Special Points and subvarieties of abelian varieties*

1. Lecture: Manin-Mumford conjecture in characteristic zero.
2. Lecture: Manin-Mumford conjecture in positive characteristic and related spaces.
3. Lecture: A combination of the Mordell-Lang and the André-Oort conjecture.

1 First Lecture

Setup: Let X denote an algebraic variety over an algebraically closed field k, Σ an infinite countable subset of "special" points. We state the following:

Question 1.1 *For an algebraic closed subvariety $Z \subset X$; what can the following be?:*

- $\Sigma' \subset \Sigma$ *arbitrary,* $Zar(\Sigma') =: Z$*?*
- $Z' \subset Z$ *irreducible component,* $Zar(Z' \cap \Sigma) = Z'$*?*

Another formulation of the previous question might be: what irreducible $Z \subset X$ fulfill $Zar(Z \cap \Sigma) = Z$?

1.1 Manin-Mumford Conjecture

Let $A = X$ be an abelian variety over $\mathbb{C}$, say $A \cong \mathbb{C}^g/\Lambda$, $\Sigma := A_{\text{tor}} = \mathbb{Q}\Lambda/\Lambda \cong (\mathbb{Q}/\mathbb{Z})^{2g}$. Recall the following:

Definition 1.2 *An irreducible closed subvariety Z of X is called special iff $Z = a + B$, with $a \in X_{tor}$ and $B \subset X$ an abelian subvariety.*

*Notes of a short-course given by Prof. Richard Pink (ETH-Zürich) at the Congress "Diophantine Geometry" held at the University of Göttingen between 17th and 22nd of Juni 2004.

Now we are in condition to state the:

Conjecture 1.3 (Manin-Mumford) *Any $Z \subset A$ irreducible closed subvariety which satisfies $Zar(Z \cap A_{tor}) = Z$ is special.*

Theorem 1.4 *The Manin-Mumford conjecture holds.*

The proof follows several steps, and what follows will be needed:

Theorem 1.5 ([3];[4]) *Let $F : A \to A$ be an endomorphism with eigenvalues not a root of unity (for the corresponding F^* in $H^1(A, \mathbb{C})$). For $Z \subset A$ irreducible and closed with $F(Z) = Z$, then Z is special.*

PROOF: [Only for the case when all eigenvalues λ have absolute values greater than 1.] Wlog (without lost of generality) we can suppose that the stabilizer of Z is trivial, since by hypothesis it will be fixed by F and hence one can quotient both A and Z by the stabilizer of Z. Since the eigenvalues are non zero, the degree of F is finite, say d. Hence $F^{-1}(Z) \supset \cup_{a \in Ker(F)} a + Z$ and all translations are different, therefore instead of $\supset$ we have an equality. Now let $c := codim Z$. Then $cl(F^{-1}(Z)) \in H^{2c}(A, \mathbb{C})$ and since the translations on the abelian variety induce the identity on the cohomology, we have:

$$d\ cl(Z) = F^*(cl(Z)),$$

which means that d is an eigenvalue of F^* in $H^{2c}(A, \mathbb{C}) = \wedge^{2c} H^1(A, \mathbb{C})$, and then must be $d = \lambda_1 \dots \lambda_{2c}$, where the $\lambda_1, \dots \lambda_{2g}$ are all the eigenvalues of F^* in $H^1(A, \mathbb{C})$. But $d = \lambda_1 \dots \lambda_{2g}$, what implies that $g = c$, hence Z is one dimensional, and then $Z = point$. □

Let's come back to the proof of the conjecture.

PROOF: [of the Manin-Mumford conjecture for $char = 0$.] As in the previous theorem, wlog we suppose that $Stab_A(Z) = 0$. Since a special subvariety is a translation of an abelian subvariety by a torsion point, it will have zero stabilizer iff it's dimension is zero. So let's prove this.
Suppose that the field of definition of the subvariety Z is a finitely generated field $K \subset \mathbb{C}$ over $\mathbb{Q}$. For any rational prime sufficiently large one has a field inclusion $K \hookrightarrow L =$ finite extension of $\mathbb{Q}_p$, such that A has good reduction over the ring of integers $\mathcal{O}_L$.
(* Missing "4th" point, not given in the lecture.*)

Let's look at the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Gal(\bar{L}/L^{nr}) & \longrightarrow & Gal(\bar{L}/L) & \longrightarrow & Gal(L^{nr}/L) & \longrightarrow & 1 \\ & & & & & & \downarrow \cong & & \\ & & & & & & Gal(\bar{l}/l). & & \end{array} \tag{1}$$

