Three Quantum Feller semigroups

3.1 C^* algebras

Definition 3.1. A *Banach algebra* is a Banach space which is also a complex associative algebra, so has a multiplication compatible with the vector-space operators and the norm, which is submultiplicative. If the Banach algebra is *unital*, so that it has a multiplicative identity 1, called its *unit*, then we require the norm $\|1\|$ to be 1.

An *involution* on a Banach algebra is an isometric conjugate-linear map which reverses products and is self inverse.

A Banach algebra with involution A is a C^* *algebra* if and only if the C^* *identity* holds:

$$\|a^*a\| = \|a\|^2 \qquad \text{for all } a \in \mathsf{A}.$$

Remark 3.2. The C^* identity connects the algebraic and analytic structure of the algebra in a very rigid way. For example, there exists at most one norm for which an associative algebra is a C^* algebra.

Theorem 3.3. (Gelfand) Every commutative C^* algebra is isometrically isomorphic to $C_0(S)$, where S is a locally compact Hausdorff space.

Theorem 3.4. (Gelfand–Naimark) Any C^* algebra is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of $B(\mathsf{H})$ for some Hilbert space H (a *concrete* C^* *algebra*).

Remark 3.5. Let A be a C^* algebra and, for all $n \in \mathbb{N}$, let $M_n(\mathsf{A})$ be the complex associative algebra of $n \times n$ matrices with entries in A, equipped with algebraic operations in the usual manner. By the Gelfand–Naimark theorem, we may assume that $\mathsf{A} \subseteq B(\mathsf{H})$ for some Hilbert space H, and so $M_n(\mathsf{A}) \subseteq B(\mathsf{H}^n)$. We equip $M_n(\mathsf{A})$ with the restriction of the operator norm on $B(\mathsf{H}^n)$, and then $M_n(\mathsf{A})$ becomes a C^* algebra.

[This observation is the root of the theory of operator spaces.]

Definition 3.6. A concrete C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$ is a *von Neumann algebra* if and only if any of the following equivalent conditions hold.

(i) Closure in the strong operator topology: if the net $(a_i) \subseteq \mathsf{A}$ and $a \in B(\mathsf{H})$ are such that $a_i x \to ax$ for all $x \in \mathsf{H}$, then $a \in \mathsf{A}$.

(ii) Closure in the weak operator topology: if $(a_i) \subseteq \mathsf{A}$ and $a \in B(\mathsf{H})$ are such that $\langle x, a_i x\rangle \to \langle x, ax\rangle$ for all $x \in \mathsf{H}$, then $a \in \mathsf{A}$.

(iii) Equality with its commutant: letting

$$S' := \{a \in \mathsf{A} : [a, b] = 0 \text{ for all } b \in S\}$$

denote the commutant of $S \subseteq \mathsf{A}$, then $\mathsf{A}'' := (\mathsf{A}')' = \mathsf{A}$ [von Neumann].

(iv) Existence of a predual: there exists a Banach space A_* such that $(\mathsf{A}_*)^* = \mathsf{A}$ [Sakai].

3.2 Positivity

Definition 3.7. In a C^* algebra A we have the notion of *positivity*: we write $a \geqslant 0$ if there exists $b \in \mathsf{A}$ such that $a = b^* b$. The set of positive elements in A is denoted by A_+, is closed in the norm topology and is a *cone*: it is closed under addition and multiplication by non-negative scalars. Note that a positive element is self adjoint.

Exercise 3.8. Let $\mathsf{A} = C_0(S)$ and prove that $f \in \mathsf{A}_+$ if and only if $f(x) \geqslant 0$ for all $x \in S$. Prove also that if the C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$, where H is a Hilbert space, then $a \in \mathsf{A}_+$ if and only if $\langle x, ax\rangle \geqslant 0$ for all $x \in \mathsf{H}$. [The existence of square roots is crucial in both cases.]

Definition 3.9. A linear map $\Phi : \mathsf{A} \to \mathsf{B}$ is *positive* if and only if $\Phi(\mathsf{A}_+) \subseteq \mathsf{B}_+$. A positive map is automatically bounded.

Exercise 3.10. Prove that a positive linear map commutes with the involution. [Hint: an arbitrary element in a C^* algebra A may be written in the form $(a_1 - a_2) + \mathrm{i}(a_3 - a_4)$, where $a_1, \ldots, a_4 \in \mathsf{A}_+$.]

