Level Structures of Drinfeld Modules – Closing a Small Gap

Stefan Wiedmann

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Introduction

Level structures play an important role in the definition of moduli spaces, because they offer a possibility to rigidify moduli problems. For example level N structures of elliptic curves over $\mathbb{C}$ are defined by an isomorphism of $\mathbb{Z}/N \times \mathbb{Z}/N$ and N-torsion points. If one replaces the base $\mathbb{C}$ by an arbitary scheme then one is forced to regard the N-torsion points as a group scheme which possibly has at some points of the base connected components. So in this case the concept of an isomorphism of a constant group scheme and a the group scheme of N-division points does not work any more. After Drinfeld one weakens this isomorphism condition to a morphism which matches the corresponding Cartier divisors properly. This idea leads to the notion of *generators* of a level structure, respectively in a more general setup to the notion of $\mathscr{A}$-structures and $\mathscr{A}$-generators [KM85, Ch. 1].

A similar situation occurs in the theory of Drinfeld modules, which can be seen as an analogue of elliptic curves in characteristic p. In particluar $\mathscr{A}$-structures are used to define level structures of Drinfeld modules. Unfortunately two slightly different definitions of level structures are found in the literature. For example [Dri76, Def. before Prop. 5.3] and [DH87, Def. 6.1]). In the case of elliptic curves this difference is discussed in [KM85, Th. 5.5.7]. The aim of this article is to prove the analogous result in the case of Drinfeld modules.

1 Drinfeld Modules

1.1 Basic Definitions

Let X be a geometrically connected smooth algebraic curve over the finite field $\mathbb{F}_q$, let $\infty \in X$ be a closed point and let $A := \Gamma(X \setminus \infty, \mathscr{O}_X)$ be the ring of regular functions outside ∞. In this case A is a Dedekind ring.

Let $S/\mathbb{F}_q$ be a scheme, $\mathscr{L}$ a line bundle over S and let $\mathbb{G}_{a/\mathscr{L}}$ be the additive group scheme corresponding to the line bundle $\mathscr{L}$. For all open subsets $U \subset S$ the group scheme is defined by

$$\mathbb{G}_{a/\mathscr{L}}(U) = \mathscr{L}(U).$$

$\mathbb{G}_{a/\mathscr{L}}$ is a smooth commutative group-scheme over S of relative dimension one.

Definition 1.1 ([Dri76])
Let $\operatorname{char}: S \longrightarrow \operatorname{spec} A$ *be a morphism over* $\mathbb{F}_q$. *A* Drinfeld module $E := (\mathbb{G}_{a/\mathscr{L}}, e)$ *consists of an additive group scheme* $\mathbb{G}_{a/\mathscr{L}}$ *and a ring homomorphism*

$$e : A \longrightarrow \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$$

such that:

1) *The morphism $e(a)$ is finite for all $a \in A$ and for all points $s \in S$ there exists an element $a \in A$ such that locally in s the rank of the morphism $e(a)$ is bigger than* 1.

2) The diagram

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "\mathrm{char}"'] & & \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}) \arrow[dl, "\partial"] \\ & \mathscr{O}_S(S) & \end{tikzcd}$$

commutes.

If $S = \mathrm{spec}\, R$ *is affine und if* $\mathscr{L}$ *is trivial, then we will simply write* $E = (R, e)$.

Proposition 1.2
Let S *be a connected scheme. Then there exists a natural number* $d > 0$*, such that for all* $0 \neq a \in A$ *we have* $\mathrm{rk}(e(a)) = q^{-d\deg(\infty)\infty(a)}$. *The number* d *is called the* rank *of the Drinfeld module.*

If $S = \mathrm{spec}(R)$ is an affine scheme and if $\mathscr{L}$ is trivial, then we can show that

$$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}) \cong \mathrm{AddPol}_q(R)$$

where $\mathrm{AddPol}_q(R)$ is the ring of $\mathbb{F}_q$-linear polynomials, i.e. every polynomial $f(X) \in \mathrm{AddPol}_q(R)$ is of the form

$$f(X) = \sum_{i=0}^{n} \lambda_i X^{q^i}.$$

In the affine situation a Drinfeld module is therefore given by a non trivial ring homomorphism e and a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "\mathrm{char}"'] & & \mathrm{AddPol}_q(R) \arrow[dl, "\partial"] \\ & R & \end{tikzcd}$$

If we define $e_a(X) := e(a)$ and if $e_a(X) = \sum_{i=0}^{n} \lambda_i X^{q^i}$ then $\partial(e_a(X)) = \lambda_0$, the coefficient $\lambda_{-d\deg(\infty)\infty(a)}$ is a unit in R and λ_i is nilpotent for $i > -d\deg(\infty)\infty(a)$. If in this case $\lambda_i = 0$ for all $a \in A$ then the Drinfeld module is called *standard*. One can show, that every Drinfeld module is isomorphic to a Drinfeld module in standard form.

