Binary quadratic forms with large discriminants and sums of two squareful numbers II

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Abstract

Let $\mathbf{F} = (F_1, \ldots, F_m)$ be an m-tuple of primitive positive binary quadratic forms and let $U_{\mathbf{F}}(x)$ be the number of integers not exceeding x that can be represented simultaneously by all the forms F_j, $j = 1, \ldots, m$. Sharp upper and lower bounds for $U_{\mathbf{F}}(x)$ are given, uniformly in the discriminants of the quadratic forms.
As an application a problem of Erdös is considered. Let $V(x)$ be the number of integers not exceeding x that are representable as a sum of two squareful numbers. Then $V(x) = x(\log x)^{-\alpha+o(1)}$ with $\alpha = 1 - 2^{-\frac{1}{3}} = 0.206\ldots$

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1 Introduction

It is well-known that an integer n is called squareful if $p \mid n$ implies $p^2 \mid n$ for all primes p. The distribution of squareful integers is quite regular, in fact, the number of squareful integers not exceeding x satisfies $\frac{\zeta(3/2)}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta(2/3)}{\zeta(2)} x^{\frac{1}{3}} + o(x^{\frac{1}{6}})$. This can be seen by writing a squareful number n uniquely as $n = a^3 b^2$ with $\mu^2(a) = 1$ (see e.g. [2]).
For a long time it has been an open problem to determine the order of

magnitude for the number $V(x)$ of integers not exceeding x that can be written as a sum of two squareful integers. This problem goes back to Erdös who conjectured (probably with Landau's two squares theorem in mind) $V(x) \asymp x(\log x)^{-\frac{1}{2}}$. However, this is far from the truth; in a recent paper [3] the auther showed $x(\log x)^{-0.253} \ll V(x) \ll x(\log x)^{-1/6} \log\log x$, thereby improving on earlier bounds by Baker-Brüdern [1] and Odoni [8]. Here we shall prove:

Theorem 1. *Let $V(x)$ denote the number of integers not exceeding x that are the sum of two squareful integers, and let $\alpha = 1 - 2^{-1/3} = 0.206\ldots$ Then*

$$\frac{x}{(\log x)^{\alpha+\varepsilon}} \ll V(x) \ll \frac{x}{(\log x)^{\alpha-\varepsilon}}$$

for any $\varepsilon > 0$, or, in other words, $V(x) = x(\log x)^{-\alpha+o(1)}$. The implied constant for the lower bound can be made effective. The same inequalities hold for sums of a square and a squareful number.

The line of attack is similar to [3] with a number of additional refinements. In particular, for the upper bound some further ideas are necessary. As in [3], Theorem 1 follows rather easily from a uniform result on the representation of integers by certain positive definite binary quadratic forms:

Theorem 2. *Let $M > 0$, $\varepsilon > 0$, $m \in \mathbb{N}$, and define for $u \geq 0$ the continuous function*

$$E(u) = \begin{cases} 1 - 2^{-m} & \text{if } u < 2^{-m} \\ 1 + u(\log(2^m u) - 1) & \text{if } 2^{-m} \leq u < 1 \\ u \log(2^m) & \text{if } u \geq 1. \end{cases}$$

Let $\boldsymbol{F} = (F_1, \ldots, F_m)$ be an m-tuple of primitive positive binary quadratic forms. For $1 \leq j \leq m$ let $D_j = D_{0,j} f_j^2$ be the discriminant of F_j, $D_{0,j}$ being a fundamental discriminant, and assume

$$(D_{0,i}, D_{0,j}) = 1 \text{ and } (D_i, D_j) \mid 4 \text{ for } i \neq j. \tag{1.1}$$

For $x \geq x_0(M, \varepsilon, m)$ let $U_{\boldsymbol{F}}(x)$ be the number of integers not exceeding x that can be represented simultaneously by all the forms F_j, $j = 1, \ldots, m$. Let

$$\kappa_j := \frac{\log|D_j|}{(2\log 2)\log\log x}, \quad \boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_m), \tag{1.2}$$

and assume $\|\boldsymbol{\kappa}\|_\infty \leq M$. *Then*

$$\frac{x}{(\log x)^{E(\|\boldsymbol{\kappa}\|_\infty)+\varepsilon}} \ll U_{\boldsymbol{F}}(x) \ll \frac{x}{(\log x)^{E\left(\frac{1}{m}\|\boldsymbol{\kappa}\|_1\right)-\varepsilon}} \tag{1.3}$$

where the implied constants depend at most *on* M, ε *and* m. *Here we write as usual* $\|\boldsymbol{\kappa}\|_\infty = \max_{1\leq j\leq m}\kappa_j$ *and* $\|\boldsymbol{\kappa}\|_1 = \sum_{j=1}^m \kappa_j$.

For one single quadratic form, i.e. $m = 1$, Theorem 2 reads as follows:

Corollary 1.1. *Let* $M > 0$, *and let* x *be sufficiently large. Let* F *be a primitive positive binary quadratic form with discriminant* D *satisfying*

$$|D| \sim (\log x)^{2\kappa \log 2}$$

for some $0 \leq \kappa \leq M$. *Then*

$$U_F(x) = x(\log x)^{-E(\kappa)+o(1)}.$$

Our notion of representation combines both proper and improper representation. Since we use Siegel's theorem [10] several times, the implied constants are at present not explicitly computable. However, this has no effect on the lower bound in Theorem 1 since we can just exclude the rare forms with "exceptional" discriminants (if there are any at all). It should be possible to replace the factors $(\log x)^{\pm\varepsilon}$ in Theorem 2 by some power of $\log\log x$ for discriminants having no Siegel zero.
It is interesting to compare the exponents in Theorem 2 with the following two "trivial" estimates: With $D := \prod_{j=1}^m D_j$ we have under the assumption (1.1)

$$U_{\mathbf{F}}(x) \ll |D|^\varepsilon x(\log x)^{\frac{1}{2^m}-1} \tag{1.4}$$

by counting numbers representable by *some* forms of discriminant $D_1, \ldots, D_m$, i.e. consisting basically of primes that are totally split in $K = \mathbb{Q}(\sqrt{D_1}, \ldots, \sqrt{D_m})$. On the other hand, the methods in [12] show under the assumption (1.1)

$$U_{\mathbf{F}}(x) \ll |D|^\varepsilon \frac{x}{\sqrt{|D|}}, \tag{1.5}$$

obtained by counting the represented numbers *with multiplicity*. Both (1.4) and (1.5) hold uniformly at least in $|D| \leq (\log x)^M$, say, as can be seen by a standard application of Perron's formula to $(\zeta_K(s))^{1/2^m} H_1(s)$ and, with the notation of section 2,

$$\frac{1}{h} \sum_{(\chi_1,\ldots,\chi_m)\in\widehat{\mathfrak{C}}} \left(\prod_{j=1}^{m} \bar{\chi}_j(C_j)\right) L_K(s, \iota(\chi_1,\ldots,\chi_m))H_2(s)$$

respectively; here we have written $\mathbf{C_F} = (C_1,\ldots,C_m)$, and H_1, H_2 are suitable holomorphic functions in $\Re s > 1/2$.

Theorem 2 shows that (1.4) gives the correct order of magnitude for $\|\boldsymbol{\kappa}\|_\infty < 2^{-m}$, thus in this range a typical number being represented at all is represented simultaneously by a large number of forms of discriminants $D_1,\ldots,D_m$. If all $\kappa_j > 1$, then we see by (1.5) that a typical number is represented with small multiplicity.