Definition 3.11. A linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is *n-positive*, for some $n \in \mathbb{N}$, if and only if the ampliation

$$\Phi^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{B});\ (a_{ij}) \mapsto \big(\Phi(a_{ij})\big)$$

is positive. [Identifying $M_n(\mathsf{A})$ with $M_n(\mathbb{C}) \otimes \mathsf{A}$, and similarly for $M_n(\mathsf{B})$, it follows immediately that $\Phi^{(n)} = \mathrm{id}_{M(\mathbb{C}^n)} \otimes \Phi$.]

If Φ is n-positive for all $n \in \mathbb{N}$ then Φ is *completely positive*.

Exercise 3.12. Prove that any $*$-homomorphism between C^* algebras is completely positive. Prove also that if $V \in B(\mathsf{H}; \mathsf{K})$ then

$$B(\mathsf{K}) \to B(\mathsf{H});\ a \mapsto V^* a V$$

is completely positive.

Exercise 3.13. (Paschke) A linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is completely positive if and only if

$$\sum_{i,j=1}^{n} b_i^* \Phi(a_i^* a_j) b_j \geqslant 0$$

for all $n \geqslant 1$, $a_1, \ldots, a_n \in \mathsf{A}$ and $b_1, \ldots, b_n \in \mathsf{B}$. [Hint: there is a faithful representation of B as a concrete C^* algebra which is a direct sum of cyclic representations.]

Theorem 3.14. A positive linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between C^* algebras is completely positive if A is commutative [Stinespring] or B is commutative [Arveson].

Theorem 3.15. (Kadison) A 2-positive unital linear map $\Phi : \mathsf{A} \to \mathsf{B}$ between unital C^* algebras is such that

$$\Phi(a)^* \Phi(a) \leqslant \Phi(a^* a) \qquad \text{for all } a \in \mathsf{A}. \tag{3.1}$$

Proof. Note first that if $a \in \mathsf{A}$ then

$$A := \begin{bmatrix} 1 & a \\ a^* & a^* a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \geqslant 0,$$

so

$$0 \leqslant \Phi^{(2)}(A) = \begin{bmatrix} 1 & \Phi(a) \\ \Phi(a)^* & \Phi(a^* a) \end{bmatrix}.$$

Suppose without loss of generality that $\mathsf{B} \subseteq B(\mathsf{H})$ for some Hilbert space H, and note that, by Exercise 3.8, if $x \in \mathsf{H}$ and

$$\xi := \begin{bmatrix} -\Phi(a)x \\ x \end{bmatrix} \in \mathsf{H}^2 \quad \text{then} \quad 0 \leqslant \langle \xi, \Phi^{(2)}(A)\xi \rangle = \langle x, \big(\Phi(a^* a) - \Phi(a)^* \Phi(a)\big) x \rangle.$$

As x is arbitrary, the claim follows. □

Remark 3.16. The inequality (3.1) is known as the *Kadison–Schwarz* inequality.

Exercise 3.17. Show that the inequality (3.1) holds if Φ is required only to be positive as long as a is *normal*, so that $a^* a = a a^*$. [Hint: use Theorem 3.14.]

3.3 Stinespring's dilation theorem

Theorem 3.18. (Stinespring) Let $\Phi : \mathsf{A} \to B(\mathsf{H})$ be a linear map, where A is a unital C^* algebra and H is a Hilbert space. Then Φ is completely positive if and only if there exists a Hilbert space K, a unital $*$-homomorphism $\pi : \mathsf{A} \to B(\mathsf{K})$ and a bounded operator $V : \mathsf{H} \to \mathsf{K}$ such that

$$\Phi(a) = V^* \pi(a) V \qquad (a \in \mathsf{A}).$$

Furthermore, $\|\Phi(1)\| = \|V\|^2$.

Proof. One direction is immediate. For the converse, let $\mathsf{K}_0 := \mathsf{A} \underline{\otimes} \mathsf{H}$ be the algebraic tensor product of A with H, considered as complex vector spaces. Define a sesquilinear form on K_0 such that

$$\langle a \otimes x, b \otimes y \rangle = \langle x, \Phi(a^* b) y \rangle_{\mathsf{H}} \qquad \text{for all } a, b \in \mathsf{A} \text{ and } x, y \in H.$$

It is an exercise to check that this form is positive semidefinite; this follows from the assumption that Φ is completely positive. Furthermore, the kernel

$$\mathsf{K}_{00} := \{\xi \in \mathsf{K}_0 : \langle \xi, \xi \rangle = 0\}$$

is a vector subspace of K_0. Let K be the completion of $\mathsf{K}_0/\mathsf{K}_{00} = \{[\xi] : \xi \in \mathsf{K}_0\}$.

