1 Introduction

The behavior of L-functions in the critical strip has received a lot of attention from the first proof of the prime number theorem up to now. In fact, the deeper arithmetic information of the coefficients of an L-function is mirrored in its analytic properties within the critical strip. For many applications in number theory, one needs information on the growth of L-functions on vertical lines. The functional equation, together with the Phragmen-Lindelöf principle, yields the so-called convexity estimates which are, however, not sufficient in many circumstances. Breaking these convexity bounds is one of the important themes in the theory of L-functions (cf. [12]). For Dirichlet L-functions, the first such subconvexity bounds were obtained by Weyl in the t-aspect and by Burgess in the conductor aspect. The currently most powerful technology for fairly general families of L-functions is the amplification method initiated by Duke, Friedlander, and Iwaniec [4] and applied in various circumstances in the past decade (cf. [6, 7, 10, 18, 19, 21], cf. [23] for a somewhat different method, and the references given there, and see [9] for an excellent survey).

In this paper we will obtain new subconvexity bounds for character twists of GL₂-automorphic L-functions in the conductor aspect. To be precise, let \( N, k \in \mathbb{N} \), let \( \chi_1 \) be a not necessarily primitive character to modulus \( N \), and let

\[
f(z) = \sum_{m=1}^{\infty} a(m)m^{(k-1)/2}e(mz) \in S_k(N, \chi_1) \tag{1.1}
\]
be a primitive cusp form (i.e., an eigenfunction for all the Hecke operators, arithmetically normalized by \( a(1) = 1 \)) of weight \( k \) and character \( \chi \) for the congruence subgroup \( \Gamma_0(N) \). Let \( \chi \) be a primitive character to modulus \( D \) with \( (D, N) = 1 \), and let

\[
L_f(s, \chi) = \sum_{m=1}^{\infty} \frac{a(m)\chi(m)}{m^s}
\]

be the normalized \( L \)-function attached to the twist of \( f \) with \( \chi \). The convexity bound gives

\[
L_f\left(\frac{1}{2} + it, \chi\right) \ll_{\epsilon} \left((1 + |t|)kD\sqrt{N}\right)^{1/2+\epsilon}.
\]

The Lindelöf hypothesis, of course, would replace the exponent \( 1/2 + \epsilon \) by \( \epsilon \). In the case of the full modular group, the exponent in the \( D \)-aspect was lowered to \( 1/2 - 1/22 \) in [4].

In the more general setting of Maass cusp forms of arbitrary level and character, Harcos [10] obtained \( 1/2 - 1/54 \) which was recently improved to \( 1/2 - 1/22 \) by Michel [21]. Our aim is to show the following theorem.

**Theorem 1.1.** Let \( f \in S_k(N, \chi_1) \) be a primitive cusp form and \( \chi \) a primitive character of conductor \( D \). Assume \( (D, N) = 1 \), and let \( \theta \) be a constant such that \( \lambda(n) \leq \tau(n)n^\theta \) for eigenvalues \( \lambda(n) \) of the Hecke operator \( T_n \) acting on the space of weight \( 0 \) Maass cusp forms of level \( N \). Then

\[
L_f\left(\frac{1}{2} + it, \chi\right) \ll_{t, k, N, \epsilon} D^{(2+20)/(5+20)+\epsilon}.
\]

The dependence on \( k, t, \) and \( N \) is polynomial.

The Ramanujan-Petersson conjecture predicts \( \theta = 0 \), Weil’s bound for Kloosterman sums gives \( \theta \leq 1/4 \). The currently best bound \( \theta \leq 7/64 \) is due to Kim and Sarnak [17]. If we choose \( \theta = 1/4 \), we get back the exponent \( 5/11 = 0.4545 \ldots \) of Duke, Friedlander, Iwaniec, and Michel.

**Corollary 1.2.** With the above notation,

\[
L_f\left(\frac{1}{2} + it, \chi\right) \ll_{t, k, N, \epsilon} D^{71/167+\epsilon},
\]

where \( 71/167 = 0.4251 \ldots \) Assuming the Ramanujan-Petersson conjecture,

\[
L_f\left(\frac{1}{2} + it, \chi\right) \ll_{t, k, N, \epsilon} D^{2/5+\epsilon}.
\]
By means of an approximate functional equation, the proof of Theorem 1.1 rests on a good bound for the character sum $B_\chi = \sum a(m)\chi(m)g(m)$ where $g$ is a smooth test function with support in $[M, 2M]$, $M \ll D^{1+\varepsilon}$, say. To obtain such a bound, we consider a weighted quadratic mean of all the $B_\psi$ for $\psi$ a primitive character modulo $D$. The positive weights will be chosen so as to amplify the contribution of $B_\chi$ in this sum. This procedure amounts to bounding shifted convolution sums, and we need to estimate

$$D_\psi(l_1, l_2, h) := \sum_{l_1 m_1 - l_2 m_2 = h} a(m_1)\overline{a(m_2)}g(m_1, m_2)$$

uniformly in $l_1, l_2, h$ for some nice function $g$, for example, smooth and compactly supported. This problem and its companion with Fourier coefficients replaced by the divisor function are interesting themselves and have a long history in analytic number theory (see, e.g., [16, 22] and the references given there). We will prove the following theorem.

**Theorem 1.3.** Let $\varepsilon > 0$ be given, and let $l_1, l_2, h$ be positive integers. Let $M_1, M_2, P_1, P_2$ be real numbers greater than 1. Let $g$ be a smooth function, supported on $[M_1, 2M_1] \times [M_2, 2M_2]$ such that $\|g^{(i)}\|_\infty \ll_{i,j} (P_1/M_1)^i(P_2/M_2)^j$ for all $i, j \geq 0$. Then

$$D_\psi(l_1, l_2, h) \ll_{\varepsilon, P_1, P_2, N, k} (l_1M_1 + l_2M_2)^{1/2 + \theta + \varepsilon}$$

for $\theta$ as in Theorem 1.1, uniformly in $l_1, l_2, h$. The dependence on $P_1, P_2, N,$ and $k$ is polynomial.

Using a suitable partition of unity, we see as in [10, page 360] that Theorem 1.3 carries over to more general test functions $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfying only

$$x^iy^jg^{(i,j)}(x, y) \ll_{i,j} \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} P_1^iP_2^j.$$  

Choosing $g(x, y) = g_1(x)g_2(y)$ with suitable, smooth, nonnegative functions $g_1, g_2$ satisfying $\text{supp } g_1 \subseteq [0, M + M^{1-\varepsilon}]$, $\text{supp } g_2 \subseteq [0, 2(M + h)]$, $g_1(x) = 1$ on $[0, M]$, and $g_2(y) = 1$ on $[0, M + h]$, we find as in [5, page 211].