Define σ as the element of $Gal(L^{nr}/L)$ which corresponds to the Frobenius of l.
Let's call l the residue field of L, q it's cardinal, $\bar{A}$ the reduction of A to this field, and L^{nr} the maximal unramified subextension of $\bar{L}/L$.
A result of A.Weil says that exists a monic polynomial $P \in \mathbb{Z}[T]$, whose roots have absolute value $\sqrt{q}$ and $P(Frob_l)$ annihilates $\bar{A}(\bar{l})$, where $Frob_l$ is the Frobenius endomorphism acting on the reduced abelian variety $\bar{A}$.
Now we want to prove the following:

Theorem 1.6 (Raynaud) *There exists n such that p^n annihilates $Ker(A(L^{nr})_{tor} \to \bar{A}(\bar{l}))$.*

Proof: Let $\mathcal{A}$ be a model of A over $\mathcal{O}_L$, smooth. It's formal completion is isomorphic to $Spf(\mathcal{O}_L[[X_1, \ldots, X_g]])$. Now denote $L' := \widehat{L^{nr}}$ the completion, and define:

$$\mathcal{U}_r := \{(x_1, \ldots, x_g) \in (L')^g \mid \forall i | \; x_i \mid < r\}.$$

They have the following properties:

1. $\mathcal{U}_1 = Ker(A(L') \to \bar{A}(\bar{l}))$.
2. $\exists r < 1 : \mathcal{U}_1 = \mathcal{U}_r$.
3. $\exists \epsilon > 0 : log : \mathcal{U}_\epsilon \to \mathcal{U}_\epsilon \subset LieA \otimes L'$.
4. $\exists n : p^n \mathcal{U}_r \subset \mathcal{U}_\epsilon$.

Once we obtained these properties, we're almost done, since:

$$p^n(\mathcal{U}_{1,\text{tor}}) = p^n \mathcal{U}_{r,\text{tor}} \subset \mathcal{U}_{\epsilon,\text{tor}},$$

and this last term is zero since in characteristic zero the Lie algebra has no torsion. The result then follows by (1). □

We continue with the proof of the conjecture by dividing it in two cases:

1. $A(L^{nr})_{\text{tor}} \cap Z$ is dense in Z.

2. $\left[A(\bar{L}) \setminus A(L^{nr})\right]_{\text{tor}} \cap Z$ is dense in Z.

Suppose (1) is true. Then put $\Sigma_0 := p^n A(L^{nr})_{\text{tor}}$. Then there exists an $a \in A$ such that $(a + \Sigma_0) \cap Z$ is dense in Z (Σ_0 is of finite index in $A(L^{nr})_{\text{tor}}$). By replacing Z by $Z - a$ we can assume that $\Sigma_0 \cap Z$ is dense in Z.
Observe that (1.6) says that $P(\sigma)$ annihilates Σ_0. If $P(T) = T^m - \sum_{i=0}^{m-1} a_i T^i$, then it's companion matrix is:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{m-1} & a_m \end{pmatrix}; \quad F : A^m \to A^m,$$

with eigenvalues λ, such that $\mid \lambda \mid > 1$. If we denote $u : A(L^{nr}) \to A^m(L^{nr})$ the "σ-normal" embedding $x \mapsto (x, \sigma(x), \dots, \sigma^{m-1}(x))$, then $F(u(x)) = u(\sigma(x)) = \sigma(u(x))$ if $x \in \Sigma_0$. Therefore $F = \sigma$ on $u(\Sigma_0)$.
Set $\tilde{Z} := Zar(u(\Sigma_0 \cap Z))$, then $\Rightarrow pr_1(\tilde{Z}) = Z$. Since F is proper and σ is a Galois automorphism, $F = \sigma$ also on $\tilde{Z}$ and $\tilde{Z}$ is invariant under σ: $F(\tilde{Z}) = \sigma(\tilde{Z}) = \tilde{Z}$. Now take an irreducible component Z' of $\tilde{Z}$. Because of the last equality, what F can do with the components of $\tilde{Z}$ is just to act as a permutation on then, and hence we find an n such that $F^n(Z') = Z'$. Finally is Z' special and therefore also $pr_1(Z')$ inside Z, but by assuption Z was irreducible; then Z is special and we finished the proof of the first case.
We're going to show that the second case cannot occur. Denoting $A(\bar{L})_{\text{tor}} \setminus A(L^{nr})$ by Σ_1, (2) assumes that $\Sigma_1 \cap Z$ is dense in Z.