The bounds (1.4) and (1.5) can be mixed: For any κ_0 with $2^{-m} \le \kappa_0 < 1$ we may estimate the exponent $E(\frac{1}{m}\|\boldsymbol{\kappa}\|_1)$ from below by the tangent line $\log(2^m\kappa_0)\frac{1}{m}\|\boldsymbol{\kappa}\|_1 + (1-\kappa_0)$ at a point κ_0. Thus we obtain by (1.2) and (1.3)

$$U_{\mathbf{F}}(x) \ll \frac{x}{(\log x)^{\log(2^m\kappa_0)\frac{1}{m}\|\boldsymbol{\kappa}\|_1+1-\kappa_0-\varepsilon}} = \frac{x}{(\log x)^{1-\kappa_0-\varepsilon}|D|^{\frac{\log(2^m\kappa_0)}{m\log 4}}},$$

and get the following hybrid bound:

Corollary 1.2. *Under the assumptions of Theorem* 2 *and with* $D = \prod_{j=1}^m D_j$ *we have*

$$U_{\boldsymbol{F}}(x) \ll \frac{x}{(\log x)^{1-\kappa_0-\varepsilon}|D|^{\frac{1}{2}\left(1+\frac{\log\kappa_0}{\log 2^m}\right)}}$$

for $2^{-m} \le \kappa_0 < 1$, *uniformly for all* $\boldsymbol{F}$ *satisfying* (1.1) and $\max_{1\le j\le m}|D_j| \le (\log x)^M$.

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Notation. The letter p is reserved for (positive) prime numbers, $\mathfrak{p}$ for nonzero prime ideals in number fields. All implicit and explicit constants throughout the paper depend *at most* on M, m and ε. All constants c_i are positive. The symbol ε denotes an arbitrarily small positive real number whose value may differ on each occurrence. For a group G and subsets $A, B \subseteq G$ we define the *product set* as

$$AB := \{ab \mid a \in A, b \in B\}. \tag{1.6}$$

This can be generalized in an obvious way to more than two subsets.

2 Representation of integers by systems of binary quadratic forms

2.1 Quadratic fields and L-functions

Let $D_0 < 0$ be a fundamental discriminant. The group $\mathfrak{C}$ of classes of primitive binary quadratic forms with discriminant $D = D_0 f^2$ can be identified with a subgroup of the ideal class group mod (f) in $\mathbb{Q}(\sqrt{D_0})$ (see [3] for proofs and references). Classes $C \in \mathfrak{C}$ with $C^2 = 1$ are called ambiguous. We set $\epsilon(C) = 1$ if C is ambiguous and $\epsilon(C) = \frac{1}{2}$ otherwise. The cardinality h of $\mathfrak{C}$ can be obtained by the class number formula, $h = L(1, \chi_D)\pi^{-1}\sqrt{|D|}$ for $D < -4$; here $\chi_D = \left(\frac{D}{\cdot}\right)$ is the Jacobi-Kronecker symbol. By Siegel's theorem we have

$$|D|^{\frac{1}{2}-\varepsilon} \ll h \ll \sqrt{|D|}\log(|D|) \ll |D|^{\frac{1}{2}+\varepsilon} \tag{2.1}$$

for any $\varepsilon > 0$. Unfortunately we do not have much information about the algebraic structure of the class group; one of the very few known results is

$$\#\{\text{ambiguous classes}\} = \#\,\mathfrak{C}\,/\,\mathfrak{C}^2 = 2^{\omega(D)+\nu} \tag{2.2}$$

where $\omega(n) = \sum_{p|n} 1$ and $\nu \in \{-2, -1, 0\}$. In particular, $\#\,\mathfrak{C}\,/\,\mathfrak{C}^2 \ll |D|^{\varepsilon}$ for any $\varepsilon > 0$. Equation (2.2) is a corollary of the following

Lemma 2.1. *All real characters $\chi : \mathfrak{C} \to \mathbb{C}^*$ are given as follows: Let $D = D'D''r^2$ be a decomposition of D into a product of two fundamental discriminants (including 1) D', D'', and a perfect square. Then $\chi(\mathfrak{p}) = \chi_{D'f^2}(N\,\mathfrak{p})$ if $(N\,\mathfrak{p}, D'f^2) = 1$ and $\chi(\mathfrak{p}) = \chi_{D''f^2}(N\,\mathfrak{p})$ if $(N\,\mathfrak{p}, D''f^2) = 1$.*

The correspondence between classes of forms and ideal classes implies the following

Lemma 2.2. *A prime p is represented by some (primitive) form of discriminant D if and only if $p \nmid f$ and $\chi_D(p) \neq -1$. In this case it is representable in the class $C \in \mathfrak{C}$ if and only if there is a prime ideal $\mathfrak{p} \in C$ lying over p, and it is represented exactly by the classes C and C^{-1} (which may be the same).*
Let $n = \prod p_j^{\alpha_j} \prod q_j^{\beta_j} \prod r_j^{\gamma_j}$ be the canonical factorization of a natural number n where the first product is taken over all primes $p \nmid f$ with $\chi_D(p) \neq -1$, the second product over all primes q with $\chi_D(q) = -1$, and the third product over all primes r with $r \mid f$. Let C_j, C_j^{-1} be the classes that represent p_j and $\mathcal{C}(r_j, \gamma_j)$ the set of classes that represent $r_j^{\gamma_j}$. Then n is represented by forms

of discriminant D if and only if all β_j and γ_j are even, namely by exactly the classes in the formal expansion $\prod\{C_j, C_j^{-1}\}^{\alpha_j}\prod \mathcal{C}(r_j,\gamma_j)$ in the sense of (1.6)*.*

We fix some notation: For $j = 1,\dots,m$ let $D_j = D_{0,j}f_j^2$ be discriminants satisfying (1.1). Let $D = \prod_{j=1}^m D_j$, $f = \prod_{j=1}^m f_j$, and denote the class group of forms of discriminant D_j by $\mathfrak{C}_j$. The group $\mathfrak{C}_j$ is isomorphic to the ideal class group (mod f_j) in $\mathbb{Q}(\sqrt{D_j})$. We write

$$\underline{\mathfrak{C}} := \mathfrak{C}_1 \times \dots \times \mathfrak{C}_m, \quad h_j := \#\,\mathfrak{C}_j, \quad h := \#\underline{\mathfrak{C}}.$$

For $\mathbf{C} = (C_1, \dots C_m) \in \underline{\mathfrak{C}}$ we write $\epsilon(\mathbf{C}) = \prod_{j=1}^m \epsilon(C_j)$. Let

$$\begin{aligned}
&K := \mathbb{Q}(\sqrt{D_{0,1}},\dots,\sqrt{D_{0,m}}),\\
&\mathbb{P} = \left\{p \mid \left(\frac{D_j}{p}\right) = 1 \text{ for all } j\right\}, \quad \mathbb{P}_Q := \{p \in \mathbb{P} \mid p > Q\}\\
&\mathcal{R}(\mathbf{C}) = \{n \in \mathbb{N} \mid n \text{ representable by } C_1,\dots,C_m\},\\
&\mathcal{P}(\mathbf{C}) = \{p^n \in \mathbb{N} \mid n \text{ representable by } \tilde{C}_1,\dots,\tilde{C}_m \text{ satisfying } \tilde{C}_j^n = C_j\}.
\end{aligned}$$

Let $\mathfrak{C}(K)$ be the group of ideal classes (mod f) in K.

Lemma 2.3. *We have an injection*

$$\begin{array}{llcl}
\iota: & \widehat{\underline{\mathfrak{C}}} & \hookrightarrow & \widehat{\mathfrak{C}(K)}\\
& (\chi_1,\dots,\chi_m) & \longmapsto & \chi := \prod_{j=1}^m \chi_j \circ N_{K/\mathbb{Q}(\sqrt{D_j})}
\end{array} \tag{2.3}$$

where we regard the members of $\widehat{\mathfrak{C}}_j$ as functions on ideals being trivial on principal ideals (α) with $\alpha \equiv 1$ mod (f_j).

