

Jean-Benoit Bost: “Foliations on algebraic varieties over number fields”

1. Foliations for Diophantine geometries
2. Algebraicity of leaves of foliations and Diophantine geometry
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1 Foliations

C^∞ -foliations:

1. “foliated atlas”: Let $d := \dim X$.

$$(\phi_i : U_i \rightarrow V_i)_{i \in I} \tag{1}$$

such that

$$\phi_j \circ \phi_i^{-1}(\Omega \times \{t_0\}) = \Omega' \times \{t'_0\}. \tag{2}$$

2. “integrable distributions”: Let $F \hookrightarrow T_x$ subvector bundle, such that for all C^∞ local sections v_1, v_2 of F the commutator $[v_1, v_2]$ is also a section of F .

These two definitions are equivalent (sketch of proof):

1 \Rightarrow 2:

$$\left(\begin{array}{ccc} F_{U_i} & \hookrightarrow & T_{U_i} \\ \cong \text{ (via } D\phi_i) & & \\ V_i \times (\mathbb{R}^f \oplus \{g\}) & \hookrightarrow & T_{V_i} = V_i \times \mathbb{R}^d \end{array} \right) \tag{3}$$

2 \Rightarrow 1:

Choose C^∞ -local coordinates x_1, \dots, x_d near P , a frame v_1, \dots, v_f of F near P , $v_i = \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j}$, $1 \leq i \leq f$.

We may assume

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 & * & \dots & * \\ 0 & 1 & \dots & 0 & * & \dots & * \\ & & \dots & & \dots & & \\ 0 & 0 & \dots & 1 & * & \dots & * \end{pmatrix}, \quad (4)$$

so

$$v_1 = \frac{\partial}{\partial x_1} + * \frac{\partial}{\partial x_{f+1}} + \dots \quad (5)$$

$$\dots \quad (6)$$

$$v_f = \frac{\partial}{\partial x_f} + * \frac{\partial}{\partial x_{f+1}} + \dots \quad (7)$$

Observations:

For all i, j , $[v_i, v_j]$ is a section of F , so $[v_i, v_j] = 0$. Hence:

$$\exp(tv_i) \circ \exp(tv_j) = \exp(tv_j) \circ \exp(tv_i). \quad (8)$$

Choose transversal $\phi_0 : T_0 \rightarrow V_0 \hookrightarrow \mathbb{R}^{d-f}$.

This gives

$$\phi : U \rightarrow]-\epsilon, \epsilon[^f \times V_0$$

via

$$\phi^{-1} : (t_1, \dots, t_f, u_1, \dots, u_{d-f}) \rightarrow \exp(t_1 v_1) \circ \dots \circ \exp(t_f v_f)(\phi_0(u_1, \dots, u_{d-f})).$$

2 Algebraic foliations

Let K be a field (often of characteristic 0), X a smooth algebraic variety over K , F a saturated coherent subsheaf of T_X closed under $[,]$. Let $\Delta \hookrightarrow X$ be closed of codimension ≥ 2 such that $F|_{X \setminus \Delta}$ is a subvectorbundle of $T_{X \setminus \Delta}$.

N.B.:

- If $\text{rk}(F) = 1$ then $[,]$ is closed automatically.
- If $U \subset X$ is open dense an algebraic foliation on U extends uniquely to X .

Let now $\text{char}K = 0$. Let $P \in (X \setminus \Delta)(K)$. Carrying over the proof of the first section we get the “formal leaf” of F through P :

$$\mathcal{F}_P \hookrightarrow \hat{X}_P,$$

it is a smooth formal subscheme of dimension f of the formal completion \hat{X}_P of X at P . It can be thought of as the infinitesimal “integral” of the algebraic foliation.

Let $\sigma_0, \sigma_1 : K \hookrightarrow \mathbb{C}$ be two complex embeddings. Then we get two foliations $F_{\sigma_0}(\mathbb{C})$ and $F_{\sigma_1}(\mathbb{C})$ of $X_{\sigma_0}(\mathbb{C})$ respectively $X_{\sigma_1}(\mathbb{C})$.

Three “examples”:

1) the “simplest” one: Let $K = \mathbb{C}$. Let $P, Q \in \mathbb{C}[X, Y]$ with no common factor, $X = \mathbb{A}^2$, $\Delta = (P = Q = 0)$. Let $F := \mathcal{O}_x(P \frac{\partial}{\partial X} + Q \frac{\partial}{\partial Y})$. This extends to \mathbb{P}^2 .