**Corollary 1.4.** Let $\varepsilon > 0$ be given, and let $h \leq M^{2/(1+2\varepsilon)}$. Then there is a $\delta > 0$ such that

$$\sum_{m \leq M} a(m)\overline{a(m + h)} \ll_{\varepsilon} M^{1-\delta}.$$  

$\square$
Note that with $\theta = 7/64$ we have $2/(1+2\theta) = 64/39 = 1.64 \ldots$. This is now of the same quality as Motohashi’s result [22] for the divisor function, and seems to be the widest range of the shifting parameter for a nontrivial upper bound that appears in the literature. Harcos’ paper [10] yields such a result for $h \leq M^{6/5-\varepsilon}$ while variants of the $\delta$-method (see [1, 5]) give uniformity up to $h \leq M^{3/2-\varepsilon}$.

For the proof of Theorem 1.1, we are interested in the case $M_1 = M_2 = M, P_1 = P_2 = 1, l_1, l_2 \leq L$. Our bound

$$D_g(l_1, l_2, h) \ll \varepsilon (LM)^{1/2+\theta+\varepsilon}$$ (1.11)

should be compared with the existing bounds

$$D_g(l_1, l_2, h) \ll \varepsilon (LM)^{3/4+\varepsilon}$$ (1.12)


$$D_g(l_1, l_2, h) \ll \varepsilon L(LM)^{1/2+\theta+\varepsilon}$$ (1.13)

by Sarnak [23], and

$$D_g(l_1, l_2, h) \ll \varepsilon L^{7/10+\varepsilon}M^{9/10+\varepsilon}$$ (1.14)

by Harcos [10].

The method of the proof of Theorem 1.3 generalizes that in [15]. We use Jutila’s variant of the circle method to detect the condition $l_1 m_1 - l_2 m_2 = h$, and combine this with the spectral large sieve inequalities of Deshouillers and Iwaniec [3] to exploit cancellation in the arising sums of Kloosterman sums. One of the new difficulties is now that we have to work in the congruence subgroup $\Gamma_0(Nl_1l_2)$ instead of the full modular group. The same approach works also in the setting of Maass cusp forms, and one can obtain analogous results for shifted convolution sums and subconvexity bounds. This has applications to the subconvexity problem for Rankin-Selberg $L$-functions, see [11].
Automorphic forms

In this section we fix the notation and compile some mostly well-known definitions, identities, and estimates involving Fourier coefficients of cusp forms and Kloosterman sums.

A complete set of inequivalent cusps for $\Gamma_0(N)$ is given by the rational numbers

$$\left\{ \frac{u}{w} : w \mid N, (u, w) = 1, u \left( \text{mod} \left( w, \frac{N}{w} \right) \right) \right\}.$$  \hfill (2.1)

For each cusp $a = \frac{u}{w}$ let $\sigma_a \in \text{SL}_2(\mathbb{R})$ be a scaling matrix, that is, $\sigma_a \infty = a$ and $\sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty$ where $\Gamma_a = \{ \sigma \in \Gamma_0(N) \mid \sigma_a = a \}$ is the stabilizer. A standard choice is

$$\sigma_a = \left( \begin{array}{cc}
\frac{u}{w} \sqrt{[w^2, q]} & 0 \\
\sqrt{[w^2, q]} & \left( \frac{u}{w} \sqrt{[w^2, q]} \right)^{-1}
\end{array} \right).$$  \hfill (2.2)

For $N, k \in \mathbb{N}$ and $\chi_1$ a character to modulus $N$, let $S_k(N, \chi_1)$ denote the space of holomorphic cusp forms of weight $k$ and level $N$. Thus $f \in S_k(N, \chi_1)$ satisfies

$$f(\gamma(z)) = \chi_1(d)(cz + d)^k f(z)$$  \hfill (2.3)

for $\gamma = \left( \begin{array}{cc} a & b \\
c & d \end{array} \right) \in \Gamma_0(N)$ and vanishes at every cusp of $\Gamma_0(N)$. The space $S_k(N, \chi_1)$ becomes a Hilbert space through the Petersson scalar product

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z) \overline{g(z)} y^k \frac{dx \, dy}{y^2},$$  \hfill (2.4)

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(N)$. The Hecke operators $T_m$ form a commutative algebra of operators acting on $S_k(N, \chi_1)$. If $f(z) = \sum a(m)m^{(k-1)/2}e(mz) \in S_k(N, \chi_1)$ is an eigenform for all the Hecke operators with eigenvalues $\lambda_f(m)$, then $a(m) = a(1)\lambda_f(m)$. An arithmetically normalized (i.e., $a(1) = 1$) eigenform of all the Hecke operators is called primitive. By Deligne’s proof \cite{2} of the Ramanujan-Petersson conjecture for holomorphic cusp forms of integral weight $k \geq 2$ (extended by Serre \cite{24} to $k = 1$), we know that

$$|a(m)| = |\lambda_f(m)| \leq \tau(m)$$  \hfill (2.5)

for all $m$ (including the case $(m, N) > 1$). In addition, Wilton’s classical estimate gives

$$\sum_{m \leq x} \lambda_f(m)e(axm) \ll \varepsilon N^{5/4}x^{1/2}(Nk)^{\varepsilon}$$  \hfill (2.6)

uniformly in $\varepsilon$. The explicit dependence on $k$ and $N$ is shown in \cite[Proposition 2.5]{11}.
For a primitive character $\chi$ having conductor $D$ with $(D, N) = 1$, the twisted $L$-function $L_f(s, \chi) = \sum a(m)\chi(m)m^{-s}$ has a functional equation

\[
\left(\frac{\sqrt{ND^2}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right)L_f(s, \chi)
= \eta(f, \chi) \left(\frac{\sqrt{ND^2}}{2\pi}\right)^{1-s} \Gamma\left(\frac{k+1}{2} - s\right)L_f(1-s, \bar{\chi})
\] (2.7)

with $|\eta(f, \chi)| = 1$. On the line $\Re s = 1/2$, the first $((|s| + k)D\sqrt{N})^{1+\varepsilon}$ terms of the Dirichlet series should therefore approximate the function well. As in [21, Lemma 3.1], one can prove the following lemma.

**Lemma 2.1** (approximate functional equation). For $\Re s = 1/2$,

\[
L_f(s, \chi) = \sum_{m=1}^{\infty} \frac{a(m)\chi(m)}{m^s} \frac{2\pi m}{\sqrt{ND^2}} V_s \left(\frac{2\pi m}{\sqrt{ND^2}} \right)
\]

\[
+ \eta(s, f, \chi) \sum_{m=1}^{\infty} \frac{a(m)\chi(m)}{m^{1-s}} V_{1-s} \left(\frac{2\pi m}{\sqrt{ND^2}} \right)
\]

with $|\eta(s, f, \chi)| = 1$ and

\[
x^j V_s^{(1)}(x) \ll_{j, A, \varepsilon} (ND|x|)^\varepsilon (|s| + k)^j \left(1 + \frac{x}{|s| + k}\right)^{-A}
\]

for any $A, \varepsilon \geq 0, j \in \mathbb{N}_0$. □

We will need the following Voronoi-type summation formula (see [19, page 184]).