Proof. If (α) is a principal ideal in K with $\alpha \equiv 1 \pmod f$, then $N_{K/\mathbb{Q}(\sqrt{D_j})}(\alpha) \equiv 1 \pmod{f_j}$ for $1 \le j \le m$, thus $\chi((\alpha)) = 1$ and hence $\chi \in \widehat{\mathfrak{C}(K)}$.
Assume that $\chi \in \widehat{\mathfrak{C}(K)}$ is the principal character. Fix a prime p and prime ideals $\mathfrak{p}_j \mid (p)$ in $\mathbb{Q}(\sqrt{D_j})$. Let $\mathfrak{P}$ be the prime ideal in K lying over all $\mathfrak{p}_j$, and $\mathfrak{P}_i$ the prime ideal lying over $\bar{\mathfrak{p}}_i$ and all $\mathfrak{p}_j$ for $j \neq i$. Then

$$1 = \chi(\mathfrak{P}_i)\bar{\chi}(\mathfrak{P}) = \chi_i^{2f(\mathfrak{P}|\mathfrak{p}_i)}(\mathfrak{p}_i)$$

where $f(\mathfrak{P} \mid \mathfrak{p}_i)$ is the residue class degree. Thus $\psi_j := \chi_j^{2^m}$ is a real character for all j. By Lemma 2.1 we have $\psi_j(\mathfrak{p}) = \left(\frac{D'_j}{N\,\mathfrak{p}}\right)$ for a fundamental discriminant $D'_j \mid D_j$ and $(\mathfrak{p}, D) = 1$. Thus, if $\mathfrak{P} \mid (p)$ and $p \in \mathbb{P}$, then

$$1 = \chi^{2^m}(\mathfrak{P}) = \prod_{j=1}^{m} \left(\frac{D'_j}{p}\right).$$

By (1.1) the D'_j are pairwise coprime whence by the Chinese Remainder Theorem we conclude that D'_j or $\frac{D_j}{D'_j}$ is a square for all j. This implies $\psi_j = 1 \in \widehat{\mathfrak{C}}_j$, i.e. $\chi_j^{2^{m-1}}$ is real. Inductively we see that all χ_j must be principal.

For a character $\chi \in \widehat{\mathfrak{C}(K)}$ we consider the Hecke L-function, given in $\Re(s) > 1$ by

$$L_K(s, \chi) = \prod_{\mathfrak{P} \nmid (f)} \left(1 - \frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^s}\right)^{-1}.$$

For $Q \geq |D|$ we define

$$L_K(s, Q, \chi) := \prod_{p \in \mathbb{P}_Q} \prod_{\mathfrak{P} \mid (p)} \left(1 - \frac{\chi(\mathfrak{P})}{p^s}\right)^{-1}. \tag{2.4}$$

Corollary 2.4. *For a real character $\chi \in \widehat{\underline{\mathfrak{C}}} \subseteq \widehat{\mathfrak{C}(K)}$ there are 2^m (possibly imprimitive) real Dirichlet characters ψ_i with period dividing $|D|$ such that*

$$L_K(s, \chi) = \prod_{i=1}^{2^m} L(s, \psi_i).$$

In particular, we have

$$L_K(s, \chi_0) = \prod_{p \mid f} \left(1 - \frac{1}{p^s}\right) \zeta(s) \prod_{\emptyset \neq I \subseteq \{1,\ldots,m\}} L(s, \psi_{D(I)}) \tag{2.5}$$

where $D(I) = \prod_{i \in I} D_i$.

Proof. Since ι is a monomorphism, all χ_j in (2.3) must be real if χ is real. Then the decomposition follows immediately from Lemma 2.1.

Since the discriminant of K is $D^{2^{m-1}}$, we obtain by choosing

$$\delta = \delta(t) := \frac{2^{1-m}}{\log(|D|(1 + |t|))}$$

in Lemma 4 of [5]

$$L_K(s,\chi) \ll \log^{2^m}(|D|(1+|t|)) \tag{2.6}$$

for $\chi \in \widehat{\mathfrak{C}(K)}$, uniformly in $1-\delta \leq \sigma \leq 1+\delta$, provided that $|s-1| > \frac{1}{8}$ when $\chi = \chi_0$. It is also shown in [5] that there is no zero of an L-function attached to a complex character in

$$\sigma \geq 1 - \frac{2^{1-m}c_1}{\log\left(|D|(1+|t|)\right)} \tag{2.7}$$

for some constant $0 < c_1 < 1$. This remains true for real characters with one possible exceptional zero β that is necessarily real and satisfies $1-\beta \geq c_2(\varepsilon)|D|^{\varepsilon}$. By Caratheodory's inequality (see e.g. [6], §§73, 80) we conclude from (2.6) and (2.7)

(2.8)

$$\begin{aligned} \log L_K(\sigma+it,\chi) &\ll \left|\log L\left(1+\frac{c_1}{3}\delta+it,\chi\right)\right| + \log\log(|D|(1+|t|)) \\ &\ll \log(|D|^{\varepsilon}) + \log\log(|D|(1+|t|)) \ll \log|D| + \log\log(3+|t|) \end{aligned}$$

by Siegel's theorem, uniformly in

$$1-\min\left(\frac{2^{1-m}c_1}{3\log(|D|(1+|t|))}, c_2|D|^{-\varepsilon}\right) \leq \sigma \leq 1 + \frac{2^{1-m}c_1}{\log(|D|(1+|t|))},$$

provided that $\chi \neq \chi_0$ if $|t| \leq 1/8$.

From (2.6) we obtain for fixed $\mu \geq 1$ by Cauchy's integral formula

$$\frac{d^\mu}{ds^\mu}L_K(s,\chi) \ll \log^{c_3}(|D|(1+|t|))$$

uniformly in $1-\frac{\delta}{2} \leq \sigma \leq 1+\frac{\delta}{2}$. Using [7], Satz 7, for complex χ, and Corollary 2.4 together with Siegel's theorem for real nonprincipal χ, we see

$$\frac{d^\mu}{ds^\mu}\log L_K(s,\chi) \ll_{\varepsilon,\mu} |D|^{\varepsilon} \tag{2.9}$$

for $\chi \neq \chi_0$ on $[1-c_2|D|^{-\varepsilon}, 1]$, possibly after replacing c_2 with a smaller constant.

2.2 Some Dirichlet series

We start by defining a Dirichlet series that counts essentially the primes $p \in \mathcal{R}(\mathbf{C})$. A standard computation using (2.4), orthogonality of characters and Lemma 2.2 yields for $\mathbf{C} = (C_1, \dots, C_m) \in \underline{\mathfrak{C}}$ in $\Re(s) > 1$

(2.10)
$$\frac{1}{2^m h} \sum_{(\chi_1,\dots,\chi_m)\in\widehat{\underline{\mathfrak{C}}}} \left(\prod_{j=1}^{m} \bar{\chi}_j(C_j)\right) \log L_K(s, Q, \chi) = \epsilon(\mathbf{C}) \sum_{\substack{p^n\in\mathcal{P}(\mathbf{C})\\ p\in\mathbb{P}_Q}} \frac{1/n}{p^{ns}} =: P_{\mathbf{C},Q}(s)$$

where χ is given by (2.3) and Q is to be specified later in (4.1). We write

$$P_{\mathbf{C},Q}(s) = \frac{1}{2^m h} \log \zeta(s) + T(s, \mathbf{C}, Q). \tag{2.11}$$

Note that the remainder term T is real on $1 - c_2|D|^{-\varepsilon} < s < 1$. We shall see in Lemma 4.3 that $|T|$ is small in this region.