By abuse of language the image of the map $\mathrm{char} : S \longrightarrow \mathrm{spec}\, A$ is called the *characteristic* of E. If it consists only of the zero ideal then we say E has *general characteristic*.

1.2 Division Points and Level Structures

Let E be a Drinfeld module of rank d over a base scheme S and let $0 \neq I \subsetneq A$ be an ideal.

Definition 1.3
The contravariant functor $E[I]$ on the category of schemes over S with image in the category of A/I modules defined by

$$T/S \longmapsto \{x \in E(T) \mid Ix = 0\} = \mathrm{Hom}_A(A/I, E(T))$$

for all schemes T/S is called the scheme of I division points.

Properties 1.4

1) $E[I] \subseteq E$ is a closed, finite and flat (sub-)group scheme over S of rank $|A/I|^d$. If $I = (a_1, \dots, a_n)$ for appropriate elements $a_1, \dots, a_n \in A$ then it is

$$E[I] = \mathrm{Ker}(E \xrightarrow{e_{a_1}, \dots, e_{a_n}} E \times_S \cdots \times_S E).$$

In the affine case $S = \operatorname{spec} R$ we have

$$E[I] = \operatorname{spec} R[X]/(e_{a_1}(X), \dots, e_{a_n}(X)).$$

2) If I, J are coprime ideals in A, then

$$E[IJ] \cong E[I] \times_S E[J].$$

3) If I is coprime to the characteristic of the Drinfeld module E then $E[I]$ is étale over S.

4) The group scheme $E[I]$ is compatible with base change, that is for each scheme T/S we have

$$E[I] \times_S T \cong (E \times_S T)[I].$$

Proof Cf. [Leh09], Ch. 2, Prop. 4.1, p. 27 et seq. □

If $S = \operatorname{spec} R$ is an affine Scheme and if $\mathscr{L}$ ist trivial we can use the following lemma to describe the group scheme $E[I]$ by a unique additive polynomial $h_I(X)$ of degree $|A/I|^d$.

Lemma 1.5
Let R be an $\mathbb{F}_q$-algebra. Let $H \subset \mathbb{G}_{a/R}$ be a finite flat subgroup scheme of rank n over R. Then there is a uniquely defined normalized additive polynomial $h \in R[X]$ of degree n such that $H = V(h)$.

Proof Cf. [Leh09], Ch. 1, Lem. 3.3, p. 9. □

If $S = \operatorname{spec} L$ is a field then the characteristic of the Drinfeld module (L, e) is a prime ideal of A. Thus we can define the *height* of (L, e), denoted by h.

In the case of an algebraically closed field we have the following explicit description of the I division points:

Proposition 1.6
Let $0 \neq \mathfrak{p} \subset A$ be a prime ideal and let $I = \mathfrak{p}^n$ for an $n > 0$. Then we have

$$E[\mathfrak{p}^n](L) \cong \begin{cases} (\mathfrak{p}^{-n}/A)^d & \text{for } \mathfrak{p} \neq \operatorname{char} E \\ (\mathfrak{p}^{-n}/A)^{d-h} & \text{for } \mathfrak{p} = \operatorname{char} E. \end{cases}$$

Proof Cf. [Leh09], Ch. 2, Cor. 2.4, p. 24. □

This result motivates the following definition:

Definition 1.7 ([Dri76])
Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a Drinfeld module of rank d over S and let $0 \neq I \subsetneq A$ be an ideal. A level I structure is an A-linear map

$$\iota : (I^{-1}/A)^d \longrightarrow E(S),$$

such that for all prime ideals $\mathfrak{p} \supseteq I$ we have an identity of Cartier divisors

$$(*) \qquad E[\mathfrak{p}] = \sum_{x \in (\mathfrak{p}^{-1}/A)^d} \iota(x).$$

Remark 1.8
1) If I is coprime to the characteristic of E then a level I structure is an isomorphism of group schemes
$$(I^{-1}/A)^d_S \simeq E[I].$$

2) If $E = (R, e)$ the equality of Cartier divisors simply means an equaltiy of the polynomials
$$h_{\mathfrak{p}}(X) = \prod_{x \in (\mathfrak{p}^{-1}/A)^d} (X - \iota(x)).$$

1.3 $\mathscr{A}$-Structures and $\mathscr{A}$-Generators

The notion of a level I structure is closely related to the notion of $\mathscr{A}$-structures and $\mathscr{A}$-generators. We will follow [KM85, Sec. 1.5.].

