Lecture 1: Introduction

The main things we will be talking about in the first part of this course are groupoids and stacks. In the second part of the course we will use this knowledge to study T-duality. As an introduction and motivation, let me quickly point out how groupoids, stacks, and T-duality are related to noncommutative geometry.

0.1. Groupoids and Noncommutative geometry. A major part of Connes’ style noncommutative geometry can be viewed through the lens of groupoids, via the correspondence

groupoids ⇝ groupoid C*-algebras.

More precisely, a locally compact Hausdorff groupoid with an appropriate notion of measure gives rise to a convolution C*-algebra, and there is a notion of Morita equivalence for groupoids which induces Morita equivalence of the corresponding C*-algebras.

The reason that this correspondence covers a “major part” of noncommutative geometry is that most of the well-known C*-algebras are in fact Morita equivalent to (twisted) groupoid algebras\(^1\). This includes for example all AF-algebras, noncommutative tori, and Cuntz algebras [Ren], and the continuous trace algebras as well [?]. In fact, at least to me, it is an open question to find a separable C*-algebra which is not Morita equivalent to a twisted groupoid algebra.

There are some genuine advantages to working directly with groupoids rather than C*-algebras, of which I will name three:

1. The passage to the groupoid algebra can lose information, for example C*-algebras cannot distinguish the non-equivalent groupoids S\(^1\) \(\rightarrow\) S\(^1\) and Z \(\rightarrow\) *.
2. If we are not intending to form C*-algebras there is no need to restrict to locally compact groupoids, thus for example it becomes possible to study noncommutative mapping spaces.
3. All of the morphisms we would want between groupoids can be expressed using functors, whereas at C*-algebra level even the simple map \(\mathbb{R} \rightarrow pt\) can only be described using bi-modules.

0.2. Stacks and noncommutative geometry. As a first approximation, stacks may be viewed as Morita equivalence classes of groupoids, and in this sense they describe noncommutative phenomena. On the other hand, we will learn that a stack is the precise 2-categorical analogue of a sheaf, thus stacks provide a link between noncommutative geometry and 2-categorical geometry. But there are stacks which cannot be viewed as equivalence classes of groupoids, so stacks provide a more general framework than groupoids do for studying noncommutative geometry. For these reasons (and also, simply because stacks, rather than groupoids, are the standard in algebraic geometry) I think it is worthwhile to go through the considerable work of learning the language of stacks.

\(^{1}\) A twisted groupoid C*-algebra is a groupoid algebra whose multiplication is modified by a groupoid 2-cocyle (see Lecture 3).
0.3. **T-duality.** The second half of these lectures is going to be applying what we have learned about groupoids and stacks to noncommutative T-duality. For now I will just say what the mathematical description of commutative T-duality is.

First, the dual of a torus $T := V/\Lambda \cong \mathbb{R}^n/\mathbb{Z}^n$ is

$$T^\vee := \hat{\Lambda} \equiv \text{Hom}(\Lambda, \mathbb{R}/\mathbb{Z}).$$

So T-duality for a single torus is in simplest form just the relation between a torus $V/\Lambda$ and the Pontryagin dual of its defining lattice $\Lambda$.

Now consider a principal $T$-bundle. One would guess that its T-dual is a principal $T^\vee$-bundle, but in fact it is a $U(1)$-gerbe over the trivial $T^\vee$-bundle. The reason for this will be only be apparent after we have described the transform which implements T-dualization; one thing we must conclude immediately, however, is that T-duality innately involves $U(1)$-gerbes over principal bundles. This complicates the picture further, because T-dualizing a non-trivial torus bundle with $U(1)$-gerbe on it can result in a noncommutative $T^\vee$-bundle.

Physics T-duality is the statement that string theory on one space is physically equivalent to (a different) string theory on the T-dual space. For this statement to make sense, one has to take account of the differential geometric structures (such as, for example, the gravitational metric) that are involved in defining a string theory. The end-goal of T-duality for mathematicians is thus not only to find what the T-dual(s) to a given space are, but also to transport geometric structure from one space to its T-dual in a meaningful way. In appropriate situations one can transport generalized complex structures, generalized Riemannian structures, and generalized Kähler structures across the duality. Since the dual can be noncommutative, this involves first *defining* what those structures are!

Okay, I think I’ve done as much motivating for the seminar as I will do. Let us get on to groupoids.

0.4. **The anatomy of a groupoid.** A groupoid is a category in which all arrows are invertible. In this seminar all groupoids are assumed small, that is, they are groupoid objects in the category of sets. If $\mathcal{G}$ is a groupoid then $\mathcal{G}_k$ denotes the $k$-tuples of composable arrows, $\mathcal{G}_0$ denotes the objects, and $\pi_0(\mathcal{G})$ refers to the isomorphism classes of objects. The source, target, multiplication, inversion and unit maps are denoted $s, t, \mu, \iota$ and $e$. In practice those structure maps are often self evident once the arrows and objects are specified; in such cases a groupoid is presented just by writing

$$\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0).$$

By default lowercase letters refer to arrows in a groupoid of the same uppercase letter, so I usually do not bother to write $g \in \mathcal{G}_1, h \in \mathcal{H}_1$, etcetera. Also, the multiplication map is usually omitted, so $g_1g_2 \equiv \mu(g_1, g_2)$, and (again by default) the pair $(g_1, g_2)$ is presumed composable whenever their product $g_1g_2$ is written down.