2) Let G be a (smooth) algebraic group over K . Then $\text{Lie}(G) = T_e G \cong \{\text{left } G\text{-invariant vector fields}\} = \Gamma(G, T_G)^G$.

Then a vector subspace $V \subset \text{Lie}(G)$ corresponds to a G -invariant sub-vector bundle $F \hookrightarrow T_G$ and a Lie subalgebra $V \subset \text{Lie}(G)$ to a G -invariant foliation.

When $K = \mathbb{C}$ there is the exponential

$$\exp_G : \text{Lie}(G) \rightarrow G(\mathbb{C}),$$

the image of V under \exp_G generates the leaf \mathcal{F} of F through e .

If $G = \mathbb{G}_m$, then $\exp_G = \exp$. If G is an elliptic curve embedded in \mathbb{P}^2 , then \exp_G is given by $(\mathcal{P}, \mathcal{P}')$, where \mathcal{P} is the Weierstra’s \mathcal{P} -function. If G is an abelian variety, then \exp_G is given in terms of “abelian functions”.

3) “linear foliations”:

Let S be a smooth algebraic variety over a field K , E a vector bundle over S , ∇ a connection on E . For a local section $s \in \Gamma(U, E)$ we have $\nabla s \in \Gamma(U, \Omega_X^1 \otimes E)$ and the rule $\nabla(fs) = f\nabla s + df \otimes s$.

For $v \in \Gamma(U, T_X)$ we have the derivation of s in the direction of v : $\nabla_v s = \langle \nabla s, v \rangle \in \Gamma(U, E)$.

On a suitable open $U \subset X$ let's write $E \cong \mathcal{O}_U^{\oplus e}$. There

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_e \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_e \end{pmatrix} + A \begin{pmatrix} f_1 \\ \vdots \\ f_e \end{pmatrix},$$

where $A \in M_e(\Gamma(U, \Omega_X^1))$.

Let X be the total space of E and $p : X \rightarrow S$ the projection. Then the connection ∇ on E corresponds to an $F \hookrightarrow T_X$ such that $T_X = F \oplus T_p$.

Locally $X = S \times \mathbb{A}^e$, and $F_{(s,v)} \subset T_{(s,v)}(S \times \mathbb{A}^e) = T_s S \oplus K^e$ is given by

$$F_{(s,v)} := \{(\delta s, \delta v) \mid \delta s + (A(s) \cdot \delta s) \cdot \delta v = 0\}. \quad ???$$

F is integrable (i.e. defines a foliation) if and only if ∇ is “integrable”, meaning that $[\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1, v_2]}$.

In this picture a formal section σ of E satisfying $\nabla \sigma = 0$ corresponds to a formal leaf of F .

3 Algebraicity of leaves of foliations

We use the following general setting:

- K a number field,
- X a smooth algebraic variety over K ,
- $F \subset T_x$ a saturated coherent subsheaf of T_X , closed under $[,]$, $\Delta \subset X$ as in the previous section (i.e. F is an algebraic foliation),
- $P \in (X \setminus \Delta)(K)$ a rational point,
- $\mathfrak{F}_P \hookrightarrow \hat{X}_P$ the “formal leaf” at P .

We pose the question if there are “local”/arithmetic conditions on F which imply that \mathcal{F}_P is algebraic.

We have the following results:

1.) “Schneider–Lang”:

Suppose F is a rank 1 foliation, i. e. a foliation by curves, let

$X \hookrightarrow \mathbb{P}_K^N$ be an embedding and $\sigma_0 : K \hookrightarrow \mathbb{C}$ a complex embedding.

Theorem 3.1. *Let $f : \mathbb{C} \rightarrow X_{\sigma_0}(\mathbb{C}) (\hookrightarrow \mathbb{P}^N(\mathbb{C}))$ be a nonconstant analytic mapping of finite order, s.t. there is a factorization $\hat{f}_0 : \hat{\mathbb{C}}_0 \rightarrow \mathcal{F}_P$.*

If \mathcal{F}_P is not algebraic, then $f^{-1}((X \setminus \Delta)(K))$ is finite.