Definition 1.9
Let E/S be a smooth commutative group-scheme over S of relative dimension one. Let $\mathscr{A}$ be a finite abelian group. A homomorphism of groups

$$\iota : \mathscr{A} \longrightarrow E(S)$$

is called an $\mathscr{A}$-structures on E/S if the Cartier divisor $G/S := \sum_{x\in\mathscr{A}} \iota(x)$ is a subgroup(-scheme) of E/S. In this case we call ι an $\mathscr{A}$-generator of G/S.

Remark 1.10
Using the definition above, a level I structure is an A-linear map $\iota : (I^{-1}/A)^d \longrightarrow E(S)$, such that $\iota|_{(\mathfrak{p}^{-1}/A)^d} : (\mathfrak{p}^{-1}/A)^d \longrightarrow E(S)$ is an $\mathscr{A}$-generator for all primes $\mathfrak{p}|I$, where $\mathscr{A} = (\mathfrak{p}^{-1}/A)^d$ and $G = E[\mathfrak{p}]$.

2 Deformation Theory of Drinfeld Modules and Level Structures

Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a Drinfeld module of rank d over S, $0 \neq I \subsetneq A$ be an ideal and

$$\iota : (I^{-1}/A)^d \longrightarrow E(S)$$

an A-linear map.

The question we want to stress is: If ι is an $\mathscr{A}$-generator for $E[I]$, does this imply that $\iota|_{(\mathfrak{p}^{-1}/A)^d}$ is an $\mathscr{A}$-generator for $E[\mathfrak{p}]$ for all primes $\mathfrak{p}|I$ and vice versa?

In the case of elliptic curves this is true ([KM85, Thm. 5.5.7]), but it is not true automatically, see loc. cit. Prop. 1.11.3 and Rmk. 1.11.4.

At least, Prop. 1.11.3 of loc. cit. tells us, that in the case of a level I structure ι is an $\mathscr{A}$-generator for $E[I]$ too. Even if $E[I]/S$ is an étale scheme we can conclude that $\iota|_{(\mathfrak{p}^{-1}/A)^d}$ is an $\mathscr{A}$-generator for $E[\mathfrak{p}]$ for all $\mathfrak{p}|I$ if ι is an $\mathscr{A}$-generator for $E[I]$, see loc. cit. Prop. 1.11.2. Unfortunately $E[I]/S$ is not étale if I meets the characteristic of the Drinfeld module. The strategy is now to construct a deformation space such that the non étale situation of a given Drinfeld module arises as a base change of the universal Drinfeld module.

In [Leh09, Ch. 3, Prop. 3.3] it is proved that a level I structure of a Drinfeld module is an $\mathscr{A}$-generator for the group scheme $E[I]$. The proof in loc. cit. is based on the construction of deformation spaces of Drinfeld modules, isogenies and level I structures. In the following we will repeat the basic definitions and results.

Definition 2.1

1) *Let $i : A \longrightarrow O$ be a complete noetherian A-algebra with residue field ℓ. Let $\mathscr{C}_O$ be the category of local artinian O-algebras with residue field ℓ and let $\hat{\mathscr{C}}_O$ be the category of noetherian complete local O-algebras with residue field ℓ.*

2) *Let E_0 be a Drinfeld module of rank d over ℓ, and let B be an algebra in $\mathscr{C}_O$. A* deformation *of E_0 over B is a Drinfeld module of rank d over* spec B *which specializes mod $\mathfrak{m}_B$ to E_0. Thus we obtain a functor:*

$$\begin{array}{rcl} \mathrm{Def}_{E_0} : \mathscr{C}_O & \longrightarrow & \text{Sets} \\ B & \longmapsto & \{\text{Isomorphyclasses of Deformations of } E_0\} \end{array}$$