**Lemma 2.2.** Let $f(z) = \sum a(m)m^{(k-1)/2}e(mz) \in S_k(N, \chi_1)$ be a primitive cusp form, and let $w : \mathbb{R}^+ \to \mathbb{R}^+_0$ be a smooth, compactly supported function. Suppose $(d, q) = 1$ and $N \mid q$. Then

\[
\sum_{m=1}^{\infty} a(m)e\left(\frac{dm}{q}\right)w(m) = \bar{\chi}_1(d) \sum_{m=1}^{\infty} a(m)e\left(-\frac{dm}{q}\right)w_d^*(m),
\]

where

\[
w_d^*(m) = \frac{2\pi i^k}{q} \int_0^{\infty} w(x) J_{k-1}\left(\frac{4\pi \sqrt{mx}}{q}\right) dx.
\] (2.11)
Let \( \Delta = y^2(\partial^2/\partial x^2 + \partial/\partial y^2) \) be the non-Euclidean Laplacian acting on the \( L^2 \)-space \( \mathcal{L}(N) \) of \( \Gamma_0(N) \)-invariant functions \( f: \mathbb{H} \to \mathbb{C} \) with respect to the Petersson scalar product. A Maass cusp form \( u \) of level \( N \) is an element of \( \mathcal{L}(N) \) that vanishes at every cusp and is an eigenfunction of \( \Delta \) to an eigenvalue \( \lambda \). It admits a Fourier expansion

\[
u(z) = \sqrt{y} \sum_{m \neq 0} \rho(m) K_{i\kappa}(2\pi|m|y)e(mx),
\]

where

\[
\kappa = \sqrt{\lambda - \frac{1}{4}}.
\]

The sign of the root of this possibly complex number is irrelevant due to the symmetry of the Bessel \( K \)-function. Often this expansion is also given in terms of the Whittaker function. The Hecke operators act on \( \mathcal{L}(N) \) and commute with \( \Delta \).

For even \( k \in \mathbb{N} \), let

\[
f_{j,k}(z) = \sum_{m=1}^{\infty} \hat{f}_{j,k}(m)m^{(k-1)/2}c(mz), \quad 1 \leq j \leq \theta_k(N) =: \dim_{\mathbb{C}} S_k(N),
\]

be an \( L^2 \)-orthonormal basis for \( S_k(N) = S_k(N,1) \). Let

\[
u_j = \sqrt{y} \sum_{m \neq 0} \rho_j(m) K_{i\kappa_j}(2\pi|m|y)e(mx)
\]

be a set of Maass cusp forms that form an orthonormal basis of the discrete spectrum of \( \Delta \) in \( \mathcal{L}(N) \), and denote by \( \lambda_1 \leq \lambda_2 \leq \cdots \) the corresponding eigenvalues so that \( \kappa_j \) is given by (2.13). The Kim-Sarnak result gives

\[
\lambda_1 \geq \frac{1}{4} - \left( \frac{7}{64} \right)^2, \quad |\Im \kappa_1| \leq \frac{7}{64}.
\]
We may assume that the $u_j$ are eigenforms of all Hecke operators $T_m$ with $(m,N) = 1$. For each cusp $a = u/w$ of $\Gamma_0(N)$, we define the Eisenstein series

$$E_a(z, s) = \sum_{\tau \in \Gamma_a \setminus \Gamma} \Im (\sigma^{-1}_a \tau z)^s$$

$$= \delta_{a\infty} y^s + \sqrt\pi \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \rho_a(s, 0) y^{1-s}$$

$$+ \frac{2\pi^s \sqrt y}{\Gamma(s)} \sum_{m \neq 0} |m|^{s-1/2} \rho_a(s, m) K_{s-1/2} (2\pi|m|y) e(mx),$$

where

$$\rho_a(s, m) = \sum_{\gamma > 0, \gamma \equiv u (w, N/w)} \frac{1}{\gamma^{2s}} e \left( \frac{m \delta}{\gamma} \right)$$

$$= \left( \frac{w, N/w}{wN} \right)^s \sum_{(\gamma, N/w) = 1} \frac{1}{\gamma^{2s}} \sum_{\delta(\gamma w) \equiv 1 (\gamma w) \equiv 1} \sum_{\delta \equiv u (w, N/w)} e \left( -\frac{m \delta}{\gamma w} \right),$$

see [3, page 247]. For a $C^3$-class function $\phi$, define the Bessel transforms

$$\hat{\phi}(k) := \int_0^\infty J_k(x) \phi(x) \frac{dx}{x},$$

$$\hat{\phi}(r) := \frac{\pi}{\sinh(\pi r)} \int_0^\infty J_{2ir}(x) - J_{-2ir}(x) \phi(x) \frac{dx}{x},$$

$$\hat{\phi}(r) := \frac{4}{\pi} \cosh(\pi r) \int_0^\infty K_{2ir}(x) \phi(x) \frac{dx}{x}.$$ (2.19)

Let

$$S(m, n, \gamma) = \sum_{d(\gamma)}^* e \left( \frac{md + nd}{\gamma} \right)$$

be the usual Kloosterman sum. Then we can state the Kuznetsov trace formula (see [3, Theorem 1]).
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**Proposition 2.3** (Kuznetzov’s trace formula). Let \(m, n \in \mathbb{N}\), let \(\phi\) be a \(C^3\)-class function with compact support in \((0, \infty)\). Then

\[
\sum_{\gamma \equiv 0(N)} \frac{1}{\gamma} S(m, n, \gamma) \phi \left( \frac{4\pi \sqrt{mn}}{\gamma} \right) = \sum_j \hat{\phi}(\kappa_j) \frac{\rho_j(m) \rho_j(n)}{\cosh(\pi \kappa_j)} + \frac{1}{2\pi} \sum_{k \equiv 0(2)} \hat{\phi}(k - 1) \frac{i^k (k - 1)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_N(k)} \hat{\rho}_{j,k}(m) \hat{\rho}_{j,k}(n)
\]

\[
+ \frac{1}{\pi} \sum_a \int_{-\infty}^{\infty} (mn)^{ir} \frac{\rho_a(\frac{1}{2} + ir, m)}{\rho_a(\frac{1}{2} + ir, n)} \hat{\phi}(r) dr,
\]

\[
\sum_{\gamma \equiv 0(N)} \frac{1}{\gamma} S(m, -n, \gamma) \phi \left( \frac{4\pi \sqrt{mn}}{\gamma} \right) = \sum_j \hat{\phi}(\kappa_j) \frac{\rho_j(m) \rho_j(n)}{\cosh(\pi \kappa_j)} + \frac{1}{\pi} \sum_{a} \int_{-\infty}^{\infty} (mn)^{ir} \frac{\rho_a(\frac{1}{2} + ir, m)}{\rho_a(\frac{1}{2} + ir, n)} \hat{\phi}(r) dr.
\]

\[
(2.21)
\]

The Bessel transforms satisfy the following estimates which can be proved as in [3, Lemma 7.1].