Comments:

1. “finite order ”: Let $f : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be analytic. We say f is of finite order $:\Leftrightarrow \exists \vec{f} = (f_0, \dots, f_N) : \mathbb{C} \rightarrow \mathbb{C}^{N+1} \setminus \{0\}$ analytic s.t. $f = (f_0, \dots, f_N)$ and there exists $\rho > 0$ such that $\|\vec{f}(\eta)\| = \mathcal{O}(e^{|\eta|^\rho})$. The order of f is defined to be the infimum of the possible ρ ’s.

2. Nevanlinna theory:

Let ω be the Fubini–Study 2–form on $\mathbb{P}^N(\mathbb{C})$. On $\mathbb{A}^N(\mathbb{C})$ we have

$$\begin{aligned}\omega &= \frac{1}{2\pi i} \partial \bar{\partial} \log \left(1 + \sum_{i=1}^N |\eta_i|^2 \right)^{-1} \\ &\stackrel{\text{at } 0}{=} \frac{1}{\pi} (dx_1 \wedge dy_1 + \cdots + dx_N \wedge dy_N).\end{aligned}$$

Let $r > 0$.

Then the characteristic function (Nevanlinna, Ahlfors) is given by

$$T_f(r) := \int_{D(0,r)} \log^+ \frac{r}{|\eta|} f^* \omega.$$

Fact: f is of finite order $\rho \Leftrightarrow \forall \epsilon > 0 \ T_f(r) = \mathcal{O}(r^{\rho+\epsilon})$.

Proof of \Rightarrow :

$$f^* \omega = \frac{i}{2\pi} \partial \bar{\partial} \log \|\vec{f}\|^2$$

$$\begin{aligned}T_f(r) &= \int_{\mathbb{C}} \log^+ \frac{r}{|\eta|} \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|\vec{f}\|^2 \right) \\ &= \int_{\mathbb{C}} \left(\frac{i}{2\pi} \partial \bar{\partial} \log^+ \frac{r^2}{|\eta|^2} \log \|\vec{f}\| \right) \quad (\text{Stokes, Green}) \\ &= \int_{\mu_r} \log \|\vec{f}\| - \log \|\vec{f}\|(0) \\ &= \mathcal{O}(r^\rho).\end{aligned}$$

Note that $\frac{1}{2\pi i} \partial \bar{\partial} \log^+ \frac{r^2}{|\eta|^2} = \delta_0 - \mu_r$, where μ_r is defined by $\int_{\mathbb{C}} \phi \mu_r = \int_0^1 \phi(re^{2\pi i t}) dt$.

Example: Assume $A \xrightarrow{i} \mathbb{P}^N(\mathbb{C})$ is an abelian variety.

Choose $v \in \text{Lie}(A)$ and let

$$\begin{aligned} f : \mathbb{C} &\rightarrow A \hookrightarrow \mathbb{P}^N(\mathbb{C}) \\ t &\mapsto \exp_A(tv) = [tv]. \end{aligned}$$

Write $A \simeq \text{Lie}(A)/\Gamma$.

Then

$$f^*\omega = ((t \mapsto tv)^* \circ \exp_A^* \circ i^*)\omega,$$

where $i^*\omega$ is a positive 2-form on A , $(\exp_A^* \circ i^*)\omega$ a translation invariant positive 2-form on $\text{Lie}(A)$. Hence

$$T_f(r) \asymp \int_{\mathbb{C}} \log^+ \frac{r}{|\eta|} dx \wedge dy = r^2 \frac{\pi}{2}$$

has order 2.

3.1 Application to algebraic groups

Theorem 3.2. (Lang): Assume given an algebraic group G over $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\text{Lie}(G)$ the $\bar{\mathbb{Q}}$ -Lie-algebra $(\text{Lie}(G))_{\mathbb{C}} \simeq \text{Lie}(G)_{\mathbb{C}} \xrightarrow{\exp_{G_{\mathbb{C}}}} G(\mathbb{C})$.

1. Let $L \subset \text{Lie}(G)$ line over $\bar{\mathbb{Q}}$.

If $\exists v \in L_{\mathbb{C}} \setminus \{0\}$ such that $\exp_{G_{\mathbb{C}}}(v) \in G(\bar{\mathbb{Q}})$, then L is an algebraic Lie-subalgebra (i. e. $\exists H \hookrightarrow G$ algebraic subgroup over $\bar{\mathbb{Q}}$, $L = \text{Lie}(H)$).