Even though a groupoid is a category, the correct intuition comes from viewing it as a common generalization of a space and a group. For instance a map between two groupoids, that is, a functor:

$$\mathcal{G} \xrightarrow{\phi} \mathcal{H}$$

is defined by what it does to arrows, so it is just a set function

$$\mathcal{G}_1 \xrightarrow{\phi} \mathcal{H}_1,$$

and functoriality is encoded is what resembles a group homomorphism property:

$$\phi(g_1g_2) = \phi(g_1)\phi(g_2).$$
Note that a set is a groupoid (with only identity arrows), and a functor between sets is simply a function. Also, a group is a groupoid (with only one object) and a functor between groups is the same thing as a group homomorphism.

It is also convenient to view a natural transformation \( \eta : \phi \Rightarrow \tau \) between two functors via its set map:

\[
\begin{array}{c}
G_0 \\ \eta \\
\downarrow \\
H_1
\end{array}
\]

which satisfies

\[ \eta(tg)\phi(g) = \tau(g)\eta(sg). \]

Note that natural transformations are automatically invertible.

By definition, a functor \( \phi : G \to H \) is an equivalence if there is another functor \( \tau : H \to G \) and natural isomorphisms \( \phi\tau \Rightarrow \text{id}_H \), \( \tau\phi \Rightarrow \text{id}_G \). One of the first theorems of category theory is that a functor is an equivalence if and only if it is fully faithful and essentially surjective. This theorem fails for topological groupoids, and equivalence is replaced by the notion of Morita equivalence to fix this problem.

The 2-category of groupoids is the 2-category in which objects are groupoids, 1-arrows are functors, and 2-arrows are natural transformations. Another way to package this is to say that this 2-category has groupoids for objects, and for every pair of objects \( G, H \), there is a category \( \text{HOM}(G, H) \) whose objects are functors and arrows are natural transformations.

One can write down so-called 2-commutative diagrams in any 2-category. By definition these look just like diagrams in a 1-category, except they include the data of a coherent collection of invertible 2-arrows between every pair of paths with the same start point and end point. Using 2-commutative diagrams it is usually straightforward to extend categorical notions (e.g. product, coproduct, limit) to the 2-categorical setting. For example in the 2-category of groupoids, the coproduct is disjoint union and the product is cartesian product.

Exercise: (Fibered product in groupoids) Given two functors \( H \xrightarrow{a} G \) and \( K \xrightarrow{b} G \), a fibered product over \( G \) is a terminal object in the 2-category of 2-commutative diagrams:

\[
\begin{array}{c}
C \\
\downarrow \\
H \\
\downarrow \\
G
\end{array} \xleftarrow{\pi_H} \begin{array}{c}
K \\
\downarrow \\
G \end{array}
\]

In other words, a fibered product is a groupoid \( H \coprod_G K \) equipped with arrows

\[
H \xleftarrow{\pi_H} H \coprod_G K \xrightarrow{\pi_K} K
\]

and a natural transformation \( a\pi_H \Rightarrow b\pi_K \), such that if \( C \) is any other groupoid equipped with this data, there exists an essentially unique arrow \( C \to H \coprod_G K \) and natural transformations rendering all possible composite diagrams 2-commutative.

The exercise is: write down a model for the fibered product, prove it is a fibered product, and prove it is essentially unique.

Next time we will start the discussion of topological groupoids and Morita equivalence.

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2Note that the 1-category 1-cat of small categories is monoidal, and a 2-category is the same thing as a (1-cat)-enriched category. To give a 2-categorical analogue of a categorical notion means, precisely, to translate it from the analogous concept in enriched category theory.

3The modifier “essential” for statements in 2-categories means up to a 2-arrow which not only exists but is given.
Lecture 2: Topological Groupoids

Last time we talked at a very general level about T-duality, and how it forces the existence of noncommutative models of spacetime. We also discussed how groupoids, $C^*$-algebras and stacks can be viewed as different approaches to NCG. We will use tools from all three areas when we study noncommutative T-duality in detail.

Today we will cover topological groupoids and various notions of equivalence between them.

Definition 0.1. A topological groupoid is a groupoid whose arrows $G_1$ and objects $G_0$ are topological spaces and for which the structure maps (source, target, multiplication, unit map, and inversion) are continuous, and for which the source and target maps are open. More simply, it is a groupoid object in topological spaces with open source and target maps.