**Lemma 2.4.** For a smooth \(\phi\) with support in \([X, 8X]\) satisfying \(\|\phi^{(j)}\|_{\infty} \ll (P/X)^j\) for \(0 \leq j \leq 2\),

\[
\hat{\phi}(ir), \hat{\phi}(ir) \ll \frac{1 + \left( \frac{X}{P} \right)^{-2r}}{1 + \frac{X}{P}}, \quad 0 \leq r \leq \frac{1}{4},
\]

\[
\hat{\phi}(r), \hat{\phi}(r) \ll 1 + \log \left( \frac{X}{P} \right), \quad r \geq 0,
\]

\[
(2.22)
\]

\[
\hat{\phi}(r), \hat{\phi}(r) \ll \left( \frac{P}{r} \right) \left( r^{-1/2} + \frac{X^{3/2}}{r} \right), \quad r \geq 1.
\]

The implied constants are absolute.
Finally, we state the differentiation and integration rules for the Bessel $J$-function [25, page 79]:

$$J_k(t)' = \frac{1}{2} (J_{k-1}(t) - J_{k+1}(t))$$  \hspace{1cm} (2.23)

and $(t^k J_k(t))' = t^k J_{k-1}(t)$, so that

$$\int_0^\infty g(t) J_k(\alpha \sqrt{t}) \, dt = \frac{2}{\alpha} \int_0^\infty \left( g'(t) \sqrt{t} - \frac{k}{2} \frac{g(t)}{\sqrt{t}} \right) J_{k+1}(\alpha \sqrt{t}) \, dt$$  \hspace{1cm} (2.24)

for $\alpha > 0$, $g \in C^1_0((0, \infty))$.

3 Further estimates

For the proof of Theorem 1.3, we apply the circle method in a variant of Jutila’s, see [14, 15].

Proposition 3.1 (Jutila’s circle method). Let $Q \geq 1$ and $Q^{-2} \leq \delta \leq Q^{-1}$ be two parameters. Let $w$ be a nonnegative function with support in $[Q, 2Q]$ satisfying $\|w\|_\infty \leq 1$ and $\sum w(q) > 0$. For $r \in \mathbb{Q}$, write $I_r(\alpha)$ for the characteristic function of the interval $[r - \delta, r + \delta]$ and define

$$\Lambda := \sum_q w(q) \phi(q), \hspace{1cm} \tilde{I}(\alpha) = \frac{1}{2\delta \Lambda} \sum_q w(q) \sum_{d(q)}^* I_{d/q}(\alpha).$$  \hspace{1cm} (3.1)

Then $\tilde{I}(\alpha)$ is a good approximation to the characteristic function on $[0, 1]$ in the sense that

$$\int_0^1 (1 - \tilde{I}(\alpha))^2 \, d\alpha \ll \delta \Lambda^2$$  \hspace{1cm} (3.2)

for any $\varepsilon > 0$. \hfill \square

The following lemma shows that for shifted convolution sums with Fourier coefficients of cusp forms, we have square root cancelation on average.

Lemma 3.2. For $X_1, X_2 \geq 0$, $l_1, l_2 \in \mathbb{Z}$,

$$\sum_{r \in \mathbb{Z}} \left| \sum_{\substack{1 \leq m_1 \leq X_1, 1 \leq m_2 \leq X_2 \\atop l_1 m_1 - l_2 m_2 = r}} a(m_1) \overline{a(m_2)} \right|^2 \ll \varepsilon (X_1 X_2 N^4 k^5)^{1+\varepsilon}$$  \hspace{1cm} (3.3)

uniformly in $l_1, l_2$. \hfill \square
Proof. This is essentially [15, Lemma 3]. Let \( R := 2(|l_1X_1| + |l_2X_2|) \) and \( \nu(r) := \max(0, 1 - |r|/R) \). Write

\[
S(X, \alpha) := \sum_{m=1}^{X} a(m)e(\alpha m). \tag{3.4}
\]

Then we have

\[
\begin{align*}
\sum_r \left| \sum_{1 \leq m_1 \leq X_j, l_1 m_1 - l_2 m_2 = r} a(m_1) \overline{a(m_2)} \right|^2 \\
\ll \sum_r \nu(r) \left| \sum_{1 \leq m_1 \leq X_j, l_1 m_1 - l_2 m_2 = r} a(m_1) \overline{a(m_2)} \right|^2 \\
= \int_{0}^{1} \left| \sum_r \sqrt{\nu(r)} e(r\alpha) \sum_{1 \leq m_1 \leq X_j, l_1 m_2 - l_2 m_2 = r} a(m_1) \overline{a(m_2)} \right|^2 d\alpha \\
= \int_{0}^{1} \left| \int_{0}^{1} S(X_1, l_1 \beta) \overline{S(X_2, l_2 \beta)} \sum_r \sqrt{\nu(r)} e(r(\alpha + \beta)) \, d\beta \right|^2 d\alpha \\
= \int_{0}^{1} \int_{0}^{1} S(X_1, l_1 \beta) \overline{S(X_2, l_2 \beta)} S(X_1, l_1 \beta) \overline{S(X_2, l_2 \beta)} \Delta_R(\beta_1 - \beta_2) \, d\beta_1 \, d\beta_2,
\end{align*}
\]

where

\[
\Delta_R(\beta) = \sum_{r=-R}^{R} \nu(r) e(r\beta) = \frac{1}{R} \left( \frac{\sin(\pi R \beta)}{\sin(\pi \beta)} \right)^2 \tag{3.6}
\]

is the Fejér kernel. Since \( \int_{0}^{1} |\Delta_R(\beta)| \, d\beta = 1 \), the lemma follows from (2.6). \( \blacksquare \)

The crucial ingredient for the proof of Theorem 1.3 is a spectral large sieve estimate of Deshouillers and Iwaniec [3, Theorem 2].
Proposition 3.3. Let \( K, M \geq 1 \). For any sequence \((b_m)\) of complex numbers,

\[
\sum_{|\kappa_j| \leq K} \frac{1}{\cosh(\pi \kappa_j)} \left| \sum_{M \leq m < 2M} b_m \rho_j(m) \right|^2
\]

\[
\sum_{k \equiv 0(2), 2 \leq k \leq K} \frac{(k - 1)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_k(N)} \left| \sum_{M \leq m < 2M} b_m \hat{\rho}_{j,k}(m) \right|^2 \ll_{\varepsilon} \left(K^2 + \frac{M^{1+\varepsilon}}{N}\right) \sum_{m=M}^{2M} |b_m|^2.
\]

\[
\sum_{a} \left| \sum_{M \leq m < 2M} b_m \rho_a \left( \frac{1}{2} + i t, m \right) \right|^2 \ll_{\varepsilon} (\varepsilon K^2 m^2 \theta k(N)).
\]

(3.7)

We need also individual bounds for the Fourier coefficients of the holomorphic cusp forms, Maass cusp forms, and Eisenstein series.