2. If for any $v \in \text{Lie}(G) \setminus \{0\}$, $\exp_{G_{\mathbb{C}}}(v) \in G(\bar{\mathbb{Q}})$, then v is “unipotent”, i.e. the map $t \mapsto \exp_{G(\mathbb{C})}(tv)$ is an algebraic map $\mathbb{G}_{a,\mathbb{C}} \rightarrow G_{\mathbb{C}}$.

Proof:

- $\mathbb{Q} \rightarrow K \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$,
 $\sigma_0 : K \rightarrow \mathbb{C}$,
 $X = G$, $F = \text{transl. inv.}$, $F_e = L$, $p = e$, $\mathfrak{F}_p = \exp L$,

$$\begin{aligned} f : \mathbb{C} &\rightarrow G(\mathbb{C}) \\ t &\mapsto \exp_{G(\mathbb{C})}(tv) \end{aligned}$$

of finite order, since $f^{-1}(G(K)) \supset \mathbb{Z}$ is infinite!

- apply 1. to $\tilde{G} = \mathbb{G}_a \times G$, $\tau = \frac{\partial}{\partial x} \oplus v$,
 \Rightarrow graph $(t \rightarrow \exp(tv))$ is algebraic.

Exercises:

- $\mathbb{G}_a \times \mathbb{G}_a$ yields Hermite–Lindemann, i. e. $\beta \in \bar{\mathbb{Q}} \setminus \{0\} \Rightarrow e^\beta \notin \bar{\mathbb{Q}}$,
- $\mathbb{G}_m \times \mathbb{G}_m$ yields Gelfond–Schneider.

3.2 Conjecture of Grothendieck–Katz

Let $X, F \hookrightarrow T_X$ over K be as before. Choose models

$$\mathcal{X} \xrightarrow{\pi} \text{Spec } \mathcal{O}_K\left[\frac{1}{N}\right] \quad (9)$$

$$\mathcal{F} \hookrightarrow T_\pi \quad (10)$$

and a point $(0) \neq \mathfrak{p} \in \text{Spec } \mathcal{O}_K\left[\frac{1}{N}\right]$. We get the reductions $\mathcal{X}_{\mathbb{F}_p} \rightarrow \text{Spec } \mathbb{F}_p$ and

$\mathcal{F}_p = \mathcal{F}_{\mathbb{F}_p}$, an algebraic foliation over \mathbb{F}_p .

Let $p = \text{char } \mathbb{F}_p$. We have:

$$\mathcal{F}_p \text{ is } p\text{-closed} \Leftrightarrow \forall U \hookrightarrow \mathcal{X}_{\mathbb{F}_p}, \forall v \in \Gamma(U, \mathcal{F}_p), v^{[p]} := v \circ \dots \circ v \in \Gamma(U, \mathcal{F}_p).$$

Fact: F is algebraically integrable \Rightarrow for almost every p , \mathcal{F}_p is p -closed (Grothendieck–Katz (GK), \sim 1970).

\Leftarrow was a conjecture, in general proven 1999 by Ekedahl, Shepherd–Barron, Taylor.

Partial results:

- Katz: linear systems, “Gau–Manin”.
- diophantine approximation, André, J. B. B.

Theorem 3.3. (*J. B. B.*) $P \in (X \setminus \Delta)(K)$, $\sigma_0 : K \hookrightarrow \mathbb{C}$. If F satisfies (GK), then the analytic leaf of F_{σ_0} through P is “Liouville” ($\hookrightarrow X_{\sigma_0}(\mathbb{C})$)(e. g. parametrized by analytic variety; $\Rightarrow \mathcal{F}_p$ is algebraic).

Application:

G algebraic group over K ,

$$\mathcal{G} \rightarrow \text{Spec} \mathcal{O}_K\left[\frac{1}{N}\right],$$

$$(\text{Lie}(\mathcal{G}))_K \cong \text{Lie}(G); (\text{Lie}(\mathcal{G}))_{\mathbb{F}_p} \cong \text{Lie}(\mathcal{G}_{\mathbb{F}_p}).$$

Let $h \subset \text{Lie}(G)$ be a Lie-subalgebra over K , then

h algebraic \Leftrightarrow for almost all \mathfrak{p} , $h_{\mathbb{F}_p}$ is p -closed.