3) *Let $\varphi_0 : E_0 \longrightarrow F_0$ be an isogeny of Drinfeld modules of rank d over ℓ. A* deformation *of φ_0 over B is an isogenie $\varphi : E \longrightarrow F$ where E, F are deformations of E_0 and F_0, such that φ specializes mod $\mathfrak{m}_B$ to φ_0. We obtain a corresponding functor:*

$$\begin{array}{rcl} \mathrm{Def}_{\varphi_0} : \mathscr{C}_O & \longrightarrow & \text{Sets} \\ B & \longmapsto & \{\text{Isomorphyclasses of Deformations of } \varphi_0\} \end{array}$$

4) *Let (E_0, ι_0) be a Drinfeld module of rank d over ℓ equipped with a level I structure ι_0. A* deformation *is a Drinfeld module (E, ι) over B of rank d equipped with an level I structure ι such that E is a deformation of E_0 and ι specializes to ι_0 mod $\mathfrak{m}_B$. We define the functor:*

$$\begin{array}{rcl} \mathrm{Def}_{(E_0,\iota_0)} : \mathscr{C}_O & \longrightarrow & \text{Sets} \\ B & \longmapsto & \{\text{Isomorphyclasses of Deformations of } (E_0, \iota_0)\} \end{array}$$

5) *We will denote the tangent spaces of the functors above by*

$$T_{E_0} := \mathrm{Def}_{E_0}(\ell[\varepsilon]),\ T_{\varphi_0} := \mathrm{Def}_{\varphi_0}(\ell[\varepsilon]),\ T_{(E_0,\iota_0)} := \mathrm{Def}_{(E_0,\iota_0)}(\ell[\varepsilon])$$

where $\ell[\varepsilon]$ is the ℓ-algebra with $\varepsilon^2 = 0$.

Results 2.2

1) The deformation functor Def_{E_0} is pro-represented by the smooth O-algebra $R_0 := O[[T_1, \dots T_{d-1}]]$.

2) The deformation functor Def_{φ_0} is pro-represented by an object in $\hat{\mathscr{C}}_O$.

3) The deformation functor $\mathrm{Def}_{(E_0,\iota_0)}$ is pro-represented by an object in $\hat{\mathscr{C}}_O$.

3 Closing the Gap

Proposition 3.1
Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a Drinfeld module of rank d over S and let $0 \neq I \subsetneq A$ be an ideal. If

$$\iota : (I^{-1}/A)^d \longrightarrow E(S),$$

is an A-linear map such that we have an identity of Cartier divisors

$$(**) \qquad E[I] = \sum_{x \in (I^{-1}/A)^d} \iota(x).$$

then ι is a level I structure, this means $\iota|_{(\mathfrak{p}^{-1}/A)^d} : (\mathfrak{p}^{-1}/A)^d \longrightarrow E(S)$ is an $\mathscr{A}$-generator for all primes $\mathfrak{p}|I$.

In particular you can replace condition (*) in the definition of a level I structure (Def. 1.7) by condition (**) and vice versa.

Proof of the Proposition

It is proved in [Wie04, Ch. 6, Prop. 6.7] by a simple counting argument that Prop. 3.1 is true if the base scheme S is reduced. Without loss of generality we will make the following assumptions:

1) As for coprime ideals I, J in A the scheme of division points splits we can assume that $I = \mathfrak{p}^n$.

2) If $\mathfrak{p}$ is outside of the characteristic of the Drinfeld module the scheme of division points $E[\mathfrak{p}]$ is étale. So we will assume that the characteristic of the Drinfeld module is a prime $\mathfrak{p} \neq (0)$.

3) $S = \operatorname{spec} B$ where B is the localization of a finitely generated A-algebra at a maximal prime Ideal, $\mathfrak{p} := A \cap \mathfrak{m}_B$ is not zero and $\ell := B/\mathfrak{m}_B$ is a finite extension of $A/\mathfrak{p}$.

4) The result is true if it is true for all quotients $B/\mathfrak{m}_B^n$. So we can assume, that B is a local artinian ring with residue field ℓ.

5) We will fix an element $\varpi_\mathfrak{p} \in A$, such that $(\varpi_\mathfrak{p}) = \mathfrak{p}J$ with $\mathfrak{p} \nmid J$. Then we have

$$A \hookrightarrow \hat{A}_\mathfrak{p} \cong A/\mathfrak{p}[[\varpi_\mathfrak{p}]] \hookrightarrow \ell[[\varpi_\mathfrak{p}]] =: O$$

such that $\mathfrak{m}_O \cap A = \mathfrak{p}$ and $\mathfrak{p}O = \mathfrak{m}_O$.