For the following we always assume $\Re(s) > 1$. Again by orthogonality we have

(2.12)
$$\prod_{\mathbf{C}\in\underline{\mathfrak{C}}} \exp(P_{\mathbf{C},Q}(s)) = \exp\left(\frac{1}{2^m} \sum_{p\in\mathbb{P}_Q} \sum_{\mathfrak{p}|(p)} \log\left(1 - \frac{1}{p^s}\right)^{-1}\right) = \sum_{p|n\Rightarrow p\in\mathbb{P}_Q} \frac{1}{n^s},$$

and clearly

$$\prod_{\mathbf{C}\in\underline{\mathfrak{C}}} \exp(P_{\mathbf{C},Q}(s)) = \exp\left(\sum_{\mathbf{C}\in\underline{\mathfrak{C}}} P_{\mathbf{C},Q}(s)\right) = \sum_{k=0}^{\infty} \sum_{(\mathbf{C}_1,\dots,\mathbf{C}_k)\in\underline{\mathfrak{C}}^k} \frac{1}{k!} \prod_{\nu=1}^{k} P_{\mathbf{C}_\nu,Q}(s).$$

In the series on the right-hand side of (2.12) we want to delete all n's that are not represented by a given m-tuple $\mathbf{C} = (C_1, \dots, C_m) \in \underline{\mathfrak{C}}$. We achieve this with the help of Lemma 2.2 by considering only a certain subset of summands in the sum over $(\mathbf{C}_1, \dots, \mathbf{C}_k) \in \mathfrak{C}^k$. Let

(2.13)
$$M_k(\mathbf{C}) := \left\{(\mathbf{C}_1, \dots, \mathbf{C}_k) \in \underline{\mathfrak{C}}^k \mid C_j \in \prod_{\nu=1}^{k} \{C_{\nu,j}^{-1}, C_{\nu,j}\} \text{ for all } 1 \le j \le m\right\}.$$

Here the product set is meant in the sense of (1.6) and we have written $\mathbf{C}_\nu = (C_{\nu,1}, \dots, C_{\nu,m})$. By Lemma 2.2 we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(\mathbf{C}_1,\dots,\mathbf{C}_k)\in M_k(\mathbf{C})} \prod_{\nu=1}^{k} P_{\mathbf{C}_\nu,Q}(s) = \sum_{\substack{p|n\Rightarrow p\in\mathbb{P}_Q\\ n\in\mathcal{R}(\mathbf{C})}} \frac{b(n)}{n^s},$$

where $0 \le b(n) \le 1$ and $b(n) = 1$ if $n \in \mathcal{R}(\mathbf{C})$ is a product of distinct primes $p \in \mathbb{P}_Q$. Let $2^{-m} \le \kappa < 1$ be a real parameter, and let

$$S_\kappa := [0, \kappa \log\log x] \cap \mathbb{N}_0 . \tag{2.14}$$

Then we define

$$A_{\mathbf{C},\kappa}(s) = \sum_{k \in S_\kappa} \frac{1}{k!} \sum_{(\mathbf{C}_1,\dots,\mathbf{C}_k) \in M_k(\mathbf{C})} \prod_{\nu=1}^{k} P_{\mathbf{C}_\nu,Q}(s), \tag{2.15}$$

obtaining a Dirichlet series $\sum a_n n^{-s}$ with the following properties: We have

- $0 \le a_n \le 1$ for all n,
- $a_n > 0$ only for $n \in \mathcal{R}(\mathbf{C})$,
- $a_n = 1$ if $n \in \mathcal{R}(\mathbf{C})$, $\Omega(n) \le \kappa \log\log x$ and n consists only of distinct prime factors $p \in \mathbb{P}_Q$.

Thus $A_{\mathbf{C},\kappa}$ is closely related to $\sum_{n \in \mathcal{R}(\mathbf{C})} n^{-s}$. The restrictions $k \le \kappa \log\log x$ and $p \ge Q$ in (2.12) and (2.15) are needed for the application of Lemma 4.3 and Lemma 4.4 later.
By Perron's formula we can express the quantity of interest, $U_{\mathbf{F}}(x)$, as a contour integral. For

$$S = \exp\left((\log x)^{\frac{1}{3}}\right) \tag{2.16}$$

and $c := 1 + (\log x)^{-1}$ we obtain

$$U_{\mathbf{F}}(x) \ge \frac{1}{2\pi i} \int_{c-iS}^{c+iS} A_{\mathbf{C}_{\mathbf{F}},\kappa}(s) \frac{x^s}{s} ds + O\left(\frac{x \log x}{S}\right) \tag{2.17}$$

where $\mathbf{C}_{\mathbf{F}}$ is the m-tuple of classes corresponding to the m-tuple of quadratic forms $\mathbf{F}$. Let

$$B_{\mathbf{C},Q}(s) := \delta_{\mathbf{C}} + \sum_{\substack{n \in \mathcal{R}(\mathbf{C}) \\ p|n \Rightarrow p \notin \mathbb{P}_Q}} \frac{1}{n^s}$$

where $\delta_{\mathbf{C}} = 1$ if $\mathbf{C} = 1 \in \underline{\mathfrak{C}}$ and else it vanishes, and let

$$\Omega^*(n, Q) := \sum_{p^{e_p} \| n, p \in \mathbb{P}_Q} e_p \le \Omega(n).$$

Then

$$U_{\mathbf{F}}^{(\kappa,Q)}(x) \leq \frac{1}{2\pi i}\int_{c-iS}^{c+iS}\sum_{\mathbf{C}\in\underline{\mathfrak{C}}} B_{\mathbf{C},Q}(s)A_{\mathbf{C_F}\mathbf{C}^{-1},\kappa}(s)\frac{x^s}{s}ds + O\left(\frac{x\log x}{S}\right) + O\left(\sum_{p>Q}\sum_{n\leq x, p^2|n} 1\right) \tag{2.18}$$

by Lemma 2.2, where $U_{\mathbf{F}}^{(\kappa,Q)}(x)$ denotes the number of integers $n \in \mathcal{R}(\mathbf{C_F})$ not exceeding x with $\Omega^*(n,Q) \leq \kappa \log\log x$. Our aim is now to estimate the right hand side of (2.17) and (2.18) which will be done in the following two sections. Let

$$\begin{aligned}
\Gamma_{1,1} &:= [1-(\log x)^{-\frac{1+\kappa}{2}} + iS, c+iS],\\
\Gamma_{2,1} &:= [1-(\log x)^{-\frac{1+\kappa}{2}}, 1-(\log x)^{-\frac{1+\kappa}{2}} + iS],\\
\Gamma_{3,1} &:= [1-\exp(-(\log\log x)^4), 1-(\log x)^{-\frac{1+\kappa}{2}}],\\
\Gamma_4 &:= \{s \in \mathbb{C} \mid |s-1| = \exp(-(\log\log x)^4)\}.
\end{aligned}$$

Let $\Gamma_{\nu,2}$ ($\nu = 1,\dots,3$) be the image of $\Gamma_{\nu,1}$ under reflection on the real axis, oriented such that

$$\Gamma := \Gamma_{1,2}\Gamma_{2,2}\Gamma_{3,2}\Gamma_4\Gamma_{3,1}\Gamma_{2,1}\Gamma_{1,1}$$

is homotopic to $[c-iS, c+iS]$. We remark that $s \in \Gamma$ implies

$$\sigma \geq 1 - \frac{1}{(\log x)^{\frac{1}{2}}}, \quad |t| \leq S. \tag{2.19}$$

Thus by (2.16) the bounds (2.6) - (2.9) are valid on Γ for $D \leq (\log x)^{2Mm}$ if ε is sufficiently small and x sufficiently large, and the functions $P_{\mathbf{C},Q}$ extend holomorphically to a neighbourhood of Γ. The main contribution to (2.17), (2.18) will arise from the integral over $\Gamma_{3,1}$ and $\Gamma_{3,2}$.

