From the number-theoretic point of view, there are two kinds of rational surface:

- Pencils of conics, given by an equation
  \[ a_0(U, V)X_0^2 + a_1(U, V)X_1^2 + a_2(U, V)X_2^2 = 0 \]
  where the \( a_i(U, V) \) are homogeneous polynomials of the same degree.

- Del Pezzo surfaces of degree \( d \), obtained over \( \mathbb{C} \) by blowing up \((9—d)\) points of \( \mathbb{P}^2 \) in general position. The Del Pezzo surfaces of degree 3 are the nonsingular cubic surfaces, which have an enormous but largely irrelevant literature; and those of degree 4 are the nonsingular intersections of two quadrics in \( \mathbb{P}^4 \).

In both these cases the main conjecture, due to Colliot-Thélène and Sansuc, is that the only obstruction to either the Hasse principle or weak approximation is the Brauer-Manin obstruction.
Suppose we are given finitely many polynomials $F_1(X), \ldots, F_n(X)$ in $\mathbb{Z}[X]$ with positive leading coefficients, is there an arbitrarily large integer $x$ at which they all take prime values? There are two obvious obstructions to this:

- One or more of the $F_i(X)$ may factorize in $\mathbb{Z}[X]$.

- There may be a prime $p$ such that for any value of $x \mod p$ at least one of the $F_i(x)$ is divisible by $p$.

Schinzel’s Hypothesis is that these are the only obstructions: in other words, if neither of them happens then we can choose an arbitrarily large $x$ so that every $F_i(x)$ is a prime.
Let $\mathcal{N}^2 = \mathcal{N}^2(k)$ be the set of $\alpha \times \beta$ where $\alpha, \beta$ are integers in $k$ coprime outside $\mathcal{B}$. Provided $F(\alpha, \beta)$ and $G(\alpha, \beta)$ are nonzero, we define the Legendre function $L(\mathcal{B}; F, G; \alpha, \beta)$:

$$
\alpha \times \beta \mapsto \prod_{p}(F(\alpha, \beta), G(\alpha, \beta))_p
$$

on $\mathcal{N}^2$, where the outer bracket is the multiplicative Hilbert symbol and the product is taken over all primes $p$ of $k$ outside $\mathcal{B}$ which divide $G(\alpha, \beta)$. Here $\mathcal{B}$ will contain all primes dividing the resultant of $F$ and $G$, so $F(\alpha, \beta)$ is a unit at every prime in the product.

**Lemma 1** Suppose that $(\deg F)(\deg G)$ is even; then $L$ is continuous in the topology induced on $\mathcal{N}^2$ by $\mathcal{B}$.
Theorem 1 Assume Schinzel’s Hypothesis. Let \( A \subset \mathcal{N} \) be the subset of \( \mathbb{P}^1(k) \) at which all the Legendre conditions hold and

\[
a_0(U, V)X_0^2 + a_1(U, V)X_1^2 + a_2(U, V)X_2^2 = 0
\]

is locally soluble at each place in \( \mathcal{B} \). Then the points \((U, V)\) in \( \mathbb{P}^1(k) \) at which this conic is solvable form a dense subset of \( A \) in the topology induced by \( \mathcal{B} \).
Theorem 2 Let $N \geq \deg(a_0a_1a_2)$ be a fixed integer. Let $a$ be a positive 0-cycle of degree $N$ on $\mathbb{P}^1$ defined over $k$, and for each place $v$ of $k$ suppose that the pencil of conics

$$a_0(U, V)X_0^2 + a_1(U, V)X_1^2 + a_2(U, V)X_2^2 = 0$$

contains a positive 0-cycle $b_v$ of degree $N$ defined over $k_v$; for $v$ in $B$ suppose further that $b_v$ is so chosen that its projection on $\mathbb{P}^1$ is $a$. If all the Legendre conditions hold, then on this pencil of conics there is a positive 0-cycle of degree $N$ defined over $k$ whose projection is arbitrarily close to $a$ in the topology induced by $B$. 

5
The existence of a positive 0-cycle of odd degree is the same as solubility in some field extension $K/k$ of the same odd degree.

Unfortunately, for pencils of conics this does not imply solubility in $k$. A simple counterexample is given by the pencil

$$7(Y_0^2 + Y_1^2) = (U^2 - UV - V^2)(U^2 + UV - V^2)(U^2 - 2V^2).$$

This is insoluble in $\mathbb{Q}$. But suppose $K = \mathbb{Q}(\rho)$ where $\rho^3 + \rho^2 - 2\rho - 1 = 0$ and $\rho = 2 \cos(2\pi/7)$. If $U = \rho^2 + 2\rho - 3$ and $V = \rho^2 + \rho - 2$ then

$$Y_0 = (\rho - 2)^2(\rho^2 - \rho + 1)/7,$$

$$Y_1 = (\rho - 2)^2(\rho^2 - 1)/7$$

gives a solution in $K$. 