6) As B is artinian, and therefore complete, there is a unique lift of the coefficient field ℓ to B and we can consider B as an object of $\mathscr{C}_O$.

7) We fix a Drinfeld module $E_0 = (e^{(0)}, \ell)$ and a level $\mathfrak{p}^n$ structure ι_0.

8) Let E be the universal deformation of E_0 and O as above. Then E is defined over
$$R_0 := O[[T_1, \dots, T_{d-1}]] \cong \ell[[T_0, \dots, T_{d-1}]]$$
for $T_0 := \varpi_{\mathfrak{p}}$. It is a complete regular local ring of dimension d. In especially R_0 is integral, the map $\mathrm{char} : A \longrightarrow R_0$ is injective and the Drinfeld module has general characteristic.

To prove the proposition we will follow the arguments of [Leh09, 3.3.1]. The main difference is now to use $\mathfrak{p}^n$ instead of $\mathfrak{p}$.

Let $p_{\mathfrak{p}^n} : E \longrightarrow E/E[\mathfrak{p}^n]$ the canonical quotient isogeny of Drinfeld modules with kernel $E[\mathfrak{p}^n]$. The corresponding polynomial $p^\sharp_{\mathfrak{p}^n} = h_{\mathfrak{p}^n} \in R_0[X]$ is an additive, normalized and separable polynomial of degree $|A/\mathfrak{p}^n|^d$. We define L to be a splitting field of $h_{\mathfrak{p}^n}$ over the field of quotients $\mathrm{Quot}(R_0)$. Using the zeros $V(h_{\mathfrak{p}^n})$ of $h_{\mathfrak{p}^n}$ in L we define the R_0-algebra $R_{h_{\mathfrak{p}^n}} := R_0[V(h_{\mathfrak{p}^n})]$ inside L. It is an integral and finite extension of R_0 because $h_{\mathfrak{p}^n}$ is normalized over R_0.

We will prove
$$R_{h_{\mathfrak{p}^n}} \cong R_n$$
where R_n is the base ring of the universal Drinfeld module of deformations of E_0 and a level $\mathfrak{p}^n$ structure over ℓ. We will use induction on n:

If $n = 0, 1$ then nothing is to prove. For $n > 1$ we have:
$$R_n := R_{n-1}[[S_1, \dots, S_d]]/\mathfrak{a}$$
where $\mathfrak{a}$ is the ideal generated by elements of the form $e^\sharp_{\varpi_{\mathfrak{p}}}(S_i) - \iota_r(x_i) + e^\sharp_{\varpi_{\mathfrak{p}}}(\tilde{y}_i)$. These elements are normalized polynomials in S_i, so
$$R_n := R_{n-1}[S_1, \dots, S_d]/\mathfrak{a}$$

We can find elements $\tilde{x}_1, \dots, \tilde{x}_d$ in $R_{h_{\mathfrak{p}^n}}$ such that $e^\sharp_{\varpi_{\mathfrak{p}}}(\tilde{x}_i) = x_i$ and $\tilde{x}_1, \dots, \tilde{x}_d$ generates $V(h_{\mathfrak{p}^n})$ as an $A/\mathfrak{p}^n$-module. We define a map of R_{n-1}-algebras:
$$\begin{array}{ccc} R_n & \longrightarrow & R_{h_{\mathfrak{p}^n}} \\ S_i & \longmapsto & \tilde{x}_i - \tilde{y}_i \end{array}$$

By assumption the map is well defined and surjective. As both rings have dimension d, it is an isomorphism.

Corollary 3.2
$R_{h_{\mathfrak{p}^n}}$ is a regular algebra in $\hat{\mathscr{C}}_O$.

Now we can use the setup of [Leh09, Prop. 3.3.1] to show that $E_{\mathfrak{p}^n} := E \otimes_{R_0} R_{h_{\mathfrak{p}^n}}$ and a corresponding lift of ι_0 is the universal deformation of "level" $\mathfrak{p}^n$, whereas condition (**) of Prop. 3.1 replaces condition (*) in the definition of a level I structure. We only have to check that [Leh09, Lem. 3.3.2] remains correct with $\mathfrak{p}$ is replaced by $\mathfrak{p}^n$, but this can be seen easily.

We have proved that $R_{h_{\mathfrak{p}^n}}$ is an integral ring and that it is the base ring of the universal deformation in the sense described above. We conclude that Prop. 3.1 is true on the level of the universal deformation. As everything commutes with base change we are done.

4 References

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