If

$$\pi(a)[b \otimes x] := [ab \otimes x] \qquad \text{for all } a, b \in \mathsf{A} \text{ and } x \in \mathsf{H},$$

then $\pi(a)$ extends by linearity and continuity to an element of $B(\mathsf{K})$, denoted in the same manner. Furthermore, the map $a \mapsto \pi(a)$ is a unital $*$-homomorphism from A to $B(\mathsf{K})$.

Finally, let $V \in B(\mathsf{H}; \mathsf{K})$ be defined by setting $Vx = [1 \otimes x]$ for all $x \in \mathsf{H}$. It is a final exercise to verify that $\Phi(a) = V^* \pi(a) V$, as required. □

Corollary 3.19. If $\Phi : \mathsf{A} \to B(\mathsf{H})$ is as in Theorem 3.18, with $\Phi(1) = I$, then

$$\sum_{i,j=1}^{n} \langle v_i, \big(\Phi(a_i^* a_j) - \Phi(a_i)^* \Phi(a_j)\big) v_j \rangle \geqslant 0$$

for all $n \geqslant 1$, $a_1, \ldots, a_n \in \mathsf{A}$ and $v_1, \ldots, v_n \in \mathsf{H}$.

Proof. Note first that $\|V^*\| = \|V\| = \|\Phi(1)\|^{1/2} = 1$. Hence

$$\begin{aligned}
\sum_{i,j=1}^{n} \langle v_i, \Phi(a_i^* a_j) v_j \rangle = \sum_{i,j=1}^{n} \langle V v_i, \pi(a_i^* a_j) V v_j \rangle &= \Big\| \sum_{i=1}^{n} \pi(a_i) V v_i \Big\|^2 \\
&\geqslant \Big\| V^* \sum_{i=1}^{n} \pi(a_i) V v_i \Big\|^2 \\
&= \Big\| \sum_{i=1}^{n} \Phi(a_i) v_i \Big\|^2 \\
&= \sum_{i,j=1}^{n} \langle v_i, \Phi(a_i)^* \Phi(a_j) v_j \rangle.
\end{aligned}$$

□

Definition 3.20. A pair (π, V) as in Theorem 3.18 is a *Stinespring dilation* of Φ. Such a dilation is *minimal* if

$$\mathsf{K} = \overline{\operatorname{lin}}\{\pi(a)Vx : a \in \mathsf{A},\ x \in \mathsf{H}\}.$$

Proposition 3.21. A completely positive map $\Phi : \mathsf{A} \to B(\mathsf{H})$ has a minimal Stinespring dilation.

Proof. One may take (π, V) as in Theorem 3.18 and restrict to the closed subspace of K containing $\{\pi(a)Vx : a \in \mathsf{A},\ x \in \mathsf{H}\}$. □

Exercise 3.22. Prove that the minimal Stinespring dilation is unique in an appropriate sense.

Definition 3.23. Let $(a_i) \subseteq \mathsf{A}$ be a net in the von Neumann algebra $\mathsf{A} \subseteq B(\mathsf{H})$. We write $a_i \searrow 0$ if $a_i - a_j \in \mathsf{A}_+$ whenever $i \geqslant j$ and $\langle x, a_i x\rangle \to 0$ for all $x \in \mathsf{H}$.

A map $\Phi : \mathsf{A} \to B(\mathsf{K})$ is *normal* if $\Phi(a_i) \searrow 0$ whenever $a_i \searrow 0$.

Corollary 3.24. If A is a von Neumann algebra and Φ is normal then π in Theorem 3.18 be may chosen to be normal also.

Proof. Let (π, V) be a minimal Stinespring dilation for Φ. If $x \in \mathsf{H}$, $a \in \mathsf{A}$ and $(a_i) \subseteq \mathsf{A}_+$ is such that $a_i \searrow 0$ then

$$\langle \pi(a)Vx, \pi(a_i)\pi(a)Vx\rangle = \langle x, V^*\pi(a^*a_i a)Vx\rangle = \langle x, \Phi(a^*a_i a)x\rangle \to 0,$$

since $a^*a_i a \searrow 0$. It follows by polarisation and minimality that $\pi(a_i) \searrow 0$, as claimed. □

3.4 The Gorini–Kossakowski–Sudershan–Lindblad theorem

Definition 3.25. A *quantum Feller semigroup* $T = (T_t)_{t\in\mathbb{R}_+}$ on a C^* algebra A is a strongly continuous contraction semigroup with T_t completely positive for all $t \in \mathbb{R}_+$.