3 The error estimate

3.1 The integral over the non-dominating parts

We first remark that for

$$\log Q \leq (\log x)^{\frac{1}{2}} \tag{3.1}$$

and $s \in \Gamma$ we have by (2.4) and (2.19)

$$\log L_K(s,\chi) - \log L_K(s,Q,\chi) \ll \log\log Q \ll \log\log x. \tag{3.2}$$

Equation (1.2) implies

$$|D| = \prod_{j=1}^{m} |D_j| \leq (\log x)^{2Mm}. \tag{3.3}$$

In view of (2.8), (2.10), (2.19), (3.2) and (3.3) we obtain

$$P_{\mathbf{C},Q}(s) \ll \log\log x + \log|D| + \log\log(3+|t|) + \max_{\tilde{s}\in\Gamma_4} |\log\zeta(\tilde{s})| \ll (\log\log x)^4$$

for $s \in \Gamma$. Together with (2.1), (2.14), (2.15) and (3.3) this implies

$$A_{\mathbf{C},\kappa}(s) \ll (c_4 h (\log\log x)^4)^{\log\log x} \ll \exp(c_5(\log\log x)^2)$$

for any $\mathbf{C} \in \underline{\mathfrak{C}}$. By a result of Pall [9] we obtain with (2.19) and (3.1) after a short calculation

$$\begin{aligned} \sum_{\mathbf{C}\in\underline{\mathfrak{C}}} B_{\mathbf{C},Q}(s) &\ll \sum_{\mathbf{C}\in\underline{\mathfrak{C}}} B_{\mathbf{C},Q}(1-(\log x)^{-1/2}) \\ &\ll \prod_{p\leq Q}\left(1+\frac{c_6}{p^{1-(\log x)^{-1/2}}}\right) \ll \prod_{p\leq Q}\left(1+\frac{c_7}{p}\right) \ll (\log Q)^{c_8}. \end{aligned} \tag{3.4}$$

for $s \in \Gamma$. Thus we see

$$\frac{1}{2\pi i}\int_{\Gamma_4}\sum_{\mathbf{C}\in\underline{\mathfrak{C}}}\left|B_{\mathbf{C},Q}(s)A_{\mathbf{C}_\mathbf{F}\mathbf{C}^{-1},\kappa}(s)\frac{x^s}{s}\right||ds| \ll x\exp(-(\log\log x)^3) \tag{3.5}$$

and

(3.6)
$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_{2,1}\Gamma_{1,1}}\sum_{\mathbf{C}\in\underline{\mathfrak{C}}}\left|B_{\mathbf{C},Q}(s)A_{\mathbf{C}_\mathbf{F}\mathbf{C}^{-1},\kappa}(s)\frac{x^s}{s}\right||ds| \\ &\qquad \ll \exp(c_9(\log\log x)^2)\left(\exp\left(\log x - (\log x)^{\frac{1-\kappa}{2}}\right)\log S + \frac{x}{S}\right) \\ &\qquad \ll x\exp\left(-(\log x)^{c_{10}}\right) \end{aligned}$$

with $c_{10} = \frac{1}{2}\min(\frac{1-\kappa}{2},\frac{1}{3}) > 0$. The same estimate holds for $\Gamma_{1,2}\Gamma_{2,2}$ so that we can restrict the integration in (2.17), (2.18) to $\Gamma_{3,1}+\Gamma_{3,2}$ with an admissible error.

3.2 An Application of the Selberg-Delange method

As before let $2^{-m} \le \kappa < 1$, and define $Z := 2^m(m(M+1)+2)$. In this section we shall establish for each

$$\kappa \log\log x \le k \le (m(M+1)+2)\log\log x$$

an upper bound for the number $B_k(x)$ of integers $n \le x$ with $\Omega^*(n, Z^2) = k$ and $n \in \mathcal{R}(\mathbf{C_F})$. To simplify matters, we first replace the condition $n \in \mathcal{R}(\mathbf{C_F})$ by requiring only that, for $1 \le j \le m$, n can be represented by *some* class of discriminant D_j. Let z be a complex parameter satisfying $|z| \le Z$, and let

$$W(s) := \prod_{p \le Z^2 \text{ or } p|D} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p \notin \mathbb{P}, p\nmid D \\ p > Z^2}} \left(1 - \frac{1}{p^{2s}}\right)^{-1} =: \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \quad \text{(say)}.$$

In $\Re s > 1$ (for the moment) we consider the function

$$B(s;z) := W(s) \prod_{\substack{p \in \mathbb{P} \\ p > Z^2}} \left(1 - \frac{z}{p^s}\right)^{-1} =: \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \sum_{n=1}^{\infty} \frac{\tilde{b}_z(n)}{n^s} =: \sum_{n=1}^{\infty} \frac{b_z(n)}{n^s} \quad \text{(say)}.$$

Since there are 2^m prime ideals $\mathfrak{P} \subseteq K$ lying over $p \in \mathbb{P}$, we have by (2.5)

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-z} = \zeta(s)^{\frac{z}{2^m}} \mathcal{L}(s)^{\frac{z}{2^m}} W_1(s,z)$$

where $\mathcal{L}(s)$ is a product of $2^m - 1$ real Dirichlet L-functions and $W_1(s,z) \ll (\log\log|D|)^{c_{11}}$ uniformly in $\{s \in \mathbb{C} \mid \sigma > 1 - (\log|D|)^{-1}\} \times \{z \in \mathbb{C} \mid |z| \le Z\}$. With

(3.7)

$$G(s,z) := W(s)W_1(s,z) \prod_{\substack{p \in \mathbb{P} \\ p \le Z^2}} \left(1 - \frac{1}{p^s}\right)^z \mathcal{L}(s)^{\frac{z}{2^m}} \prod_{\substack{p \in \mathbb{P} \\ p > Z^2}} \left(1 - \frac{1}{p^s}\right)^z \left(1 - \frac{z}{p^s}\right)^{-1}$$

we obtain

$$B(s;z) = \zeta(s)^{\frac{z}{2^m}} G(s,z).$$

Note that the Dirichlet series of the second product in (3.7) is absolutely convergent and uniformly bounded in $\{s \in \mathbb{C} \mid \Re s > 1 - c_{12}\} \times \{z \in \mathbb{C} \mid |z| \le Z\}$, cf. [11], p. 202. Thus the function $G(., z)$ can be extended holomorphically into a region

$$\Re s > 1 - \frac{c_{13}|D|^{-\varepsilon}}{\log(2+|t|)}$$

and satisfies there $G(s,z) \ll |D(1+t)|^{\varepsilon}$. The same holds true for the at $s=1$ holomorphic function $\tilde{G}(s,z) := \zeta(s)^{-|z|}B(s;|z|)$, and we have with the above notation

$$|b_z(n)| = \left|\sum_{d|n} w(d)\tilde{b}_z(n/d)\right| \leq \sum_{d|n} w(d)|\tilde{b}_z(n/d)| = b_{|z|}(n).$$

By an application of Perron's formula we thus obtain (see [11], ch. II.5, Theorem 3)

$$\sum_{n\leq x} b_z(n) = x(\log x)^{\frac{z}{2^m}-1}\left(\frac{G(1,z)}{\Gamma(\frac{z}{2^m})} + O\left(\frac{|D|^{\varepsilon}}{\log x}\right)\right)$$

uniformly in $|z| \leq Z$.