Proposition 1.7 (Boxall) *Assume $A[p] \subset A(L^{nr})$ (and $A[4]$ if $p = 2$). Then $\forall x \in \Sigma_1\ \exists \tau \in Gal(\bar{L}/L^{nr}) : \tau(x) - x \in A[p] \setminus \{0\}$.*

PROOF: If we decompose A_{tor} as $A_{\text{p-tor}} \oplus A_{\text{non p-tor}}$, it is clear that it suffices prove the statement for $x \in A_{\text{p-tor}} \setminus A(L^{nr})$. Setting $n := min_n\{p^n x \in A(L^{nr})\}$, we can find an $\bar{L}$-automorphism $\tau \in Gal(\bar{L}/L^{nr})$ such that $\tau(p^{n-1}x) - p^{n-1}x \in A[p] \setminus \{0\}$ and consequently $\tau^{p^n-1}(x) - x \in A[p] \setminus \{0\}$. □

Take $x \in \Sigma_1 \cap Z$ and τ as in the previous proposition. Then:

$${}^{\tau}Z = Z \ni \tau(x) = (\tau(x) - x) + x \in (\tau(x) - x) + Z,$$

and $\tau(x) - x$ belongs to the finite set $A[p] \setminus \{0\}$, and then there exists $y \in A[p] \setminus \{0\}$ with the property $Z = y + Z$, but this would imply that $y (\neq 0)$ stabilizes Z, contradicting our initial assumtpion. Then the (2) never occurs, and the theorem is then proved. □

2 Second Lecture

In this section the field K will be of positive characteristic p, and algebraically closed.

2.1 Proof of the Manin-Mumford conjecture for prime characteristic

Let's state the first result valid in characteristic zero, also for characteristic p:

Theorem 2.1 ([4]) *Let $Z \subset A$ as in the first section (but now over K with $char(K) = p > 0$) with the extra condition, that it has no nontrivial subfactor defined over a finite field where Z and A are. Let $F : A \to A$ be an endomorphism such that no eigenvalue of F^* is a root of unity. Then Z is special.*

Before we proceed to give the proof, we want to make some remarks about the extra condition added to this $char(p)$-version of (1.5). And this extra assumption is just to avoid the following simple argument. Suppose that $Z \subset A$ are both defined over $\mathbb{F}_q$. Then taking $F = Frob_q$, we have that $F(Z) = Z$, independently of what Z defined over $\mathbb{F}_q$ we take. Hence it will be in general false that this should imply the speciality of Z. Once we removed this case, we give the:
PROOF: [Idea] Suppose F is a separable isogeny.
Take $\hat{A}$ the formal completion of A at 0, and $\hat{Z}$ the corresponding for Z^{reg}. Then up to isogeny we obtain the following decomposition: $\hat{A} = \prod_{\text{finitely many}} \hat{A}_{r/s}$; $r/s \in \mathbb{Q}$ with $F^s = Frob_p^r \circ$(formal isomorphism on $\hat{A}_{r/s}$). This induces a decomposition $\hat{Z} = \prod_{\text{finitely many}} \hat{Z}_{r/s}$. □

Corollary 2.2 *Let A be an abelian variety in characteristic p, $Z \subset A$ irreducible, closed, $pZ = Z$ and suppose A has no nontrivial supersingular factor. Then Z is special.*

2.2 Back to characteristic 0.

Let A be an abelian variety over a number field K, $Z \subset A$ a closed but not necessarily irreducible subvariety, $\mathfrak{p}$ a place of K, $Z_\mathfrak{p} \subset A_\mathfrak{p}$ the corresponding reductions, $k_\mathfrak{p}$ the residue class field, and p it's characteristic.

Theorem 2.3 ([4]) *If $\forall' \mathfrak{p} : pZ_\mathfrak{p} \subset Z_\mathfrak{p}$. Then Z is special.*

PROOF: If Z is absolutely irreducible, then $\forall' \mathfrak{p}$: $Z_{\mathfrak{p}}$ is absolutely irreducible. So in order to use (2.2) we have to know for how many primes, the reduction has no supersingular factors. We state the following without proof:

Theorem 2.4 (R.Pink) *There exists a finite extension K' of K, such that the reduction of A at $\mathfrak{p}'$ has no supersingular factor for a set of $\mathfrak{p}'$ of Dirichlet density* 1.