Lemma 3.4. For \( K, m \geq 1, t \in \mathbb{R}, \)

\[
\sum_{\kappa_j \leq K} \frac{\rho_j(m)\rho_j(m)}{\cosh(\pi \kappa_j)} \ll_{\varepsilon} (mNK)^{1+\varepsilon} K^2 m^2 \theta k(N),
\]

and

\[
\sum_{k \equiv 0(2), 2 \leq k \leq K} \frac{(k - 1)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_k(N)} \left| \hat{\rho}_{j,k}(m) \right|^2 \ll_{\varepsilon} (mNK)^{1+\varepsilon} K^2,
\]

and

\[
\sum_{a} \left| \rho_a \left( \frac{1}{2} + it, m \right) \right|^2 \ll_{\varepsilon} (\varepsilon K^2 m^2 \theta k(N)).
\]

Here all implied constants depend at most on \( \varepsilon \).

Proof. The first two estimates are equations (2.29) and (2.31) in [21] (note the different normalization there). To prove the last statement, we fix a cusp form \( \psi \) and write \( w = w^* w' w'' \) with \( w^* \mid (w^*)^\infty \) and \( (w'', w^*) = 1 \). Denoting as usual \( L(N,s,\psi) = \prod_p (1 - \psi(p)p^{-s})^{-1} \), we get by (2.18)
\[
\rho_a \left( \frac{1}{2} + it, m \right) 
= \left( \frac{w, \frac{N}{w}}{wN} \right)^{1/2+it} \frac{1}{\phi \left( \frac{w, \frac{N}{w}}{w} \right)} \sum_{w' \mid m} \frac{w' \psi \left( \frac{-um}{w'} \right) \psi(w'')}{L^{[N]}(1 + 2it, \psi^2)}
\]
\[
\times \sum_{x \left( \frac{w^*}{w} \right)} \psi(x) e \left( \frac{x}{w^*} \right) \sum_{\gamma \mid N^\infty} \frac{\psi^2(\gamma)}{\gamma^{1+2it}} \sum_{y \left( \frac{\gamma w''}{w} \right)} e \left( \frac{-my}{\gamma w''} \right) \sum_{z \mid m} \psi^2(z) \frac{1}{\gamma^{2it}}
\]
\[
\ll \varepsilon \left( (|t| + 1) mN \right)^{\varepsilon} \left( \frac{w, \frac{N}{w}}{wN} \right)^{1/2} \frac{1}{\phi \left( \frac{w, \frac{N}{w}}{w} \right)} \sum_{w' \mid m} w' \sqrt{w'} \frac{1}{\sqrt{w^*}}
\]
\[
\leq \left( (|t| + 1) mN \right)^{\varepsilon} \left( \frac{w \left( \frac{N}{w} \right)}{N} \right)^{1/2} \frac{1}{\phi \left( \frac{w, \frac{N}{w}}{w} \right)} \sum_{w' \mid m} \frac{1}{\sqrt{w^*}}
\]
\[
\ll \left( (|t| + 1) mN \right)^{\varepsilon} \frac{w}{N}^{1/2}.
\]
(3.10)

By (2.1) we obtain
\[
\sum_a \left| \rho_a \left( \frac{1}{2} + it, m \right) \right|^2 \ll \left( (|t| + 1) mN \right)^{\varepsilon} \sum_{w \mid N} \left( \frac{w, \frac{N}{w}}{w} \right) w \frac{w}{N} \ll \left( (|t| + 1) mN \right)^{\varepsilon}. \tag{3.11}
\]

Propositions 2.3 and 3.3, and Lemma 3.4 yield good estimates for sums of Kloosterman sums.

**Proposition 3.5.** Let \( P_1, P_2, S, Q \geq 1 \) be real numbers and let \( u \) be a smooth function with support in \([S, 2S] \times [Q, 2Q]\) satisfying \( \|u^{(ij)}\|_\infty \leq (P_1/S)^i(P_2/Q)^j \) for \( 0 \leq i, j \leq 2 \). Let \( N, h \) be positive integers, and let \( b(s), S \leq s \leq 2S, \) be a sequence of complex numbers. Then
\[
\sum_{N\mid q} \sum_{s} S(\pm h, s, q)b(s)u(s, q)
\ll \varepsilon Q \left( \sum_s \left| b(s) \right|^2 \right)^{1/2} \left( 1 + \frac{hS}{Q^2} + \frac{S}{N} \right)^{1/2} h^0 \left( 1 + \left( \frac{hS}{Q^2} \right)^{-\theta} \right) P_1 P_2^{5/2} (NhSQP_2)^{\varepsilon}.
\]
(3.12)
The dependence on $P_1, P_2$ can easily be improved.

Proof. We consider only the case of $S(h, s, q)$, the other case being similar. Let

$$U_h(t, q) := \int_{-\infty}^{\infty} u \left( x, \frac{4\pi \sqrt{hx}}{q} \right) \frac{4\pi \sqrt{hx}}{q} e(-tx) dx, \quad (3.13)$$

and

$$c_t(s) := b(s) e(ts). \quad (3.14)$$

By Fourier inversion, we have

$$qu(s, q) = \int_{-\infty}^{\infty} U_h \left( t, \frac{4\pi \sqrt{hx}}{q} \right) e(ts) dt, \quad (3.15)$$

so that

$$\sum_{N|q} \sum_s S(h, s, q) b(s) u(s, q) = \int_{-\infty}^{\infty} \sum_{N|q} \sum_s \frac{1}{q} S(h, s, q) c_t(s) U_h \left( t, \frac{4\pi \sqrt{hx}}{q} \right) dt. \quad (3.16)$$

By Proposition 2.3 and Cauchy's inequality,

$$\left( \sum_q \sum_s \right)^2 \ll \sum_j \frac{|\hat{U}_h(t, \kappa_j)|^2}{\cosh(\pi \kappa_j)} \sum_{s=S}^{2S} c_t(s) \rho_j(s) \left( \sum_j \frac{|\hat{U}_h(t, \kappa_j)|^2}{\cosh(\pi \kappa_j)} \right) \rho_j(h) \quad (3.17)$$