Expanding $\sum_{n\leq x} b_z(n)$ in a power series of z, the coefficient of z^k is an upper bound for $B_k(x)$. We can proceed as in [11], section II.6, Theorem 3. We write $h_0(z) = G(1,z)(z\Gamma(\frac{z}{2^m}))^{-1}$ and note that $h_0''(z) \ll |D|^{\varepsilon}$ for $|z| \leq Z$. Cauchy's integral formula with radius $r = (k-1)2^m(\log\log x)^{-1}$ yields

$$\begin{aligned} B_k(x) &\leq \frac{x}{\log x}\frac{(2^{-m}\log\log x)^{k-1}}{(k-1)!}\left(h_0\left(\frac{2^m(k-1)}{\log\log x}\right) + O\left(\frac{1}{\log\log x}\right)\right) \\ &\ll \frac{x}{(\log x)^{1-\varepsilon}}\frac{(\log\log x)^{k-1}}{2^{m(k-1)}(k-1)!} \end{aligned}$$

uniformly in $1 \leq k \leq (m(M+1)+2)\log\log x$. For $2^{-m} \leq \kappa < 1$ and sufficiently large x we obtain

$$\begin{aligned} U_{\mathbf{F}}^{(m(M+1)+2,Z^2)}(x) - U_{\mathbf{F}}^{(\kappa,Z^2)}(x) &\leq \sum_{k=\lfloor\kappa\log\log x\rfloor}^{\lceil(m(M+1)+2)\log\log x\rceil} B_k(x) \\ &\ll \frac{x}{(\log x)^{1-\varepsilon}} \sum_{k=\lfloor\kappa\log\log x\rfloor}^{\infty} \frac{(\log\log x)^{k-1}}{2^{m(k-1)}(k-1)!}. \end{aligned}$$

By partial integration one verifies inductively

$$\Gamma(K)\sum_{k<K}\frac{x^k}{k!} = e^x\int_x^{\infty} t^{K-1}e^{-t}dt = e^x\Gamma(K,z) \tag{3.8}$$

for $K \in \mathbb{N}$ and $x > 0$ where $\Gamma(K,z)$ denotes the incomplete Γ-function.

Therefore

$$\sum_{k=\lfloor\kappa\log\log x\rfloor}^{\infty}\frac{(\log\log x)^{k-1}}{2^{m(k-1)}(k-1)!}=\frac{(\log x)^{2^{-m}}}{\Gamma(\lfloor\kappa\log\log x\rfloor-1)}\int_0^{2^{-m}\log\log x}t^{\lfloor\kappa\log\log x\rfloor-2}e^{-t}dt$$

$$\ll(\log x)^{2^{-m}+\varepsilon}\left(\frac{e}{\kappa\log\log x}\right)^{\kappa\log\log x}\frac{(2^{-m}\log\log x)^{\kappa\log\log x}}{e^{2^{-m}\log\log x}}$$

by Stirling's formula so that we conclude

$$U_{\mathbf{F}}^{(m(M+1)+2,Z^2)}(x)-U_{\mathbf{F}}^{(\kappa,Z^2)}(x)\ll\frac{x}{(\log x)^{1+\kappa(\log(2^m\kappa)-1)-\varepsilon}}. \tag{3.9}$$

4 The main term

The evaluation of the main term follows to large extent the lines in [3]. For $\mathbf{C}\in\underline{\mathfrak{C}}$ and $\mu\geq 0$ let

$$\pi_{\mathbf{C}}(\xi):=\epsilon(\mathbf{C})\sum_{\substack{p\leq\xi,p\in\mathbb{P}\\ p\in\mathcal{P}(\mathbf{C})}}1,\quad\tilde{\pi}_{\mathbf{C}}(\xi):=\epsilon(\mathbf{C})\sum_{\substack{p^n\leq\xi,p\in\mathbb{P}\\ p^n\in\mathcal{P}(\mathbf{C})}}1,$$

$$\rho_{\mathbf{C}}^{(\mu)}(\xi):=\epsilon(\mathbf{C})\sum_{\substack{p^n\leq\xi,p\in\mathbb{P}_Q\\ p^n\in\mathcal{P}(\mathbf{C})}}\frac{(-\log p^n)^{\mu}}{np^n}.$$

We start with the following variant of the Siegel-Walfisz theorem.

Lemma 4.1. *For any $A>0$ there is a constant $c=c_{A,m}>0$ such that*

$$\pi_{\boldsymbol{C}}(\xi)=\frac{1}{2^m h}\int_2^{\xi}\frac{dt}{\log t}+O\left(\xi\exp(-c\sqrt{\log\xi})\right)$$

for $\boldsymbol{C}\in\underline{\mathfrak{C}}$, uniformly in $|D|\leq(\log\xi)^A$. The same holds for $\tilde{\pi}_{\boldsymbol{C}}(\xi)$.

Proof. This is straightforward by applying Perron's formula to

$$\Psi_{\mathbf{C}}(s):=-\frac{1}{2^m h}\sum_{(\chi_1,\dots\chi_m)\in\widehat{\underline{\mathfrak{C}}}}\left(\prod_{j=1}^{m}\bar{\chi}_j(C_j)\right)\frac{L_K'(s,\chi)}{L_K(s,\chi)}$$

with χ as in (2.3) which gives as in (2.10) an explicit formula for $\epsilon(\mathbf{C})\sum_{n\leq\xi,n\in\mathcal{P}(\mathbf{C})}\Lambda(n)$. Note that we can absorb the contribution of the p^n, $n>1$, and the contribution of the non-split primes into the O-term. To calculate the residues of $\Psi_{\mathbf{C}}(s)x^s s^{-1}$, we observe that by Lemma 2.4 the real characters can be handled like real Dirichlet characters. For the complex characters the distribution of zeros is given by [5], Main Theorem and Lemma 5. Note that by Lemma 2.3 the residue of $\Psi_{\mathbf{C}}(s)$ at $s=1$ is $(2^m h)^{-1}$.

From now on we fix, for given $\varepsilon > 0$,

$$Q := \max\big(\exp(|D|^{\varepsilon}), Z^2\big) \tag{4.1}$$

(with Z as in section 3.2) which satisfies (3.1).

Corollary 4.2. *For* $\boldsymbol{C} \in \underline{\mathfrak{C}}$ *and* $\xi \geq Q$ *we have*

$$\rho_{\boldsymbol{C}}^{(0)}(\xi) = \frac{1}{2^m h}\left(\log\log\xi - \varepsilon\log|D| + O(1)\right)$$

and

$$\rho_{\boldsymbol{C}}^{(\mu)}(\xi) = \frac{(-1)^{\mu}}{2^m h\mu}\left((\log\xi)^{\mu} - |D|^{\varepsilon} + O(1)\right)$$

for $\mu \geq 1$.

Proof. This is immediate by partial summation using (4.1) and the preceding lemma with $A = \frac{1}{\varepsilon}$ for $\pi_{\mathbf{C}}$ and $\tilde{\pi}_{\mathbf{C}}$ and observing that

$$\epsilon(\mathbf{C}) \sum_{\substack{Q<p\leq\xi\\ p\in\mathbb{P}, p\in\mathcal{P}(\mathbf{C})}} \frac{(\log p)^{\mu}}{p} \leq (-1)^{\mu}\rho_{\mathbf{C}}^{(\mu)}(\xi) \leq \epsilon(\mathbf{C}) \sum_{\substack{Q<p^n\leq\xi\\ p\in\mathbb{P}, p^n\in\mathcal{P}(\mathbf{C})}} \frac{(\log p^n)^{\mu}}{p^n}.$$

Here it is important to sum only over primes or prime powers larger than Q.

Lemma 4.3. *For* $\boldsymbol{C} \in \underline{\mathfrak{C}}$, $1-(\log x)^{-1/2} \leq \sigma \leq 1$ *and* $\varepsilon > 0$ *we have*

$$|T(\sigma, \boldsymbol{C}, Q)| \leq \frac{\varepsilon\log|D| + O(1)}{2^m h}$$

where T *is defined in* (2.11).