If A is unital, with unit 1, and $T_t 1 = 1$ for all $t \in \mathbb{R}_+$ then T is *conservative*.

Theorem 3.26. Let T be a uniformly continuous quantum Feller semigroup on the unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$. Its generator $\mathcal{L}$ is bounded, $*$-preserving and *conditionally completely positive*: if $n \geqslant 1$, $a_1, \ldots, a_n \in \mathsf{A}$ and $v_1, \ldots, v_n \in \mathsf{H}$ then

$$\sum_{i,j=1}^{n} \langle v_i, \mathcal{L}(a_i^* a_j)v_j\rangle \geqslant 0 \quad \text{whenever} \quad \sum_{i=1}^{n} a_i v_i = 0.$$

Proof. The boundedness of $\mathcal{L}$ follows immediately from Theorem 1.21, and if $a \in \mathsf{A}$ then

$$\mathcal{L}(a)^* = \lim_{t\to 0+} t^{-1}(T_t(a) - a)^* = \lim_{t\to 0+} t^{-1}(T_t(a^*) - a^*) = \mathcal{L}(a^*),$$

by continuity of the involution and the fact that positive maps are $*$-preserving.

To see that conditional complete positivity holds, let $a_1, \ldots, a_n \in \mathsf{A}$ and $v_1, \ldots, v_n \in \mathsf{H}$. By Corollary 3.19, if $t > 0$ then

$$t^{-1} \sum_{i,j=1}^{n} \langle v_i, \big(T_t(a_i^* a_j) - T_t(a_i)^* T_t(a_j)\big) v_j\rangle \geqslant 0.$$

Letting $t \to 0+$ gives that

$$\sum_{i,j=1}^{n} \langle v_i, \big(\mathcal{L}(a_i^* a_j) - \mathcal{L}(a_i)^* a_j - a_i^* \mathcal{L}(a_j)\big) v_j\rangle \geqslant 0,$$

and if $\sum_{i=1}^{n} a_i v_i = 0$ then the second and third terms vanish. □

Theorem 3.27. (Lindblad, Evans) Let $\mathcal{L}$ be a $*$-preserving bounded linear map on the unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{H})$. The following are equivalent.

(i) $\mathcal{L}$ is conditionally completely positive.

(ii) $(zI - \mathcal{L})^{-1}$ is completely positive for all sufficiently large $z > 0$.

(iii) $T_t = \exp(t\mathcal{L})$ is completely positive for all $t \in \mathbb{R}_+$.

The semigroup T which arises is conservative if and only if $\mathcal{L}(1) = 0$.

Proof. The equivalence of (ii) and (iii) is a consequence of Theorem 1.37 and (2.4), and Theorem 3.26 gives that (iii) implies (i). That (i) implies (iii) is an exercise, as is the final remark. □

Remark 3.28. Since completely positive unital linear maps between unital C^* algebras are automatically contractive, this characterises the generators of uniformly continuous conservative quantum Feller semigroups on unital C^* algebras.

Theorem 3.29. (Lindblad, Christensen) If $\mathsf{A} \subseteq B(\mathsf{H})$ is a von Neumann algebra then $\mathcal{L}$ is conditionally completely positive and normal if and only if there exists a normal completely positive map $\Psi : \mathsf{A} \to \mathsf{A}$ and an element $G \in \mathsf{A}$ such that

$$\mathcal{L}(a) = \Psi(a) + G^* a + aG \qquad \text{for all } a \in \mathsf{A}.$$

Remark 3.30. If A is just a C^* algebra then Christensen and Evans have showed that Theorem 3.29 remains true with $\mathcal{L}$ and Ψ no longer required to be normal, but G and the range of Ψ must be taken to lie in the ultraweak closure of A.

Theorem 3.31. (Kraus) Suppose $\mathsf{A} \subseteq B(\mathsf{H})$ is a von Neumann algebra. A linear map $\Psi : \mathsf{A} \to B(\mathsf{K})$ is normal and completely positive if and only if there exists a family of operators $(L_i)_{i\in\mathbb{I}} \subseteq B(\mathsf{K};\mathsf{H})$ such that

$$\Psi(a) = \sum_{i\in\mathbb{I}} L_i^* a L_i \qquad \text{for all } a \in \mathsf{A},$$

with convergence in the strong operator topology. The cardinality of the index set $\mathbb{I}$ may be taken to be no larger than $\dim \mathsf{K}$.