N.B.: $G = \mathbb{G}_m \times \mathbb{G}_m$, take for $h = K \cdot (x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y})$; is algebraic $\Leftrightarrow \lambda \in \mathbb{Q}$,
since $(\)^{[p]} = x \frac{\partial}{\partial x} + \lambda^p y \frac{\partial}{\partial y}$ and

for almost all \mathfrak{p} , $h_{\mathfrak{p}}$ p -closed \Leftrightarrow for almost all p , $[\lambda]^p = [\lambda] \Leftrightarrow$ for almost all p , $[\lambda] \in \mathbb{F}_p$.

4 About the proofs:

Show formal leaf \mathfrak{F}_P (of dimension 1 for simplicity) is algebraic:

Level 1: “Auxiliary polynomials”:

$V \hookrightarrow \mathbb{C}^N$ any subset!

$$\begin{aligned} \eta_D : \mathbb{C}[X_1, \dots, X_N]_{\leq D} &\rightarrow \mathbb{C}^V \\ p &\mapsto p|_V. \end{aligned}$$

Key observation: For $D \rightarrow \infty$, $\text{rk}(\eta_D) \sim *D^{\dim \bar{V}^{\text{Zar}}}$.

Proof:

$$\begin{aligned} \mathbb{C}[X_1, \dots, X_N]_{\leq D} &\cong \Gamma(\mathbb{P}^N, \mathcal{O}(D)) \xrightarrow{\eta_D} \mathbb{C}^V, \\ \mathbb{C}[X_1, \dots, X_N]/I_+(\bar{V}) &\underset{D \gg 0}{\simeq} \Gamma(\bar{V}^{\text{Zar}}, \mathcal{O}(D)). \end{aligned}$$

$\text{rk}(\eta_D) = \dim \Gamma(\bar{V}^{\text{Zar}}, \mathcal{O}(D))$ (Hilbert).

Extends to \hat{V} , the germ of the analytic or formal subvariety.

algebraicity $\Leftrightarrow \text{rk}(\eta_D) \sim *D^{\dim \hat{V}} (\leq \dim \hat{V})$.

Level 2: Algebraic varieties over function fields

Let C be a curve over k , $K = k(C)$, X projective variety over K , $P \in X(K)$ a rational point, $\hat{V} \subset \hat{X}_P$ smooth formal curve.

We choose again models \mathfrak{X} , $\hat{\mathcal{V}}$ and \mathcal{P} .

Proposition 4.1. *If \hat{V} extends to a smooth formal subscheme $\hat{\mathcal{V}} \hookrightarrow \hat{\mathfrak{X}}_P$ and if $N := N_p \hat{\mathcal{V}}$ (line bundle) has positive degree, then $\hat{V}/\hat{\mathcal{V}}$ is algebraic.*

Proof:

Choose $\mathcal{O}(1)$ over \mathfrak{X} , consider

$$\begin{array}{ccc} \eta_D : & \Gamma(\mathfrak{X}, \mathcal{O}(D)) & \rightarrow \Gamma(\hat{\mathcal{V}}, \mathcal{O}(D)) \\ & \parallel & \downarrow \\ A^i D \subset & A_D & \Gamma(\mathcal{V}_i, \mathcal{O}(D)) \end{array}$$

where $A_D^i := \{s \in A_D \mid s|_{\mathcal{V}_{i-1}} = 0\}$ and

$$\mathcal{V}_0 = \mathcal{P} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots$$

Want $\text{rk}(\eta_D) \sim *D^2$ for $D \rightarrow \infty$.

Observe

$$\begin{array}{ccc} \eta_D(A_D^i)/\eta_D(A_D^{i+1}) & \hookrightarrow & \Gamma(\mathbb{P}, \check{N}^{\otimes i} \otimes \mathcal{O}(D)) \\ \dim_k \text{---} \text{---} & \leq & \dim \text{---} \text{---} \leq c(i+D) \\ & = & 0 \text{ if } \frac{c}{D} > \lambda \end{array}$$

$$\text{rk}(\eta_D) \leq \sum_{i \geq 0} \dots$$

Comments:

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$$\begin{array}{ccc} \eta_D : & \pi_* \mathcal{O}(D) & \rightarrow \pi_* \mathcal{O}(D)|_{\hat{\mathcal{V}}} \\ & \parallel & \\ & E_D & \text{v. b. over } C, \quad A_D = \Gamma(C, E_D) \\ & \cup & \\ & E_D^i & \text{filtration} \end{array}$$

- $E_D^i/E_D^{i+1} \xrightarrow{\eta_D^i} \mathcal{P}^*(\tilde{N}^{\otimes i} \otimes \mathcal{O}(D))$
- we assume wlog \hat{V} Zariski-dense in X , want $d := \dim X = 1$.
 $\Rightarrow \eta_D$ injective.