Proof. The proof is essentially the same as in [3], Proposition 3.3. Since for any $\mu \geq 0$ the Dirichlet series for $T^{(\mu)}(s, \mathbf{C}, Q) = \sum_{\nu=1}^{\infty} t_{\mu}(\nu)\nu^{-s}$, say, converges (conditionally) at $s=1$, we have

$$T^{(\mu)}(1, \mathbf{C}, Q) = \lim_{\xi\to\infty}\sum_{\nu=1}^{\xi} t_{\mu}(\nu)\nu^{-1}.$$

On the other hand, by (2.10) and (2.11) we have

$$\sum_{\nu=1}^{\infty} \frac{t_\mu(\nu)}{\nu^s} = \epsilon(\mathbf{C}) \sum_{\substack{p^n \in \mathcal{P}(\mathbf{C}) \\ p \in \mathbb{P}_Q}} \frac{(-\log p^n)^\mu}{np^{ns}} - \frac{1}{2^m h} \frac{d^\mu}{ds^\mu} \log \zeta(s),$$

so we see by the preceding corollary

$$|T(1, \mathbf{C}, Q)| \leq \frac{\varepsilon \log|D| + O(1)}{2^m h} \quad \text{and} \quad |T^{(\mu)}(1, \mathbf{C}, Q)| \leq \frac{|D|^\varepsilon + O(1)}{2^m h}$$

for $\mu \geq 1$. We apply now Taylor's formula up to degree $\mu_0 := \lceil 2(M+1)m \rceil$, say, and use the trivial estimation

$$T^{(\mu_0)}(s, \mathbf{C}, Q) \ll \max_{\chi \neq \chi_0} \left| \frac{d^{\mu_0}}{ds^{\mu_0}} \log L_K(s, Q, \chi) \right| \ll (\log x)^\varepsilon$$

for $|s-1| \leq (\log x)^{-1/2}$ by (2.9) and (4.1).

We focus now on the integral

(4.2)
$$I := \frac{1}{2\pi i} \int_{\Gamma_{3,1}+\Gamma_{3,2}} K_{\mathbf{C}_1}(s) A_{\mathbf{C}_2,\kappa}(s) \frac{x^s}{s} ds = -\frac{1}{\pi} \Im \int_J K_{\mathbf{C}_1}(s) A_{\mathbf{C}_2,\kappa}(s) \frac{x^s}{s} ds$$

where $(\mathbf{C}_1, \mathbf{C}_2) \in \underline{\mathfrak{C}}^2$, $K_{\mathbf{C}_1}(s) = B_{\mathbf{C}_1,Q}(s)$ or 1 and

$$J = -\Gamma_{3,1} = [1 - (\log x)^{-\frac{1+\kappa}{2}}, 1 - \exp(-(\log\log x)^4)].$$

Note that the integrand in $\int_{\Gamma_{3,2}}$ is the complex conjugate of the integrand in $\int_{\Gamma_{3,1}}$. As in [3] we appeal to the following simple lemma that allows us to replace all O-terms entering $A_{\mathbf{C}_2,\kappa}(s)$ via (2.11), (2.15) and Lemma 4.3 by the same constant c_{14}.

Lemma 4.4. *Let z_ν, $\nu = 1, \ldots, k$, be k complex numbers with $\Im(z_\nu) < 0 < \Re(z_\nu)$ and let $z := \prod_{\nu=1}^k z_\nu$. Then $-\Im(z)$ is positive and increasing in all $\Re(z_\nu)$ as long as*

(4.3)
$$k \frac{\Im(z_\nu)}{\Re(z_\nu)} > -\pi$$

for all ν.

By Lemma 4.3, (2.11), (2.14) and (3.3) condition (4.3) is satisfied for $z_\nu = P_{\mathbf{C}_\nu,Q}(s)$ where $\mathbf{C}_\nu \in \underline{\mathfrak{C}}$, $s \in J$, $\varepsilon < \frac{1-\kappa}{2Mm}$ and x sufficiently large.
For the proof of Theorem 2 we need bounds for the quantity $\#M_k(\mathbf{C})$, defined in (2.13). The following lemma is a direct consequence of the result in [3], section 5, applied to each factor $\mathfrak{C}_j$ of $\underline{\mathfrak{C}}$ (the upper bound is trivial).

Lemma 4.5. *We have*

$$h^k \prod_{j=1}^m \min\left(\frac{1}{\#\,\mathfrak{C}_j\,/\,\mathfrak{C}_j^2}, \frac{2^k}{h_j}\right) \ll \#M_k(\boldsymbol{C}) \leq h^k \prod_{j=1}^m \min\left(\frac{1}{\#\,\mathfrak{C}_j\,/\,\mathfrak{C}_j^2}, \frac{2^k}{h_j}\right).$$

We are now prepared to prove Theorem 2. Let $\mathbf{F}$ satisfying (1.1) be given, and let $\mathbf{C}_\mathbf{F} \in \underline{\mathfrak{C}}$ be the corresponding m-tuple of classes. Define $\boldsymbol{\kappa}$ by (1.2).

We start with the lower bound in (1.3). We write $\kappa^* := \max(2^{-m}, \|\boldsymbol{\kappa}\|_\infty)$ and $\kappa := \min(\kappa^*, 1-\varepsilon)$. By (1.2), (2.1), (2.2), (2.14), (3.3) and Lemma 4.5 we have

$$\#M_k(\mathbf{C}_\mathbf{F}) \gg \frac{h^k}{|D|^\varepsilon} \prod_{j=1}^m \min\left(1, \frac{2^k}{(\log x)^{\kappa^* \log 2}}\right) \gg \frac{(2^m h)^k}{(\log x)^{m\kappa^* \log 2 + \varepsilon}}$$

for $k \in S_\kappa$. Together with Lemmata 4.3, 4.4, (2.11), (2.15) and (4.2) this yields

$$I \gg -\Im\left(\frac{1}{(\log x)^{m\kappa^* \log 2+\varepsilon}} \int_J \frac{x^s}{s} \sum_{k \in S_\kappa} \frac{1}{k!}\left(\log\frac{1}{1-s} - \varepsilon \log|D| - c_{14} - i\pi\right)^k ds\right).$$

Writing

$$z_1(s) = \left(\log\frac{1}{s} - \varepsilon\log|D| - c_{14} - i\pi\right), \quad \tilde{z}_1(s) = \Re z_1(s)$$

and $L = \lfloor \kappa \log\log x \rfloor$ we infer by (3.8) after a change of variables

$$I \gg \frac{x}{(\log x)^{m\kappa^* \log 2+\varepsilon}} \int_{\exp(-(\log\log x)^4)}^{(\log x)^{-\frac{1+\kappa}{2}}} \frac{x^{-s}}{s} \Im\left(\frac{\Gamma(L+1, z_1(s))}{\Gamma(L+1)}\right) ds \tag{4.4}$$

where $\Gamma(k,z)$ is the incomplete Γ-function (see [3], section 3.2). Note that (4.5)

$$\Im(\Gamma(L+1, z_1(s))) = \tilde{z}_1(s)^L e^{-\tilde{z}_1(s)} \int_0^\pi \left(1 + \frac{t^2}{\tilde{z}_1(s)^2}\right)^{\frac{L}{2}} \cos\left(t - L\arctan\frac{t}{\tilde{z}_1(s)}\right) dt.$$

As in [3], Lemma 3.4, we see that the right hand side of (4.5) is positive for $s \in [\exp(-(\log\log x)^4), (\log x)^{-\frac{1+\kappa}{2}}]$, hence we may restrict the integral in (4.4) to the interval $[(\log x)^{-1}, 2(\log x)^{-1}]$ and use Stirling's formula to estimate $\Gamma(L+1)$. Together with (2.17), (3.5) and (3.6) this proves the desired lower bound

$$I \gg \frac{x}{(\log x)^{\kappa^* \log(2^m) - 1 + \kappa(1-\log\kappa)+\varepsilon}} = \frac{x}{(\log x)^{E(\|\boldsymbol{\kappa}\|_\infty)+\varepsilon}}.$$