A groupoid is called étale if the source map is a local homeomorphism, and is called proper if the map $G \to G_0 \times G_0$ is proper. Taking the perspective that a groupoid is a generalized group, étale corresponds to discrete groups, while proper corresponds to compact groups. Let us look at some examples:

- A topological space $X$ can be viewed as a groupoid, which we denote $X \to X$.
- A topological group $G$ can be viewed as a groupoid, which we denote $G \to \text{pt}$.
- If a group $G$ acts continuously on a space $X$, (continuously in the sense that $G \times X \to X$ is continuous) we can form the associated transformation groupoid (also sometimes called the action groupoid). By definition, this is the groupoid $G \rtimes X := (G \times X \to X)$ with $s(g,x) := x$, $t(g,x) := gx$, $(g,g') \cdot (g',x) := (gg',x)$. Note that a group action is proper if and only if the action groupoid is proper, and a group is discrete if and only if the action groupoid is étale. The source map is a projection in the product topology, which is always an open map (so the target $t = s \circ i$ is open as well).
- Given a surjective map $Y \to X$, you can make the relative pair groupoid $Y \times_X Y \to Y$, whose source and target are the two projections to $Y$. Equivalently, the subset $R = Y \times_X Y \subset Y \times Y$ is a relation on $Y$, so the relative pair groupoid $R \to Y$ is sometimes called a relation groupoid. We will prove that the source is open if and only if $Y \to X$ is open, and that in this case the topological quotient space $Y/R$ (viewed as a groupoid) is Morita equivalent to the relation groupoid. An important special case is the pair groupoid of an open cover $Y = \coprod U_\alpha \to X$. This is called the Čech groupoid of the cover:

$$Y \times_X Y \simeq (\coprod_{(\alpha,\beta)} U_\alpha \cap U_\beta \to U_\alpha).$$

Here $(\alpha, \beta)$ is an ordered pair and the source and target maps are the sum of the inclusions:

$$s : U_\alpha \cap U_\beta \hookrightarrow U_\beta, \quad t : U_\alpha \cap U_\beta \hookrightarrow U_\alpha.$$

- Consider a homomorphism $G \to H$ of abelian groups. Then $G$ acts on $H$ via the homomorphism so you can form the action groupoid $G \rtimes H$. The product group structure on $(G \times H)_1 = G \times H$ is compatible with all of the groupoid structure maps, and makes $G \rtimes H$ a strict abelian group object in topological groupoids. Such things are called Picard groupoids, or abelian strict 2-groups. More generally, a crossed module of topological
groups gives rise to a (not necessarily abelian) group object in topological groupoids, which is the same as a strict 2-group. A favorite example is the homomorphism \[ Z \cdot \theta \to \mathbb{R}/\mathbb{Z} \quad n \mapsto \theta n, \ \theta \in \mathbb{R}. \]

The resulting groupoid \( Z_{\theta} \ltimes \mathbb{R}/\mathbb{Z} \) has as its groupoid algebra (see Lecture 4) the 2-dimensional noncommutative torus with parameter \( \theta \).

More examples will come when we discuss groupoid cohomology.

Remark 0.2. When \( \theta \) is irrational, then the homomorphism \( Z_{\theta} \cdot \to \mathbb{R}/\mathbb{Z} \) is injective so the resulting action of \( Z \) if free and there is a continuous bijection \( Z \times \mathbb{R}/\mathbb{Z} \to (\mathbb{R}/\mathbb{Z}) \times X_{bad} \) \( (n, t) \mapsto (n\theta + t, t) \), where \( X_{bad} = (\mathbb{R}/\mathbb{Z})/\mathbb{Z} \theta \) denotes the topological quotient. This should not be taken to mean that the action groupoid is isomorphic to the relation groupoid of the quotient map \( \mathbb{R}/\mathbb{Z} \to X_{bad} \).

Indeed the continuous bijection is not continuously invertible, which is equivalent to the fact that the quotient map to \( X_{bad} \) is not open. This is of course why the transformation groupoid shines: it is a well-behaved replacement for a bad quotient space.

0.5. Principal \( G \)-modules and Morita equivalence.
Recall that two rings \( R \) and \( S \) are Morita equivalent when there is an \( R \)-\( S \) bimodule \( M \) and a \( S \)-\( R \)-bimodule \( N \) and isomorphisms \( M \otimes_R N \simeq S \) and \( N \otimes_S M \simeq R \). The point of Morita equivalence is that it induces an equivalence of categories:
\[ R-\text{Mod} \to S-\text{Mod}. \]

Adding topological considerations gives the analogue, strong Morita equivalence, for \( C^* \)-algebras. Morita equivalence is considered the correct notion of equivalence for NCG because, among other things, it induces a homeomorphism between the underlying spectra which are thought of as the underlying noncommutative spaces. In fact, you can say a property of a \( C^* \)-algebra \( A \) is in the realm of noncommutative geometry if it only depends on the category of Hilbert \( A \)-modules. (e.g. K-theory, cyclic theories)

Morita equivalence for groupoids can be viewed as an analogue of this. That is, we can come up with a category of modules over a topological groupoid and define Morita equivalence bi-modules which implement equivalences between the module categories. The underlying “noncommutative space” which is preserved by Morita equivalence of groupoids is the stack associated to the groupoid, and in applicable cases, the \( C^* \)-algebra of the groupoid.