Obviously, $U_h(t, \cdot)$ is supported on $[2\pi \sqrt{hSQ}^{-1}, 4\sqrt{2\pi} \sqrt{hSQ}^{-1}]$. Integrating $i$ times by parts, we see

$$\frac{\partial^n}{\partial q^n} U_h(t, q) \ll \left( \frac{P_1}{S|t|} \right)^i \left( \frac{QP_2}{\sqrt{hS}} \right)^n S Q \quad (3.18)$$
for $t \neq 0$. We split the summation/integration over $j$, $k$, and $r$ into $j, k, r \leq T$, and $j, k, r > T$ for some $T > 1$ to be chosen in a minute. By (2.16), Lemma 2.4, Proposition 3.3, and Lemma 3.4, we obtain

$$
\sum_q \sum_s \ll_{\varepsilon} SQ\left(\frac{P_1}{S|t|}\right)^i \left(\sum_s |b(s)|^2\right)^{1/2} \left(T^2 + \frac{S}{N}\right)^{1/2} \left(Th^0\right)(STh)^\varepsilon
\times \left(1 + \log \frac{\sqrt{hS}}{QP_2} + \left(\frac{\sqrt{hS}}{QP_2}\right)^{-2\theta} + \frac{P_2^{3/2}}{T^{3/2}} \left(\frac{1}{\sqrt{T}} + \frac{(hS)^{3/4}}{TQ^{3/2}}\right)\right).
$$

Choosing for example

$$
T = 1 + P_2 \frac{\sqrt{hS}}{Q},
$$

we see

$$
\sum_q \sum_s \ll_{\varepsilon} SQ\left(\frac{P_1}{S|t|}\right)^i \left(\sum_s |b(s)|^2\right)^{1/2} \left(1 + P_2 \frac{\sqrt{hS}}{Q} + \frac{S^{1/2}}{N^{1/2}}\right)h^0
\times \left(1 + \left(\frac{\sqrt{hS}}{QP_2}\right)^{-2\theta}\right)P_2^2 (NhSQP_2)^\varepsilon.
$$

Finally, we choose $i = 2$ if $|t| \geq P_1/S$ and $i = 0$ else. After integrating over $t$ we get the proposition.

## 4 Proof of Theorem 1.3

We start by transforming the sum in question with Jutila’s circle method. Let $Q > Nl_1l_2$ be a parameter to be chosen later, and let $\delta = Q^{-1}$. Let $\tilde{w}$ be a smooth function with support in $[Q, 2Q]$ satisfying $\|q^{(j)}\|_{\infty} \asymp Q^{-j}$ for all $j$, and let $w(q) := \tilde{w}(q)X_{(q \equiv 0 (Nl_1l_2))}(q)$. With the notation as in Proposition 3.1, we have

$$
\Lambda \asymp Q^2 (Nl_1l_2)^{-1}
$$

(4.1)
and

\[
D_g(l_1, l_2, h) = \int_0^1 \sum_{m_1} \sum_{m_2} a(m_1) \overline{a(m_2)} e(l_1 m_1 \alpha) e(-l_2 m_2 \alpha) g(m_1, m_2) e(-\alpha h) d\alpha
\]

\[
= \frac{1}{2\delta^2} \sum_{Nl_1 l_2 | q} \tilde{w}(q) \sum_{d(q)} \int_{-\delta}^{\delta} \sum_{m_1} \sum_{m_2} a(m_1) \overline{a(m_2)} e\left(l_1 m_1 \left(\frac{d}{q} + \eta\right)\right) e\left(-l_2 m_2 \left(\frac{d}{q} + \eta\right)\right) g(m_1, m_2) e\left(-h \left(\frac{d}{q} + \eta\right)\right) d\eta + E;
\]

by Cauchy's inequality and Proposition 3.1, the error E is bounded by

\[
E \ll \varepsilon Q^{1+\varepsilon} \max_{\delta^{1/2}} \sum_{m_1} \sum_{m_2} a(m_1) \overline{a(m_2)} e(\alpha_1 m_1) e(\alpha_2 m_2) g(m_1, m_2)
\]

\[
\ll \varepsilon (Q\sqrt{M_1 M_2 N^{2(1/2)^2}})^{1+\varepsilon} p_1 p_2
\]

\[
\ll \sqrt{M_1 M_2 N^{1/2} p_1 p_2} (N K Q M_1 M_2)^{\varepsilon}.
\]

Here we used partial summation together with (2.6). For \( N l_1 l_2 | q \), we can transform the double sum over \( m_1, m_2 \) by means of Lemma 2.1 getting

\[
\sum_{m_1} \sum_{m_2} a(m_1) \overline{a(m_2)} e\left(\frac{d(l_2 m_2 - l_1 m_1)}{q}\right) g_{q, l_1, l_2, \eta}^*(m_1, m_2) =: \Sigma(q, d, \eta),
\]

where

\[
g_{q, l_1, l_2, \eta}(x_1, x_2) = \frac{4\pi^2 l_1 l_2}{q^2} \int_0^\infty \int_0^\infty \tilde{g}(t_1, t_2) e(l_1 t_1 \eta - l_2 t_2 \eta)
\]

\[
\times J_{k-1} \left(\frac{4\pi l_1 \sqrt{x_1 t_1}}{q}\right) J_{k-1} \left(\frac{4\pi l_2 \sqrt{x_2 t_2}}{q}\right) dt_1 dt_2
\]

so that

\[
\sum_{\text{d}} \sum_{N l_1 l_2 | q} \tilde{w}(q) \int_{-\delta}^{\delta} \Sigma(q, d, \eta) \cdot e\left(-h \left(\frac{d}{q} + \eta\right)\right) d\eta
\]

\[
= \int_{-\delta}^{\delta} e(-\eta h) \sum_{N l_1 l_2 | q} \tilde{w}(q)
\]

\[
\times \sum_{m_1, m_2} S\left(-h, l_2 m_2 - l_1 m_1, q\right) a(m_1) \overline{a(m_2)} g_{q, l_1, l_2, \eta}^*(m_1, m_2) d\eta.
\]
The double sum over \( m_1, m_2 \) equals

\[
\sum_{r \in \mathbb{Z}} S(-h, r, q) \sum_{l_2 m_2 - l_1 m_1 = r} a(m_1) \overline{a(m_2)} g^*_{q, l_1, l_2, \eta}(m_1, m_2) \\
= S(-h, 0, q) \sum_{l_2 m_2 = l_1 m_1} a(m_1) \overline{a(m_2)} g^*_{q, l_1, l_2, \eta}(m_1, m_2) \\
- \int_{1/2}^{\infty} \sum_{r > 0} S(-h, r, q) \sum_{m_1 \leq x, l_2 m_2 - l_1 m_1 = r} a(m_1) \overline{a(m_2)} h^+_{l_1, l_2}(r, q; x, \eta) dx \\
- \int_{1/2}^{\infty} \sum_{r < 0} S(-h, r, q) \sum_{m_2 \leq x, l_2 m_2 - l_1 m_1 = r} a(m_1) \overline{a(m_2)} h^-_{l_1, l_2}(r, q; x, \eta) dx
\]  