Hence

$$\begin{aligned}
0 \underset{D \gg 0}{\leq} \deg E_D &= \sum_{i \geq 0} \deg E_D^i/E_D^{i+1} \\
&\leq \sum_{i \geq 0} \text{rk}(E_D^i/E_D^{i+1})(-i \deg N + D \deg \mathcal{P}^* \mathcal{O}(1))^+ \\
&\sim *D^2.
\end{aligned}$$

This implies $d = 1$.

Arakelov geometry:

$\mathfrak{X} \rightarrow \text{Spec} \mathbb{Z}$ finite type, $\mathfrak{X}_{\mathbb{Q}}$ smooth.

Definition 4.2. A hermitian vector bundle over \mathfrak{X} is $\bar{E} = (E, \|\cdot\|)$ ($\|\cdot\|$ C^∞ -hermitian metric on $E_{\mathbb{C}}$, invariant under complex conjugation). Pullback tensor operation are defined as expected.

For K a number field, $\mathfrak{X} = \text{Spec } \mathcal{O}_K$, $\mathfrak{X}(\mathbb{C}) = \{\sigma : K \hookrightarrow \mathbb{C}\}$, then a hermitian vector bundle is given by $\bar{E} = (E; \|\cdot\|_{\sigma}$ hermitian metric on E_{σ} for every $\sigma : K \hookrightarrow \mathbb{C}$).

Definition 4.3. Arakelov degree: $\hat{\deg} \bar{E} := * \log \#(E/\mathcal{O}_K \cdot s) - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|s\|_{\sigma}$ if $\text{rk}(E) = 1$, $s \in E \setminus \{0\}$, in general $\hat{\deg} \bar{E} = \hat{\deg} \wedge^{\text{rk}(E)} \bar{E}$.

- \bar{L}_1, \bar{L}_2 of rk 1, $\hat{\deg} \bar{L}_1 \otimes \bar{L}_2 = \hat{\deg} \bar{L}_1 + \hat{\deg} \bar{L}_2$,
- $0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$
 $\Rightarrow \hat{\deg} \bar{E} = \hat{\deg} \bar{S} + \hat{\deg} \bar{Q}$,
- \bar{L}_1, \bar{L}_2 rk 1, $\phi : L_{1,K} \rightarrow L_{2,K}$, $\phi \neq 0 \Rightarrow \hat{\deg} \bar{L}_1 = \hat{\deg} \bar{L}_2 + h(\phi)$,
 $h(\phi) := \sum_{\mathcal{P} \neq (0) \text{ in } \mathcal{O}_K} \log \|\phi\|_{\mathcal{P}} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\phi\|_{\sigma}$.

Level 3: Over a number field

K number field, X over K , $P \in X(K)$, $\hat{V} \subset \hat{X}_P$ formal curve.

Choose $\mathcal{O}(1)/X$.

Consider

$$\eta_{D,K} : \Gamma(X, \mathcal{O}(D)) \rightarrow \Gamma(\hat{V}, \mathcal{O}(D)),$$

\hat{V} Zariski-dense in X , $\hat{V} = \varinjlim V_i$.

Let $N_K = T_P \hat{V}$.

Choose

$$\begin{array}{c} \mathfrak{X} \\ \pi \downarrow \uparrow \mathcal{P} \\ \text{Spec } \mathcal{O}_K \end{array} .$$

Let $\bar{E}_D = (\pi_* \mathcal{O}(D), \|\cdot\|_{L^2})$ and

$$\begin{array}{ccc} E_D^i & \subset & E_D \\ \ni & & \ni \\ s|_{V_{i-1}} = 0 & & s \end{array} ,$$

$$\eta_{D,K}^i : E_{D,K}^i / E_{D,K}^{i+1} \hookrightarrow \check{N}_K^{\otimes i} \otimes \mathcal{O}(D)|_{\mathcal{P}}.$$

As above

$$\begin{aligned} \hat{\text{deg}} \bar{E}_D &= \sum_{i \geq 0} \hat{\text{deg}}(E_D^i / E_D^{i+1}) \\ &= \sum_{i \geq 0} \text{rk}(E_D^i / E_D^{i+1}) (-i \hat{\text{deg}} \bar{N} + D \hat{\text{deg}} \mathcal{P}^* \mathcal{O}(1) + h(\eta_D^i)). \end{aligned}$$

Basic principle: If $\frac{1}{D}(\dots)$ is $\ll 0$ when $\frac{i}{D} \gg 0$, this equality leads to a contradiction when $D \rightarrow \infty$ if $\dim X \geq 2$.