To obtain the upper bound, we write $\kappa := \frac{1}{m}\,\|\boldsymbol{\kappa}\|_1$. In view of (1.4) and (1.5) we may assume $2^{-m} \leq \kappa \leq 1-\varepsilon$ and hence apply (3.9) later. We have by Lemma 4.5, (1.2) and (2.1)

$$\#M_k(\mathbf{C}) \leq \frac{(2^m h)^k}{h} \ll \frac{(2^m h)^k}{(\log x)^{m\kappa \log 2 - \varepsilon}}$$

for any $\mathbf{C} \in \underline{\mathfrak{C}}$. Let

$$z_2(s) = \left(\log\frac{1}{s} + \varepsilon \log|D| + c_{14} - i\pi\right), \quad \tilde{z}_2(s) = \Re z_2(s).$$

In the same way as (4.4) we obtain
(4.6)

$$I \ll \frac{xB_{\mathbf{C}_1,Q}(1-(\log x)^{-1/2})}{(\log x)^{m\kappa\log 2-\varepsilon}} \int_{\exp(-(\log\log x)^4)}^{(\log x)^{-\frac{1+\kappa}{2}}} \frac{x^{-s}}{s} \Im\left(\frac{\Gamma(L+1, z_2(s))}{\Gamma(L+1)}\right) ds.$$

To estimate the integral in (4.6), we use

$$\Im(\Gamma(L+1, z_2(s))) \ll \tilde{z}_2(s)^L e^{-\tilde{z}_2(s)}$$

for $s < (\log x)^{-\frac{1+\kappa}{2}}$ and break the integral into I_1 and I_2 where I_1 is the integral over $J_1 := [\exp(-(\log\log x)^4), (\log x)^{-1+\varepsilon}]$ and I_2 is the integral over $J_2 := [(\log x)^{-1+\varepsilon}, (\log x)^{-\frac{1+\kappa}{2}}]$. On J_1 we have

$$\frac{\Im(\Gamma(L+1, z_2(s)))}{\Gamma(L+1)} \ll (\log x)^{\kappa(1-\log\kappa)-1+\varepsilon},$$

while for $s \in J_2$ we have

$$\frac{\Im\Gamma(L+1, z_2(s))}{s\Gamma(L+1)} \ll (\log x)^{c_{15}}.$$

Since $\int_J s^{-1} ds \ll (\log\log x)^4$, we see by (2.16), (2.18), (3.5), (3.6), (4.1) - recall our assumption $\kappa \geq 2^{-m}$ - and (4.6) that

$$U_{\mathbf{F}}^{(\kappa,Q)}(x) \ll \frac{x}{(\log x)^{E(\kappa)-\varepsilon}} \sum_{\mathbf{C}\in\underline{\mathfrak{C}}} B_{\mathbf{C},Q}(1-(\log x)^{-1/2}).$$

From this, (3.4) and (3.9) we infer the upper bound (1.3) for $U_{\mathbf{F}}^{(m(M+1)+2,1)}(x)$. Finally by [4], Corollary 1, the number of integers n not exceeding x with $\Omega(n) \geq (m(M+1)+2)\log\log x$ is $\ll x(\log x)^{-m(M+1)\log 2} \ll x(\log x)^{-E(\kappa)}$. This completes the proof of Theorem 2.

5 Application: Sums of two squareful numbers

We may now apply Theorem 2 to prove Theorem 1. Since every squareful number n can be written as $n = a^3b^2$ and numbers of this type are squareful, we have

$$V(x) = \#\{1 \leq n \leq x \mid \exists\, \mathbf{a} \in \mathbb{N}^2 : a_1^3x_1^2 + a_2^3x_2^2 \text{ represents } n\}.$$

Let $\mathcal{P}$ be the set of all primes $p \equiv 3 \pmod 4$ such that $L(s, \chi_{-4p})$ has no Siegel zero. For $\varepsilon > 0$ and large x we consider the set $\mathcal{F}$ of forms $x_1^2 + p^3x_2^2$ with $p \in \mathcal{P}$ and

$$(\log x)^{\frac{2^{2/3}}{3}\log 2} \leq p \leq 2(\log x)^{\frac{2^{2/3}}{3}\log 2},$$

i.e. $\kappa = 2^{-\frac{1}{3}}$. It is clear that for these forms all estimates in Theorem 2 are effective, and we have $\#\mathcal{F} \gg (\log x)^{\frac{2^{2/3}}{3}\log 2 - \varepsilon}$ by the prime number theorem and the fact that there are only $O(\log\log\log x)$ exceptional discriminants up to $(\log x)^2$. Two different forms $F_1, F_2 \in \mathcal{F}$ satisfy (1.1), thus we obtain by Theorem 2

$$\begin{aligned} V(x) &\gg \sum_{F\in\mathcal{F}} U_F(x) - \sum_{F_1\neq F_2\in\mathcal{F}} U_{(F_1,F_2)}(x) \\ &\gg \frac{x(\log x)^{\frac{2^{2/3}}{3}\log 2-\varepsilon}}{(\log x)^{1-2^{-\frac{1}{3}}+\frac{2^{2/3}}{3}\log 2+\varepsilon}} - \frac{x(\log x)^{\frac{2^{5/3}}{3}\log 2}}{(\log x)^{1-2^{-\frac{1}{3}}+\frac{5\cdot 2^{2/3}}{6}\log 2-\varepsilon}} \gg \frac{x}{(\log x)^{1-2^{-\frac{1}{3}}+\varepsilon}}. \end{aligned}$$

To obtain an upper bound for $V(x)$, we proceed as follows: Let $\mathcal{F}_1$ be the set of all forms $F_{a_1,a_2}(\mathbf{x}) = a_1^3x_1^2 + a_2^3x_2^2$ with $(a_1a_2)^3 \geq (\log x)^3$. Let $\tilde{U}_F(x) := \{n \leq x \mid n = F(x_1, x_2) \text{ for some } x_1, x_2 \geq 1\}$. Then obviously $\tilde{U}_{F_{a_1,a_2}}(x) \leq \frac{x}{(a_1a_2)^{3/2}}$, and therefore

$$\begin{aligned} \sum_{F\in\mathcal{F}_1} U_F(x) &\leq \#\{\text{squareful numbers } \leq x\} + \sum_{F\in\mathcal{F}_1} \tilde{U}_F(x) \\ &\ll x^{1/2} + \sum_{\Delta\geq\log x} \frac{x\tau(\Delta)}{\Delta^{3/2}} \ll \frac{x}{(\log x)^{\frac{1}{2}-\varepsilon}} \end{aligned}$$

where τ is the divisor function. Let $\mathcal{F}_2$ be the set of all forms $F_{a_1,a_2}(\mathbf{x}) = a_1^3x_1^2 + a_2^3x_2^2$ with $(a_1a_2)^3 \leq (\log x)^3$. If we write $a_i = \tilde{a}_i d$ for $i = 1, 2$ with $(\tilde{a}_1, \tilde{a}_2) = 1$, then Corollary 1.2 with $\kappa_0 = 2^{-\frac{1}{3}}$ yields

$$U_{F_{a_1,a_2}}(x) \ll \frac{xd^{-3}}{\tilde{a}_1\tilde{a}_2(\log(xd^{-3}))^{1-2^{-1/3}-\varepsilon}} \ll \frac{x}{a_1a_2(\log x)^{1-2^{-1/3}-\varepsilon}}$$

so that

$$\sum_{F \in \mathcal{F}_2} U_F(x) \ll \sum_{\Delta \leq \log x} \frac{x\tau(\Delta)}{(\log x)^{1-2^{-1/3}-\varepsilon}\Delta} \ll \frac{x}{(\log x)^{1-2^{-1/3}-\varepsilon}}.$$

This completes the proof.

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