(4.7)

with

\[
h^+_{l_1, l_2}(r, q; x, \eta) := \frac{d}{dx} g^*_{q, l_1, l_2, \eta}(x, \frac{r + l_1 x}{l_2}),
\]
\[
h^-_{l_1, l_2}(r, q; x, \eta) := \frac{d}{dx} g^*_{q, l_1, l_2, \eta}(\frac{l_2 x - r}{l_1}, x).
\]  

(4.8)

We impose the condition

\[
d \leq \min \left( P_1 \left( l_1 M_1 \right)^{-1}, P_2 \left( l_2 M_2 \right)^{-1} \right).
\]  

(4.9)

Integrating by parts \( j_1 \) times with respect to \( t_1 \) and \( j_2 \) times with respect to \( t_2 \), and using \( |J_k(x)| \leq 1 \), we find by (2.24)

\[
g^*_{q, l_1, l_2, \eta}(x_1, x_2) \ll_{j_1, j_2} \frac{M_1 M_2 l_1 l_2}{q^2} \left( \frac{P_1 q k}{l_1 \sqrt{M_1 x_1}} \right)^{j_1} \left( \frac{P_2 q k}{l_2 \sqrt{M_2 x_2}} \right)^{j_2}.
\]  

(4.10)

Write \( l_i' := l_i/(l_1 l_2) \) for \( i = 1, 2 \). By (2.5), the total contribution \( D_1 \) to \( D_g(l_1, l_2, h) \) of the first term on the right-hand side of (4.7) is bounded by

\[
\frac{\tau(l_1') \tau(l_2')}{\Lambda} \sum_{Q \leq q \leq 2Q} \sum_{N l_1 l_2 | q} (h, q) \tau((h, q)) \sum_m \tau^2(m) |g^*_{q, l_1, l_2, \eta}(l_2', m, l_1' m)|.
\]  

(4.11)

We choose \( j_1 = j_2 = 0 \) if

\[
m \leq \frac{P_1 P_2 Q^2 k^2}{(l_1' l_2')^{3/2} (M_1 M_2)^{1/2}}
\]  

(4.12)
and \( j_1 = j_2 = 2 \) otherwise so that

\[
D_1 \ll_{\varepsilon} \frac{\tau(l_1^0)\tau(l_2^0)\tau(h)^2}{Q} \left( \frac{M_1 M_2 l_1 l_2}{Q^2} \right)^{1+\varepsilon} \left( \frac{P_1 P_2 Q^2 k^2}{(l_1^0 l_2^0)^{3/2}} \right)^{1/2},
\]

\[
\ll_{\varepsilon} \frac{(l_1 l_2)^2}{Q} \sqrt{M_1 M_2} N k^2 P_1 P_2 (k P_1 P_2 l_1 l_2 Qh)^{\varepsilon}.
\]

The second and the third terms on the right-hand side of (4.7) can be treated similarly. We consider only the second. We have

\[
h_{1_1,1_2}^+(r, q; x, \eta) = g_{q,1_1,1_2,n}^{(1,0)} \left( \frac{x+r+l_1 x}{l_2} \right) + \frac{l_1}{l_2} g_{q,1_1,1_2,n}^{(0,1)} \left( \frac{x+r+l_1 x}{l_2} \right). \tag{4.14}
\]

Integrating by parts \( j_1 \) times with respect to \( t_1 \) and \( j_2 \) times with respect to \( t_2 \), together with (2.23) and (2.24), gives

\[
\ll_{j_1,j_2} \frac{\partial^{i_1} \partial^{i_2}}{\partial t_1 \partial q^{i_2}} (\tilde{w}(q) h_{1_1,1_2}^+(r, q; x, \eta))
\ll_{i_1,i_2,j_1,j_2} \frac{l_1^2 l_2 M_1 M_2}{Q^2} \left( \frac{\sqrt{M_1}}{\sqrt{x}} + \frac{\sqrt{l_2 M_2}}{\sqrt{r+l_1 x}} \right)
\times \left( \frac{l_1}{l_2} \right)^{i_1} \left( \frac{l_1}{l_2} \right)^{i_2}
\times \left( \frac{k Q P_1}{l_1 \sqrt{x M_1}} \right)^{i_1} \left( \frac{k Q P_2}{\sqrt{l_2 (r+l_1 x) M_2}} \right)^{i_2}.
\tag{4.15}
\]

Let \( \varepsilon > 0 \) be given. Then we can choose \( j_1, j_2 \) sufficiently large, so that the contribution of

\[
r \geq \left( \frac{Q^2 P_2^2 k^2}{l_2 M_2} \right)^{1+\varepsilon}
\]

\[
\ll_{i_1,i_2} \frac{l_1^2 l_2 M_1 M_2}{Q^2} \left( \frac{\sqrt{M_1}}{\sqrt{x}} + \frac{\sqrt{l_2 M_2}}{\sqrt{r+l_1 x}} \right)
\times \left( \frac{l_1}{l_2} \right)^{i_1} \left( \frac{l_1}{l_2} \right)^{i_2}
\times \left( \frac{1}{r} P_2 k (l_1 l_2 M_2 Q P_2 k)^{\varepsilon} \right)^{i_1} \left( \frac{1}{Q} (P_1 + P_2) k (l_1 l_2 M_1 M_2 P_1 P_2 Q k)^{\varepsilon} \right)^{i_2}.
\tag{4.16}
\]

to (4.7) is negligible. For the remaining \( x \) and \( r \), we have

\[
\ll_{i_1,i_2} \frac{l_1^2 l_2 M_1 M_2}{Q^2} \left( \frac{\sqrt{M_1}}{\sqrt{x}} + \frac{\sqrt{l_2 M_2}}{\sqrt{r+l_1 x}} \right)
\times \left( \frac{l_1}{l_2} \right)^{i_1} \left( \frac{l_1}{l_2} \right)^{i_2}
\times \left( \frac{1}{r} P_2 k (l_1 l_2 M_2 Q P_2 k)^{\varepsilon} \right)^{i_1} \left( \frac{1}{Q} (P_1 + P_2) k (l_1 l_2 M_1 M_2 P_1 P_2 Q k)^{\varepsilon} \right)^{i_2}.
\tag{4.17}
\]
Using a smooth partition of unity, we can restrict the support of $h_1^+(\cdot, q; x, \eta)$ to dyadic intervals $[R, 2R]$ with $R \leq (Q^2P^2k^2/l_2M_2)^{1+\varepsilon}$. By Lemma 3.2,

$$
\sum_r \left| \sum_{l_2m_2 - l_1m_1 = r} a(m_1)\overline{a(m_2)} \right|^2 \ll \left( \frac{Q^4P^2P^2N^4k^2}{l_1^2l_2^2M_1M_2} \right)^{1+\varepsilon}
$$