Definition 0.3. A (right) \( G \)-module is a space \( P \) with a continuous map \( P \to G_0 \) called the moment map and a continuous action map:
\[ P \times_{G_0} G_1 \to P \quad (p,g) \mapsto pg. \]

Here \( P \times_{G_0} G_1 := \{ (p,g) \mid tg = \varepsilon p \} \) is the fibered product and by “action” we mean that \( (pg_1)g_2 = p(g_1g_2) \). The base of a \( G \)-module is the quotient space \( P/G \). A morphism of \( G \)-modules is a continuous \( G \)-equivariant map.

Note that \( G \) itself is a \( G \)-module. A \( G \)-module is called trivial if it is isomorphic to a pullback of the \( G \)-module \( G \) via some map \( X \to G_0 \). In other words a trivial module is one of the form \( X \times_{f,G_0} G \) with moment map \( (x,g) \mapsto sg \).

A left module is defined similarly, and one can convert a left module \( P \) to a right module \( P^{op} \) by setting \[ p \cdot g := g^{-1}p, \quad g \in G, \ p \in P^{op}. \]
A $G$-module $P$ is called **principal** if
\[ P \times_{G_0} G \to P \times_{P/G} P; \quad (p, g) \mapsto (p, pg) \]
is a homeomorphism.

**Proposition 0.4.** A $G$-module $P$ is principal if and only if the action is free (meaning $(pg = pg') \Rightarrow (g = g')$) and
\[ P \times_{G_0} G \to P \times_{P/G} P \]
is a continuous closed (or open, or proper) map.

**Proof.** It is easy to check that freeness is equivalent to bijectivity of the map, and a map is a continuous closed (or open or proper) bijection if and only if it is a homeomorphism.

The following lemma is useful in a variety of contexts:

**Lemma 0.5.**
\begin{enumerate}
\item A pullback of an open map is open.
\item A surjection $X \to Y$ is open if and only if the projections $X \times Y \to X$ are open.
\item For any principal $G$-module $P$, the base map $P \to P/G$ is open.
\end{enumerate}

**Proof.** To see that a pullback of an open map is open, consider the diagram
\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_2} & Z \\
\downarrow{\pi_X} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]
in which $g$ is an open map. A basis for the opens in $X \times Y Z$ is given by sets of the form $\pi_X^{-1}(W) \cap \pi_Z^{-1}(U)$ for open subsets $U \subset Z, W \subset X$, so the openness of $\pi_X$ follows from the two equalities:
\begin{enumerate}
\item $\pi_X(\pi_Z^{-1}(U)) = f^{-1}(g(U))$,
\item $\pi_X(\pi_X^{-1}(W) \cap \pi_Z^{-1}(U)) = W \cap \pi_X(\pi_Z^{-1}(U))$.
\end{enumerate}

Now we prove the second statement of the proposition. Suppose that $X \xrightarrow{f} Y$ is a surjection, then it is easy to see that $f$ is open if and only if the saturation $f^{-1}(f(U))$ of any open $U \subset X$ is itself open. But if $X \times_Y X \xrightarrow{\pi_1, \pi_2} X$ denote the two projections, then the saturation of $U \subset X$ is equal to $\pi_1(\pi_2^{-1}(U))$. So $f$ is open if and only $\pi_1$ is open as desired.

Now let $P$ be a principal $G$-module and write $B = P/G$. Then the third statement follows from the second as long as $P \times_B P \xrightarrow{\pi_2} P$ is open. But the homeomorphism $P \times_B P \simeq P \times_{G_0} G$ identifies $\pi_1$ with the map
\[ P \times_{G_0} G \xrightarrow{id \times t} P \times_{G_0} G_0 \simeq P \]
which is open since the target map $G \xrightarrow{t} G_0$ is open.

**Proposition 0.6.** Every morphism of principal $G$-modules which restricts to the identity on the base is an isomorphism.

**Proof.** First we consider the case of trivial bundles. So let $\alpha, \beta : B \to G_0$ be continuous maps. Then a morphism
\[ B \times_{\alpha, G_0, t} G \xrightarrow{\phi} B \times_{\beta, G_0, t} G \]
which induces the identity on the base $B$ is, by $G$-equivariance, necessarily of the form

$$(b, g) \mapsto (b, \eta(b)g)$$

where $\eta : X \to G$ is a continuous map satisfying $t\eta(b) = \beta(b)$ and $s\eta(b) = \alpha(b)$. But then the inverse $\iota \circ \eta$ induces a continuous inverse for $\phi$, so $\phi$ is an isomorphism.