Remark 3.32. With Ψ and $(L_i)_{i\in\mathbb{I}}$ as in Theorem 3.31, we may write

$$\Psi(a) = L^*(a \otimes I_{\mathsf{K}_1})L$$

for some $L \in B(\mathsf{K}; \mathsf{H} \otimes \mathsf{K}_1)$; suppose K_1 has orthonormal basis $(e_i)_{i\in\mathbb{I}}$ and let

$$L : \mathsf{K} \to \mathsf{H} \otimes \mathsf{K}_1;\ x \mapsto \sum_{i\in\mathbb{I}} x \otimes e_i.$$

Lemma 3.33. Let T be a uniformly continuous semigroup on a von Neumann algebra with generator $\mathcal{L}$. Then $\mathcal{L}$ is normal if and only if T_t is normal for all $t \in \mathbb{R}_+$.

Theorem 3.34. (Gorini–Kossakowski–Sudarshan, Lindblad) A bounded linear map $\mathcal{L}$ on a von Neumann algebra $\mathsf{A} \subseteq B(\mathsf{H})$ is the generator of a uniformly continuous conservative quantum Feller semigroup composed of normal maps if and only if

$$\mathcal{L}(a) = -\mathrm{i}[H, a] - \tfrac{1}{2}\big(L^*La - 2L^*(a \otimes I)L + aL^*L\big) \qquad \text{for all } a \in B(\mathsf{H}),$$

where $H = H^* \in B(\mathsf{H})$ and $L \in B(\mathsf{H}; \mathsf{H} \otimes \mathsf{K})$ for some Hilbert space K.

Proof. If $\mathcal{L}$ has this form then it is straightforward to verify that the semigroup it generates is as claimed.

Conversely, suppose $\mathcal{L}$ is the generator of a semigroup as in the statement of the theorem. Then Theorem 3.27 gives that $\mathcal{L}$ is conditionally completely positive and $\mathcal{L}(1) = 0$. Moreover, $\mathcal{L}$ is normal, by the preceding lemma, and so Theorem 3.29 gives that

$$\mathcal{L}(a) = \Psi(a) + G^*a + aG \qquad \text{for all } a \in \mathsf{A},$$

where Ψ is completely positive and normal, and $G \in \mathsf{A}$. Taking $a = 1$ in this equation shows that $G^* + G = -\Psi(1)$, so $G = -\frac{1}{2}\Psi(1) + \mathrm{i}H$ for some self adjoint $H \in \mathsf{A}$. The result now follows by Theorem 3.31. ∎

3.5 Quantum Markov processes

Remark 3.35. Let S be a compact Hausdorff space. If X is an S-valued random variable on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ then

$$j_X : \mathsf{A} \to \mathsf{B};\ f \mapsto f \circ X$$

is a unital $*$-homomorphism, where $\mathsf{A} = C(S)$ and $\mathsf{B} = L^\infty(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 3.36. A *non-commutative random variable* is a unital $*$-homomorphism j between unital C^* algebras.

A family $(j_t : \mathsf{A} \to \mathsf{B})_{t \in \mathbb{R}_+}$ of non-commutative random variables is a *dilation* of the quantum Feller semigroup T on A if there exists a conditional expectation $\mathbb{E}$ from B onto A such that $T_t = \mathbb{E} \circ j_t$ for all $t \in \mathbb{R}_+$.

Many authors have tackled this problem of constructing such dilations: Evans and Lewis; Davies; Accardi, Frigerio and Lewis; Vincent-Smith; Kümmerer; Sauvageot; Bhat and Parthasarathy; ....

Essentially, one attempts to mimic the functional-analytic proof of Theorem 2.18. Given an initial 'measure' μ, which is a state on the C^* algebra A, the sesquilinear form

$$\mathsf{A}^{\underline{\otimes} n} \times \mathsf{A}^{\underline{\otimes} n} \to \mathbb{C};\ (a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) \mapsto \mu\big(T_{t_1}(a_1^* \ldots (T_{t_n - t_{n-1}}(a_n^* b_n)) \ldots b_1)\big)$$

must be shown to be positive semidefinite, and the key to this is the complete positivity of the semigroup maps. There are many technical details to be addressed.