Consequently there exist infinitely many primes $\mathfrak{p}'$ such that $Z_{\mathfrak{p}'} = B_{\mathfrak{p}'} + a_{\mathfrak{p}'}$. Then we can lift these equalities to $Z = B + a$ with a a rational point, but don't know a priori if a is torsion or not.
Before we prove this, dividing by B we can suppose that $Z = \{a\}$. Then from the hypothesis follows that $\forall' \mathfrak{p}$: $(p-1)a_{\mathfrak{p}} = 0 \in A_{\mathfrak{p}}$. It's known also that the number of rational points in the reduction $\mid A_{\mathfrak{p}}(k_{\mathfrak{p}}) \mid$ equals $det(Frob_{\mathfrak{p}} - 1) = \prod(\lambda_i - 1)$. So applying the following result we finished the proof:

Theorem 2.5 (R.Pink) *Let $P \in \mathbb{Z}[T]$ be a product of cyclotomic polynomials, $a \in A(K)$ such that $\forall'$ $P(p_{\mathfrak{p}})$ $a_{\mathfrak{p}} = 0 \Rightarrow a$ is torsion.*

□

2.3 A conjecture of Beauville and Catanese

Take X a smooth, projective algebraic variety over a number field K. Fix i, j, m and define:

$$S_m^{i,j} := \{\mathcal{L} \in Pic^0(X)_{\bar{K}} \mid h^i(X, \Omega^j \otimes \mathcal{L}) \geq m\};$$

which are closed sets in $Pic^0(X)$ (by semicontinuity).

Theorem 2.6 (Conjectured by B-C, and proved by Green-Lazarfeld; Simpson) *The $S_m^{i,j}$ are special.*

PROOF: [R.Pink] Define $S_m^r := \{\mathcal{L} \in A \mid \sum_{i+j=r} h^i(X, \Omega^j \otimes \mathcal{L}) \geq m\}$. By a result of Deligne-Illusie, we have that $(Frob_p)_* \Omega^* \cong (\Omega^*, 0)$ (here $\Omega = \Omega_{dR}$). Hence $\oplus_{i+j=r} H^i(X_{\mathfrak{p}}, \Omega^j \otimes \mathcal{L}) = \mathbb{H}^r(X_{\mathfrak{p}}, ((Frob_p)_* \Omega^*) \otimes \mathcal{L}) = \mathbb{H}^r(X_{\mathfrak{p}}, \Omega^* \otimes Frob_p^* \mathcal{L})$ and $H^i(X_{\mathfrak{p}}, \Omega^j \otimes \mathcal{L}^p) \Rightarrow \mathbb{H}^r(X_{\mathfrak{p}}, \Omega^* \otimes \mathcal{L}^p) \rightsquigarrow p(S_m^r)_{\mathfrak{p}} \subset (S_m^r)_{\mathfrak{p}}$. □

3 Third Lecture

This part of the lecture will be soon writen in whole extent by Prof. R. Pink.

3.1 Shimura Varieties

Let G be a connected reductive algebraic group over the rationals $\mathbb{Q}$.

Definition 3.1 *A Shimura datum is a pair (G, X), where $X \subset Hom(\mathbb{C}^*, G(\mathbb{R}))$ is a $G(\mathbb{R})$-conjugacy class, such that for all $x \in X$ the following are satisfyed:*

1. *the eigenvalues of $x(z)$ for $z \in \mathbb{C}^*$ on $Lie(G) \otimes \mathbb{R}$ are $z/\bar{z}, 1, \bar{z}/z$;*
2. *$int(x(\sqrt{-1}))$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$;*
3. *$G^{ad}_{\mathbb{R}}$ has no compact factors defined over $\mathbb{Q}$;*
4. *G/G^{der} is an extension of a $\mathbb{Q}$-split torus by a torus of compact type.*

Remarks 3.2 *$T_{X,x} \cong z/\bar{z}$ eigenspace on $Lie(G) \otimes \mathbb{C}$.*
X has a left invariant complex structure.

One can also define a *connected Shimura datum* just by replacing in the definition $G(\mathbb{R})$-conjugacy class by a $G(\mathbb{R})^{\circ}$-conjugacy class.
Take $\Gamma \subset G(\mathbb{R})$ an arithmetic subgroup, which acts properly discontinuously on X. This gives rise to a complex analytic space $\Gamma \backslash X$ called the *(connected) Shimura variety* associated to the (connected) Shimura datum.

Theorem 3.3 (Satake, [1]) *$S := \Gamma \setminus X$ is an algebraic variety.*

Archetype: $G = CSp_{2g,\mathbb{Q}} = \{U \in GL_{2g} \mid^t UEU = \text{scalar}E; E = \begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix}\}$.
X is generated by $\mathbb{C} \ni z = a + ib \mapsto \begin{pmatrix} aId_g & bId_g \\ -bId_g & aId_g \end{pmatrix}$. $\Gamma = \{\gamma \in CSp_{2g}(\mathbb{Z}) \mid \gamma \equiv Id_{2g}(mod n)\} \rightsquigarrow \Gamma \setminus X$ an irreducible component of $\mathcal{A}_{g,1,n}$.
A morphism of connected Shimura data from (H, Y) to (G, X) is a homomorphism $\phi : H \to G$, leading a mapping $Y \to X : \ y \mapsto \phi ø y$ which is complex analytic. This gives us a definition of a morphism between Shimura varieties $\Delta \setminus Y \to \Gamma \setminus X$. These morphisms are algebraic.