Now let $P \xrightarrow{\phi} Q$ be any morphism of principal $G$-modules which induces the identity on the base $B$. It is easy to verify using $G$-equivariance that $\phi$ is a bijection, so we only need to show that it is an open map.

Let $X := P \times_B Q$ and note that the map $X \to B$ is open since $P \to B$ is so. Pulling back the isomorphisms $P \times_B P \simeq P \times_{G_0} G$ and $Q \times_B Q \simeq Q \times_{G_0} G$ give trivializations:

$$X \times_B P \simeq X \times_{G_0} G, \quad X \times_B Q \simeq X \times_{G_0} G,$$

Thus we have a pullback diagram:

$$
\begin{array}{ccc}
X \times_B P & \xrightarrow{\pi_P} & P \\
id_X \times \phi & \downarrow & \phi \\
X \times_B Q & \xrightarrow{\pi_Q} & Q
\end{array}
$$

in which the horizontal arrows are open surjections and the left vertical arrow is an isomorphism because it is a morphism of trivializable $G$-modules. Thus for an open set $U \subset P$, $\phi(U) = \pi_Q((id_X \times \phi)(\pi_P^{-1}(U)))$ is open as desired.

\begin{prop}
A principal $G$-module is trivializable if and only if admits a section.
\end{prop}

\begin{proof}
If a bundle is isomorphic to $X \times_f G$ for some map $X \xrightarrow{f} G_0$, then $X \xrightarrow{id \times f} X \times_{G_0} G$ induces a section. On the other hand, let $P \xrightarrow{\sigma} B$ be a principal $G$-module over $B = P/G$. If $\sigma$ is a section $B \xrightarrow{\sigma} P$, then composing $\sigma$ with the moment map $P \to G_0$ gives a map $X \to G_0$, and we have a morphism of principal $G$-bundles:

$$X \times_{G_0} G \to P \quad (x, g) \mapsto \sigma(x)g$$

which is necessarily an isomorphism.
\end{proof}

Note that when $G = (G \rightrightarrows *)$ is a group, a principal $G$-module $P$ is exactly a principal $G$-bundle whose base is the quotient space $P/G$. There is however some ambiguity in the literature as to the definition of a principal $G$-bundle, and it is common in modern texts to require that principal $G$-bundles admit local trivializations. When comparing groupoid Morita equivalence with $C^*$-algebra Morita equivalence, however, local triviality is too stringent, so it will not be included.

In case you are looking for an example of a principal bundle without local trivializations, here is one:

\begin{example}
The squaring map

$$\prod_\infty S^1 \to \prod_\infty S^1 \quad (z_i) \mapsto (z_i^2)$$

is a principal $\prod_\infty \mathbb{Z}/2\mathbb{Z}$-bundle in the product topology and has no local sections. Indeed, any neighborhood of a point in the base contains entire circles, and so any section would induce a continuous section of the squaring map $S^1 \to S^1$.

Now we come to the important notion of Morita equivalence for groupoids. A Morita morphism $G \xrightarrow{P} H$ is a $G$-$H$-bimodule $P$ such that
(1) The actions of $\mathcal{G}$ and $\mathcal{H}$ commute.
(2) The right $\mathcal{H}$ action is principal and induces an isomorphism $\mathcal{G}_0 \simeq P/\mathcal{H}$.

The composition $\mathcal{G} \xrightarrow{P \ast Q} \mathcal{K}$ of two Morita morphisms $\mathcal{G} \xrightarrow{\beta} \mathcal{H}$ and $\mathcal{H} \xrightarrow{\gamma} \mathcal{K}$ is given by the $\mathcal{G}$-$\mathcal{K}$-bimodule:

$$P \ast Q := P \times_{\mathcal{H}_0} Q/(ph, q) \sim (p, hq).$$

Exercise: Show that $\mathcal{G} \xrightarrow{P \ast Q} \mathcal{K}$ is indeed a Morita morphism.

A Morita morphism is called a Morita equivalence if the left action is also principal and induces an isomorphism $\mathcal{G}\backslash P \simeq \mathcal{H}_0$. A Morita morphism $\mathcal{G} \xrightarrow{\beta} \mathcal{H}$ gives rise to a functor

$$\text{Prin}_{\mathcal{G}} \ast \mathcal{F} \rightarrow \text{Prin}_{\mathcal{H}},$$

which is an equivalence of categories if and only if $\mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is a Morita equivalence. Indeed, this follows immediately from:

**Proposition 0.9.** For a Morita equivalence $\mathcal{G} \xrightarrow{\beta} \mathcal{H}$, we have canonical isomorphisms of $\mathcal{G}$-$\mathcal{G}$ and $\mathcal{H}$-$\mathcal{H}$-bimodules:

$$P \ast P^{\text{op}} \simeq \mathcal{G} \text{ and } P^{\text{op}} \ast P \simeq \mathcal{H}.$$ 