(4.18)

in the considered range of $x$. Thus, by Proposition 3.5 applied to the congruence subgroup $\Gamma_0(Nl_1l_2)$, the total contribution $D_2$ to $D_g(l_1, l_2, h)$ of the second term on the right-hand side of (4.7) is bounded by

$$
\frac{l_1^2l_2M_1M_2}{Q^4A} \left( \sqrt{M_1} + \sqrt{\frac{l_2}{l_1}M_2} \right) QX^{1/2} \frac{Q^2}{l_1l_2\sqrt{M_1M_2}} \left( 1 + \frac{h}{l_2M_2} + \frac{Q^2}{l_1l_2M_2} \right)^{1/2} h^\theta
$$

$$
\times \left( 1 + \left( \frac{h}{l_2M_2} \right)^{-\theta} \right) (hQM_1M_2l_1l_2)^\varepsilon,
$$

(4.19)

up to a constant that depends polynomially on $P_1, P_2, N, k, N$. Choosing $Q$ sufficiently large, we see that the contribution of (4.3) and (4.13) is negligible. Since $h \leq l_1M_1 + l_2M_2$, we obtain after simplifying

$$
D_2 \ll_{p_1, p_2, N, k, \varepsilon} \left( l_1M_1 + l_2M_2 \right)^{1/2+\theta+\varepsilon},
$$

(4.20)

uniformly in $h$.

5 Proof of Theorem 1.1

By the approximate functional equation and a smooth partition of unity, we have to bound sums

$$
B_X = \sum_m a(m)x(m)g(m)
$$

(5.1)

with a smooth, compactly supported function $g$. To this end, we use the amplification technique of [4].

**Lemma 5.1.** Let $M \ll D^{3/2}$, and let $g$ be a smooth function supported in $[M, 2M]$ such that $\|g^{(j)}\|_\infty \ll (P/M)^j$ for all $j$ and some $P \geq 1$. Then

$$
B_X \ll_{\varepsilon, P, k, N} (DM^2)^{(3+2\theta)/(10+4\theta)+\varepsilon}
$$

(5.2)

for any $\varepsilon > 0$. The dependence on $P, k$, and $N$ is polynomial.

□
Proof. We consider the sum

\[ S := \sum_{\psi \pmod{D}} \left| \sum_m a(m) \psi(m) g(m) \right|^2 \sum_{l=1}^L \bar{\chi}(l) \psi(l) \]  

for a parameter \( L \gg D^{\delta} \) with some \( \delta > 0 \). Then

\[ \# \{ 1 \leq L | (l, D) = 1 \} \gg \epsilon LD^{-\epsilon} \]  

(see, e.g., [13]). Taking only the contribution of \( \chi \) to \( S \), we see

\[ B_\chi \ll \epsilon S^{1/2} L^{-1} D^\epsilon. \]  

On the other hand, opening the square gives by orthogonality of characters

\[ S \leq \phi(D) \sum_{1 \leq l_1, l_2 \leq L} \bar{\chi}(l_1) \chi(l_2) \sum_{l_1, m_1 \equiv l_2 m_2 (D)} a(m_1) \overline{a(m_2)} g(m_1) g(m_2) \]

\[ = \phi(D) \sum_h \sum_{1 \leq l_1, l_2 \leq L} \bar{\chi}(l_1) \chi(l_2) D_{g \times g}(l_1, l_2, hD). \]  

The diagonal term \( h = 0 \) can be estimated from above by

\[ \phi(D) \sum_{\substack{l_1, m_1 = l_2 m_2 \leq L \\text{ M} \leq m_1, m_2 \leq 2M}} |a(m_1)a(m_2)| \ll \phi(D) \sum_{t \leq L} \sum_{M \leq m \leq 2M} |a(m)|^2 \tau(ml) \]

\[ \ll \epsilon D(LM)^{1+\epsilon}. \]  

Clearly \( D_{g \times g}(l_1, l_2, hD) = 0 \) for \( hD > 2LM \). Thus, by Theorem 1.3, the off-diagonal contribution can be bounded by

\[ \phi(D) \frac{LM}{D} L^2 (LM)^{1/2+\theta+\epsilon}, \]

and we get

\[ S \ll \epsilon D(LM)^{1+\epsilon} + L^{7/2+\theta+\epsilon} M^{3/2+\theta+\epsilon}. \]  

Using (5.5) and (5.9), we find

\[ B_\chi \ll \epsilon \left( (DML)^{-1/2} + (LM)^{3/4+\theta/2} \right) (DML)^{\epsilon}. \]  

Choosing \( L := D^{2/(5+2\theta)} M^{-(1+2\theta)/(5+2\theta)} > 1 \), we obtain the lemma. \( \square \)
**Theorem 1.1** follows now easily from **Lemma 2.1.** By (2.9), the contribution of \( m \geq (D \sqrt{N}(|s| + k))^{1+\varepsilon} \) to the sum in (2.8) is negligible. By a smooth partition of unity, we break \( V_s \) into functions \( V_{s,M} \) supported in dyadic boxes \([M, 2M]\) with \( M \ll N, k, t, \varepsilon \ D^{1+\varepsilon} \), and apply **Lemma 5.1** with \( g(m) := m^{-it}V_{s,M}(2\pi m(ND^2)^{-1/2}) \).

Note added in proof. **Theorem 1.3** can be applied to many more subconvexity problems. For two primitive (holomorphic or Maass or one of each) cusp forms \( f \) and \( g \) of level \( N \) let \( L(f \otimes g, s) = \zeta(2s) \sum \lambda_f(n)\lambda_g(n)n^{-s} \) be the Rankin-Selberg L-function. Let \( \kappa \) denote the weight of \( f \) if \( f \) is holomorphic and the spectral parameter if \( f \) is Maass. Then **Theorem 1.3** (replacing [23, Theorems A1 and A2]) and the same reasoning as in [20] show

\[
L(f \otimes g, 1/2 + it) \ll_{\varepsilon, N, t, g} k^{(3+2\theta)/4+\varepsilon}. \tag{5.11}
\]

The details may appear elsewhere. However, at the Zeta 2004 meeting at Oberwolfach, Professors Jutila and Motohashi announced the exponent \( 2/3 + \varepsilon \), if \( g \) is holomorphic and \( N = 1 \), and gave some evidence to perhaps tackle the general case.

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**References**


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