Observations:

\hat{V} formal leaf $\Rightarrow A^D i! \eta_D^i$ are defined over \mathcal{O}_K .

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\mathcal{P}} \log \|\eta_D^i\| \leq i \log i + \gamma D,$$

\hat{V}_σ analytic $\Rightarrow \|\eta_D^i\|_\sigma \leq r^{-i} C^D$ “Cauchy”,

$$\log \|\eta_D^i\| \leq \alpha D + \beta i.$$

1. Schneider–Lang:

$$\mathbb{P}^N(\mathbb{C}) \leftarrow X, F, \sigma_0 : K \hookrightarrow \mathbb{C}, f : \mathbb{C} \rightarrow X_{\sigma_0}(\mathbb{C}),$$

$$(\xi_\alpha)_{1 \leq \alpha \leq k}, f(\xi_\alpha) = P_\alpha \in X(K), \hat{V}_\alpha = \mathfrak{F}_{P_\alpha},$$

$$\eta_{D,K} : \Gamma(X, \mathcal{O}(D)) \rightarrow \bigoplus_{\alpha=1}^k \Gamma(\hat{V}_\alpha, \mathcal{O}(D)).$$

Additional fact:

$$\begin{aligned} T_f(r) = \mathcal{O}(r^\rho) &\Rightarrow \log \|\eta_D^i\| \leq -\frac{k}{\rho} i \log \frac{i}{D} + c(i + D), \\ \frac{1}{D} h(\eta_D^i) &\leq ([K : \mathbb{Q}] - \frac{k}{\rho}) \frac{i}{D} \log \frac{i}{D} + c(\frac{i}{D} + 1), \end{aligned}$$

gets very negative for $k \gg 0$.

2. Grothendieck–Katz:

X, F, σ_0 as before, f (GK)

Additional fact:

(GK) formal leaf extends “almost smoothly” $\sum_{\mathcal{P}} \log \|\eta_D^i\| \leq c(i + D)$
(weak smooth)

$$\forall \lambda > 0 \exists c(\lambda) : \forall \lambda > 0 \exists c(\lambda \log \|\phi_D^i\|) \leq -\lambda i + c(\lambda) D$$

$$\begin{aligned} -i \hat{\deg} \bar{N} + h(\eta_D^i) &\leq c'(i + D) + c(\lambda) D - \lambda i \\ &\leq -\epsilon i + c'' D, \lambda > c'. \end{aligned}$$

Theorem 4.4. (*C. Gasbarri + J. B. B.*):

X , F rank 1, $P \in X(K)$, $\sigma_0 : K \hookrightarrow \mathbb{C}$, $\bar{\mathbb{Q}} \subset \mathbb{C}$,

$$f : C \rightarrow X_{\sigma_0}(\mathbb{C})$$

analytic along a leaf of F_{σ_0} , of finite order, C parabolic Riemann surface (affine algebraic curve).

$f(C)$ is not algebraic

$$\Rightarrow \sum_{\xi \in C, f(\xi) \in (X \setminus \Delta)(\bar{\mathbb{Q}})} \frac{1}{[K(f(\xi)) : K]} < \infty.$$

Example: M affine curve over K , (E_1, ∇_1) , (E_2, ∇_2) .

Assume: $\exists \phi_{\mathbb{C}} : (E_{1,\mathbb{C}}, \nabla_{1,\mathbb{C}})^{\text{an}} \cong (E_{2,\mathbb{C}}, \nabla_{2,\mathbb{C}})^{\text{an}}$ (i.e. same monodromy).

If $\phi_{\mathbb{C}}$ is not algebraic

$$\Rightarrow \sum_{\substack{P \in C(\bar{\mathbb{Q}}), \\ \phi(P) \text{ defined over } \bar{\mathbb{Q}}}} \frac{1}{[K(P, \phi(P)) : K]} < \infty.$$