**Proof.** The isomorphism $P \times_{\mathcal{G}_0} P \simeq P \times_{\mathcal{H}_0} \mathcal{H}$ induces

$$P^{\text{op}} \ast P = \mathcal{G}\backslash (P \times_{\mathcal{G}_0} P) \simeq (\mathcal{G}\backslash P) \times_{\mathcal{H}_0} \mathcal{H} \simeq \mathcal{H}.$$

Similarly, $P \ast P^{\text{op}} \simeq \mathcal{G}$. $\square$

Here are some examples:

1. A surjective open map $Y \rightarrow X$ induces a Morita equivalence $(Y \times_X Y \rightrightarrows Y) \rightarrow (X \rightrightarrows X)$. In particular, a space $X$ is Morita equivalent to the Čech groupoid of each of its covers.
2. If $H$ and $K$ are closed subgroups of a locally compact Hausdorff group $G$, then the quotients $G \rightarrow G/K$ and $G \rightarrow H\backslash G$ are open and $H \ltimes G/K \rightarrow H\backslash G \ltimes K$ is a Morita equivalence. In particular $\mathbb{Z} \rightrightarrows pt$ is Morita equivalent to $\mathbb{R} \ltimes \mathbb{R}/\mathbb{Z}$. 

Lecture 3: Groupoid Cohomology and Twisted Groupoid Algebras

Last time we got acquainted with Morita equivalence. Today we will finish that discussion, showing the relationship between Morita equivalence and functors. Then we move on to groupoid cohomology, in particular the degree 1 and 2 cohomology, which correspond to groupoid presentations of principal bundles and gerbes. Finally, we will discuss the relationship between twisted groupoid algebras and gerbes.

0.6. Morita equivalences from functors. A functor $G \xrightarrow{\phi} H$ gives rise to Morita morphism whose underlying bi-module is:

$$P_\phi := G_0 \times_{H_0} H_1$$

with the obvious right action of $H$ and the left $G$-action $g \cdot (s, g) := (tg, \phi(g)h)$.

When is $P_\phi$ a Morita equivalence? Well, it is a Morita equivalence precisely when the $G$-action on $P_\phi$ is principal. We can characterize this in terms of properties of $\phi$. The functor $\phi$ is called topologically essentially surjective if $G_0 \times_{H_0} H_1 \rightarrow H_0$, $(x, h) \mapsto sh$ is surjective (note that this map is automatically open as well), and $\phi$ is called topologically fully faithful if $G_1 \rightarrow G_0 \times_{H_0} H_1 \times_{H_0} G_0$, $g \mapsto (tg, \phi(g), sg)$ is an isomorphism.

**Proposition 0.10.** $P_\phi$ is a Morita equivalence if and only if $\phi$ is topologically fully faithful and essentially surjective. (In this case $\phi$ is called an essential Morita equivalence.)

We will not prove this now because it is straightforward and we will prove something more general later.

Another natural question is: to what extent are Morita morphisms induced by functors?

**Proposition 0.11.** Let $G \xrightarrow{G} H$ be a Morita morphism. Then there is a canonically associated groupoid $\mathcal{G}(P)$, and functors $G \xleftarrow{\phi} \mathcal{G}(P) \xrightarrow{\eta} H$

such that $\phi$ is an essential Morita equivalence and $P$ is isomorphic as a bi-module to $(P_\phi)^{op} \ast P_\eta$.

Furthermore, $\phi$ is an open surjection on objects.

**Proof.** (sketch) Write $\alpha : P \rightarrow G_0$ for the $G$-moment map and set $\mathcal{G}(P) = \alpha^* \mathcal{G} := (P \times_{G_0} G_1 \times_{G_0} P \Rightarrow P)$

This is just the pullback of $\mathcal{G}$ via the moment map $\alpha : P \rightarrow G_0$, which by the principality of the $H$-action is an open surjection. Using this property of $\alpha$, it follows immediately that the quotient map $\alpha^* \mathcal{G} \xrightarrow{\tau} \mathcal{G}$ is an essential Morita equivalence.

Now the action of $\mathcal{G}$ on $P$ induces an action of $\alpha^* \mathcal{G}$ on $\alpha^* P = P \times_{G_0} P \simeq P \times_{H_0} H$, which can only be of the form:

$$\alpha^* \mathcal{G} \times_P (P \times_{H_0} H) \rightarrow (P \times_{H_0} H), \quad (p, g, p') \cdot (p', h) := (p, \tau(p, g, p')h)$$

for a continuous functor $\alpha^* \mathcal{G} \xrightarrow{\tau} \mathcal{H}$.