6
Let $V$ be a Del Pezzo surface of degree 4 (that is, the smooth intersection of two quadrics in $\mathbb{P}^4$) defined over an algebraic number field $k$. Salberger and Skorobogatov have shown that the only obstruction to weak approximation on $V$ is the Brauer-Manin obstruction. More precisely:

**Theorem 3** Suppose that $V(k)$ is not empty. Let $A$ be the subset of the adelic space $V(A)$ consisting of the points $\prod P_v$ such that

$$\sum \text{inv}_v(A(P_v)) = 0 \text{ in } \mathbb{Q}/\mathbb{Z}$$

for all $A$ in the Brauer group $\text{Br}(V)$. Then the image of $V(k)$ is dense in $A$. 


Write the Del Pezzo surface $V$ as $Q_1 \cap Q_2$ where $Q_1, Q_2$ are quadrics in $\mathbb{P}^4$. Choose coordinates so that the given point of $V(k)$ is $(1, 0, 0, 0, 0)$ and the tangents to $Q_1, Q_2$ at this point are $X_1 = 0, X_2 = 0$ respectively. The equations of $Q_1$ and $Q_2$ take the form

$$X_0X_1 + f_1(X_1, \ldots, X_4) = 0,$$
$$X_0X_2 + f_2(X_1, \ldots, X_4) = 0$$

where $f_1$ and $f_2$ are homogeneous quadratic. Thus $V$ is birationally equivalent to the cubic surface $X_2f_1 = X_1f_2$, which is indeed obtained by blowing up the given point of $V(k)$; and this cubic surface is birationally equivalent to the pencil of affine conics

$$Vf_1(U, V, X_3, X_4) = Uf_2(U, V, X_3, X_4),$$

which can be parametrized by the points $(U, V)$ of $\mathbb{P}^1$. 
Diagonalizing this equation and then making it homogeneous gives a pencil of conics of the form

\[ Z_0^2 g_1(U, V) + Z_1^2 \frac{g_2(U, V)}{g_1(U, V)} + Z_2^2 \frac{g_5(U, V)}{g_2(U, V)} = 0, \]

where \( g_r \) is homogeneous of degree \( r \). Writing

\[ Z_0 = g_2 Y_0, \quad Z_1 = g_1 Y_1, \quad Z_2 = g_1 g_2 Y_2 \]

and dividing by \( g_1 g_2 \) we obtain

\[ g_2 Y_0^2 + Y_1^2 + g_1 g_5 Y_2^2 = 0. \]
Lemma 2 Let $V$ be a Del Pezzo surface of degree 4, defined over a field $L$ of characteristic 0. If $V$ contains a positive 0-cycle of degree 2 and a positive 0-cycle of odd degree $n$, both defined over $L$, then $V(L)$ is not empty.

Theorem 4 Let $V$ be a del Pezzo surface of degree 4, defined over a field $L$ of characteristic 0. If $V$ contains a 0-cycle of odd degree defined over $L$ then $V(L)$ is not empty.
Now consider pencils of 2-coverings of elliptic curves, where the underlying pencil of elliptic curves has the form
\[ E : Y^2 = \prod_{i=1}^{3} (X - c_i(U, V)). \]
Here the \( c_i(U, V) \) are homogeneous polynomials in \( \mathcal{O}[U, V] \) all having the same even degree.

**Lemma 3** Suppose that the Tate-Shafarevich group of \( E/k \) is finite and the 2-Selmer group of \( E \) has order 8. Then every curve representing an element of the 2-Selmer group contains points defined over \( k \).
We can apply these ideas to K3 surfaces defined over \( \mathbb{Q} \) whose equation has the form

\[
V : a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0.
\]

There is an obvious map from \( V \) to the quadric surface

\[
W : a_0Y_0^2 + a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 = 0.
\]

If \( a_0a_1a_2a_3 \) is a square then each of the two pencils of lines on \( W \) is defined over \( \mathbb{Q} \), and a general line of either pencil pulls back to a curve of genus 1 on \( V \) which is a 2-covering of its Jacobian.

**Theorem 5** Assume Schinzel’s Hypothesis and the finiteness of all relevant Tate-Shafarevich groups. Let \( V \) be everywhere locally soluble and such that \( a_0a_1a_2a_3 \) is a square. Suppose also that no \( -a_ia_j \) is in \( \mathbb{Q}^*2 \). If \( A \) is not empty and Condition D holds, then \( V \) contains rational points.