb{Q}}$. A *T-special subvariety* is associated to $(H_{\mathbb{Q}}, X_H)$ with $H_{\mathbb{Q}} = T \cdot H_{\mathbb{Q}}^{\mathrm{der}}$.

In the case where $T = \{1\}$, T-special is the same as strongly special.

Theorem 2.12 (Clozel, Ullmo)**.** *Let Z_n be a sequence of strongly special (or T-special) subvarieties. Then there exists a strongly special (or T-special) subvariety Z such that there exists a subsequence Z_{n_k} with*

1. *$\mu_{Z_{n_k}}$ converges weakly to μ_Z: i.e., for all $f \in C_b^0(S)$, $\int_S f\mu_{Z_{n_k}} \to \int_S f\mu_Z$.*
2. *$Z_{n_k} \subset Z$ for $k \gg 1$.*

Galois orbits: Let F be a number field over which S is defined.

Theorem 2.13 (Ullmo, Yafaev)**.** *There exists $B > 0$ such that for all $N > 0$ and all $(H, X_H) \subset (G, X)$ with $H_{\mathbb{Q}} = T_{\mathbb{Q}} H_{\mathbb{Q}}^{\mathrm{der}}$ with $\dim T > 0$, and every associated sub-Shimura variety Z,*

$$\#\{Z^{\sigma} : \sigma \in \mathrm{Gal}(\overline{F}/F)\} \gg B^{i(T)} |K_T^m / K_T| (\log|\mathrm{disc}(L_T)|)^N.$$

$K = \prod K_p$, $K_T = T(\mathbf{A}_f) \cap K = \prod K_{T,p}$, K_T^m is the maximal compact open subgroup of $T(\mathbf{A}_f)$, $i(T) = \#\{p$ such that $K_{T,p}^m \neq K_{T,p}\}$, L_T is the splitting field of T.

Theorem 2.14. *Let Z_n be a sequence of special subvarieties of S such that $\#\{Z_n^{\sigma} : \sigma \in \mathrm{Gal}(\overline{F}/F)\}$ is uniformly bounded. Then there exists a torus T and a subsequence Z_{n_k} such that Z_{n_k} is a T-special subvariety.*

Corollary 2.15 (Alternative: Galois/ergodic)**.** *Let $Z_n \subset Y \subset S$ be a Zariski dense (in Y) sequence of special subvarieties. Then after passing to a subsequence, at least one of the following occurs:*

1. *$\mu_{Z_n} \to \mu_Z$, $Z_n \subset Z \subset Y$ for all $n \gg 1$. Then $Y = Z$ is special.*
2. *$\#\{Z_n^{\sigma} : \sigma \in \mathrm{Gal}(\overline{F}/F)\} \to \infty$. Then Klinger-Yafaev prove (under GRH) that for $n \gg 1$, there exists a subvariety Z_n' with $Z_n \subsetneq Z_n' \subset Y$.*

*Proof of Theorem **??**: Ratner's theory.* Fix $G_{\mathbb{Q}}$ a semisimple group over $\mathbb{Q}$. Let $\Omega = \Gamma\backslash G(\mathbb{R})^+$ for an arithmetic lattice $\Gamma \subset G(\mathbb{Q})$. We define $\mathcal{X}$ to be the set of connected closed Lie subgroups $H \subset G(\mathbb{R})^+$ such that

1. $H \cap \Gamma$ is a lattice in H, so in particular $\Gamma\backslash\Gamma H \simeq H \cap \Gamma\backslash H$; $H \cap \Gamma\backslash H$ is endowed with a canonical H-invariant probability measure μ_H.
2. Let $L(H)$ be the subgroup of H generated by unipotent 1-parameter subgroups of H. Then $L(H)$ acts ergodically on $H \cap \Gamma\backslash H$ with respect to μ_H: i.e., any L-invariant μ_H-measurable set in $H \cap \Gamma\backslash H$ has measure 0 or 1.

Properties:

1. Let $H \in \mathcal{X}$. Then there exists a $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}} \subseteq G_{\mathbb{Q}}$ such that $H = H_{\mathbb{Q}}(\mathbb{R})^+$: $H_{\mathbb{Q}}$ is the Mumford-Tate group of H (the smallest $\mathbb{Q}$-subgroup of $G_{\mathbb{Q}}$ such that $H \subset H_{\mathbb{Q}}(\mathbb{R})^+$).
2. If $H = H_{\mathbb{Q}}(\mathbb{R})^+ \in \mathcal{X}$ ($H_{\mathbb{Q}} \simeq R.H_1.\cdots.H_r$), then R is unipotent (for all $i \in \{1, \ldots, n\}$) $H_i(\mathbb{R})$ is not compact.
3. Let L be a subgroup generated by unipotent 1-parameter subgroups. Then there exists a unique $H \in \mathcal{X}$ such that $L \subset L(H) \subset H$ and L acts ergodically on $H \cap \Gamma\backslash H$.

□

Theorem 2.16 (Ratner)**.** *Let L be a group generated by unipotents. Let $H = \mathrm{MT}(L)(\mathbb{R})^+$. Then $\overline{\Gamma\backslash\Gamma L} = \Gamma\backslash\Gamma H = H \cap \Gamma\backslash H$.*

Let $P(\Omega)$ be the set of probability measure on $\Omega = \Gamma\backslash G(\mathbb{R})^+$ and let $Q(\Omega) = \{\mu_H : H \in \mathcal{X}\} \subset P(\Omega)$.

Theorem 2.17 (Mozer-Shah)**.** *$Q(\Omega)$ is compact: If $\mu_n \in Q(\Omega)$, there exists $H \in \mathcal{X}$ and a subsequence $\mu_{n_k} \to \mu_H$ and $H_{n_k} \subset H$ for all $k \gg 1$.*

Principle of proof. Let $S = \Gamma\backslash G(\mathbb{R})^+/K_\infty$. Let $\Omega = \Gamma\backslash G(\mathbb{R})^+ \leftrightarrow \mathcal{X}$. Let Z_n be a sequence of strongly special subvarieties corresponding to $(H_{n,\mathbb{Q}}, X_n)$. (Deligne's definition) $\implies$ $H_{n,\mathbb{Q}}(\mathbb{R})^+ \in \mathcal{X}$. $\mu_{H_n} \to \mu_H$ for some $H \in \mathcal{X}$, $H_n \subset H$ for $n \gg 1$.

Problem: Prove that $H_{\mathbb{Q}}$ is related to Shimura varieties. $\square$

Lemma 2.18. *One can reconstruct (H, X_H) from $H^{\mathrm{der}}_{\mathbb{Q},n} \subset H_{\mathbb{Q}}$.*

Pass from $\Gamma\backslash G(\mathbb{R})^+ \to S = \Gamma\backslash G(\mathbb{R})^+$. $\mu_{Z_n} = \pi_{\mathcal{X}_n} * \mu_{H_n}$, $x_n \in X_{H_n}$ for all $x \in X$.

Université Paris-Sud