Now recall $(\alpha^* \mathcal{G})_0 = P$, so $P_\phi = P \times_{G_0} \mathcal{G}$ and $P_\tau = P \times_{H_0} H \simeq P \times_{G_0} P$. Thus

$$(P_\phi)^{op} \times_P P_\tau \simeq P \times_{G_0} \mathcal{G} \times_P P \times_{G_0} P \simeq (\alpha^* \mathcal{G})_1 \times_P P$$

Then dividing out the action of $\alpha^* \mathcal{G}$ gives $(P_\phi)^{op} \ast P_\tau \simeq P$. 

□
The above proposition can be developed into an interpretation of the 2-category of topological groupoids, Morita morphisms, and bi-module isomorphisms as a left localization of the 2-category of topological groupoids, functors, and natural transformations at the class of essential Morita equivalences. It is also possible to develop a right localization, using instead of $\alpha^*G$, a groupoid $L$ called the linking groupoid of a Morita morphism $G \to \mathcal{H}$, which has arrows

$$G \to L \leftarrow \mathcal{H},$$

in which the left facing arrow is an essential Morita equivalence that is furthermore, injective on objects.

**Exercises:**

(1) Let $G$ be a groupoid and $X \xrightarrow{f} G_0$ an open surjection, and let

$$f^*(G) := (X \times_{G_0} G \times_{G_0} X \to X)$$

denote the pullback groupoid. Show that the obvious functor $f^*G \to G$ is an essential Morita equivalence.

(2) Define a composition for morphisms of the form $G \xleftarrow{\phi} G' \xrightarrow{\tau} H$ with $\phi$ an essential Morita equivalence. (Do this directly, without replacing the functors with bimodules.)

(3) Describe the linking groupoid of a Morita equivalence.

0.7. Groupoid cohomology.

Given a bundle of abelian groups $B \xrightarrow{b} G_0$ and continuous groupoid action $G \times_{G_0} B \to B$ by group automorphisms, we can form the groupoid cohomology with $B$ coefficients. It is given by a complex $(C^*(G; B), \delta)$, which looks just like the complex computing continuous group cohomology:

$$C^k(G; B) := \{ \text{continuous maps } f : G_k \to B \mid b(f(g_1, \ldots, g_n)) = tg_1 \}$$

and for $f \in C^k(G; B)$,

$$\delta f(g_1, \ldots, g_{k+1}) := g_1 \cdot f(g_2, \ldots, g_{k+1}) + \sum_{i=1 \ldots k} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} f(g_1, \ldots, g_{k+1}).$$

As is common, we tacitly restrict to the quasi-isomorphic subcomplex

$$\{ f \in C^k \mid f(g_1, \ldots, g_k) = 0 \text{ if some } g_i \text{ is a unit } \}$$

except for 0-cochains which have no such restriction. When the $G$-module is $B = G_0 \times A$, where $A$ is an abelian group, we write $A$ for the cohomology coefficients.

**Facts to check on your own:**

(1) This generalizes Čech cohomology, indeed, when $G$ is a Čech groupoid acting trivially on $A$, the complex $(C^k(G; A), \delta)$ is identical to the Čech complex of the cover with coefficients in the sheaf of sections of $G_0 \times A \to G_0$.

(2) This reduces to continuous group cohomology when $G$ is a group.

(3) If $G$ acts trivially on $A$, then $A$-valued 1-cocycles can be interpreted as functors $G \to (A \Rightarrow \text{pt})$, and 1-coboundaries can be interpreted as natural transformations between such functors. In degree 1 $A$ need not be abelian.
(4) Alternatively, 1-cocycles can be viewed as transition functions for principal $A$-modules. Indeed, given a 1-cocycle $\mathcal{G} \to A$, one forms the groupoid:

$$\mathcal{G}_\phi := (A \times \mathcal{G}_1 \to A \times \mathcal{G}_0)$$

with $s(a,g) = (a\phi(g), sg)$ and $t(a,g) = (a, tg)$. $A$ acts on the left by groupoid automorphisms, and this can be viewed as the “good” presentation of a principal $A$-bundle on $\pi_0(\mathcal{G})$. You should verify that when $\mathcal{G}$ is a Čech groupoid, then $\mathcal{G}_\phi$ is $A$-equivariantly Morita equivalent to the actual principal $A$-bundle

$$\pi_0(\mathcal{G}_\phi) \equiv (A \times \mathcal{G}_0)/(a, tg) \sim (a\phi(g), sg)$$