Definition 3.4 *The image of a Shimura morphism is called special subvariety of $\Gamma \setminus X$.*

Remarks 3.5 • *Special subvarieties of dimension* 0 *are called special points.*

- $Y = point \leftrightarrow H$ *is a torus.*
- $\Gamma_X \in \Gamma \setminus X$ *is special* $\Leftrightarrow x \cdot \mathbb{C}^* \to G(\mathbb{R})$ *factors through a torus over* $\mathbb{Q}$.

3.2 André-Ooort Conjecture

Let $S = \Gamma \setminus X$ be a Shimura variety over $\mathbb{C}$; $Z \subset S$ an irreducible closed subvariety; $Z \cap$ (special points) is Zariski dense in Z. Then the conjecture asserts that Z is special.
It's known to be true for $G = GL_{2,\mathbb{Q}} \times GL_{2,\mathbb{Q}}$ (André); when $dim(Z) = 1$ ([2]).

3.3 Connected mixed Shimura subvarieties

Let V be a $\mathbb{Q}$-representation of G such that:

$$\forall x \in X : \mathbb{C}^* \xrightarrow{x} G(\mathbb{R}) \text{ acting on } V \otimes \mathbb{R},$$

which gives us a complex structure on $V \otimes \mathbb{R}$. Hence taking a $\mathbb{Z}$-lattice Γ_V in V gives us an abelian variety $\Gamma_V \setminus V \otimes \mathbb{R}$.
With the representation V of G one can define $\tilde{G}$ as the semi-direct product of G and V. Then $\tilde{X}$ as the $\tilde{G}(\mathbb{R})^{\circ}$-conjugacy generated by X. The induced pair $(\tilde{G}, \tilde{X})$ is called a *connected mixed Shimura datum.*
Setting $\tilde{\Gamma} := \Gamma \ltimes \Gamma_V$, one has the following:

Theorem 3.6 $\tilde{\Gamma} \setminus \tilde{X}$ *is an algebraic variety.*

One defines also as before the special (mixed) subvarieties as the images of Shimura morphisms between connected mixed Shimura datum. Therefore the special points are the special varieties in this context, of dimension zero.

Remarks 3.7 *1. One has the natural morphism* $\tilde{\Gamma} \setminus \tilde{X} \to \Gamma \setminus X$.

2. The special (mixed) points lay over the special points of $\Gamma \setminus X$, *which are torsion points on the corresponding CM-fibers.*

3. *Special subvarieties in a fiber are translates of abelian subvarieties by torsion points.*

Definition 3.8 *Consider $T' \leftarrow T \xrightarrow{\phi} S$ Shimura morphisms. An irreducible component of $\phi(\psi^{-1}(t'))$ is called a weakly special subvariety.*

Fact: Weakly special subvarieties contained in an abelian fiber are precisely the translates of abelian subvarieties.

3.4 Generalized Hecke orbit conjecture

Let S be a connected mixed Shimura variety over $\mathbb{C}$, $s \in S$ and Σ it's generalized Hecke orbit. Take $Z \subset S$ any closed irreducible algebraic subvariety, such that $Z \cap \Sigma$ is Zariski dense in Z. Then this conjecture says that Z is weakly special.
Evidences:

- [2]: $dimZ = 1$ and S pure.
- For $Z \subset$ (abelian fiber). Then this conjecture is equivalent to the Mordell-Lang conjecture.
- Equidistribution: Clozel, Oh, Ullmo.

References

[1] W. Baily, A. Borel, *On the compactification of arithmetically defined quotients of bounded symmetric domains*, Bull. Amer. Math. Soc. **70** (1964), 588-593.

[2] Bas Edixhoven, Andreii Yafaev, *Subvarieties of Shimura Varieties*, Ann. of Math. 2 **157**, (2003), no. 2, 621-645.

[3] E. Hrushovsky, *The Manin-Mumford conjecture and the model theory of difference fields*, Ann. Pure Appl. Logic **112** (2001), no. 1,43-115.

[4] Richard Pink, Damian Roessler, *On Hrushovski's proof of the Manin-Mumford conjecture*, Preprint, `www.math.ethz.ch/~pink`.