**Example 0.12. Nonabelian gerbes.** Let $\mathcal{G}$ be a groupoid and $B \to \mathcal{G}_0$ a bundle of not necessarily abelian groups over $\mathcal{G}_0$. Suppose we have two continuous functions

$$\mathcal{G}_2 \times_{\mathcal{G}_0} B \to B ; \ (g_1, g_2, p) \mapsto \sigma(g_1, g_2)p \quad \text{and} \quad \mathcal{G} \times_{\mathcal{G}_0} B \to B ; \ (g, p) \mapsto \tau(g)(p),$$

such that $\sigma(g_1, g_2)$ is an element of the fiber of $B$ over $rg_1$, $\tau(g)$ is an isomorphism from the fiber over $sg$ to the fiber over $rg$, and the following equations are satisfied:

\begin{align*}
\tau(g_1) \circ \tau(g_2) &= \text{ad}(\sigma(g_1, g_2)) \circ \tau(g_1 g_2) \\
(\tau(g_1) \circ \sigma(g_2, g_3))\sigma(g_1, g_2 g_3) &= \sigma(g_1, g_2)\sigma(g_1 g_2, g_3)
\end{align*}

where $\text{ad}(p)(q) := pqp^{-1}$ for elements $p, q \in B$ that both lie in the same fiber over $\mathcal{G}_0$. We will write $g(p) := \tau(g)(p)$. The pair $(\sigma, \tau)$ can be thought of as a 2-cocycle in “nonabelian cohomology” with values in $B$, and when $B$ is a bundle of abelian groups, $\tau$ is simply an action and $\sigma$ a 2-cocycle.

From the data $(\sigma, \tau)$ we form the groupoid

$$\mathcal{G}^\sigma := (B \times_{b, \mathcal{G}_0, t} \mathcal{G}_1 \to \mathcal{G}_0)$$

with source, target and multiplication maps

\begin{align*}
(1) \ s(p, g) := sg & \quad t(p, g) := tg \\
(2) \ (p_1, g_1) \circ (p_2, g_2) &= (p_1 g_1 p_2)\sigma(g_1, g_2), g_1 g_2
\end{align*}

An important special case of this is $B = U(1) \times \mathcal{G}_0$, with $U(1)$ acted upon trivially by $\mathcal{G}$. Then $\sigma$ reduces to a $U(1)$-valued 2-cocycle on $\mathcal{G}$, and we will see that $\mathcal{G}^\sigma$ is a groupoid presentation of a $U(1)$-gerbe. On the other hand, the pair $(\mathcal{G}, \sigma)$ can be used to make a twisted groupoid algebra $C^*(\mathcal{G}; \sigma)$.

**0.8. (twisted) Groupoid algebras.**

Concepts: Haar system, $L^2(\mathcal{G})$, $C^*(\mathcal{G})$ and $C^*(\mathcal{G}, \sigma)$, Renault’s theorem on representations, groupoid Morita equivalence implies $C^*$-Morita equivalence.

Read Renault’s exposition [Ren] of twisted $C^*$-algebras for the full details. Here I will just mention that the **twisted groupoid algebra** $C^*(\mathcal{G}, \sigma)$ of a locally compact Hausdorff groupoid with left Haar system and 2-cocycle $\sigma : \mathcal{G}_2 \to U(1)$ is a $C^*$-completion of the vector space $C_c(\mathcal{G}_1)$ of compactly supported functions on $\mathcal{G}_1$, with multiplication:

$$a \ast b(g) := \int_{g_1 g_2 = g} a(g_1)b(g_2)\sigma(g_1, g_2), \quad a, b \in C_c(\mathcal{G}).$$

When $\sigma \equiv 1$ this is just called the groupoid algebra, and in this case Renault has proved that (under mild conditions) every representation of $C^*(\mathcal{G})$ is induced from a representation of the groupoid. Furthermore, we have:
Theorem 0.13. [MRW] Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids with Haar systems which are locally compact Hausdorff, second countable, and have paracompact unit spaces. Then a Morita equivalence $\mathcal{G} \xrightarrow{P} \mathcal{H}$ induces a Morita equivalence $C^*(\mathcal{G}) \sim C^*(\mathcal{H})$, whose Morita equivalence bi-module is $C_c(P)$ with the induced actions of $C_c^*(\mathcal{G})$ and $C_c^*(\mathcal{H})$.

There is also a twisted groupoid algebra version of this theorem, and in particular, the Morita class of $C^*(\mathcal{G}, \sigma)$ is independent of the choice of Haar system.

Examples: Noncommutative tori and continuous trace algebras are examples of twisted groupoid algebras. The $C^*$-algebras of abelian gerbes are sums of twisted groupoid algebras.
Lecture 4: Stacks

I will take the approach in which a stack is viewed as the 2-categorical analogue of a sheaf. From this perspective stack theory can be viewed as a natural arena for the 2-categorical analogue of geometry, just as sheaf theory been understood as a natural arena for geometry. Aside from having a pleasing geometric interpretation, the 2-categorical viewpoint has the benefits of being the most general and of leading the way towards higher stacks (as in [To]).

I will state the definition of a stack immediately, and then work towards understanding the definition by recalling some concepts from category theory and giving some examples.

Recall that a presheaf on a site $S$ with values in a category $C$ is nothing more than a functor $F : S^{op} \to C$.

A sheaf is a presheaf satisfying the sheaf axiom, which says that for every cover $\{U_i \to X\}_{i \in I}$ in $S$, $F(X)$ is the equalizer of the diagram:

\[
\prod_{i \in I} F(U_i) \Rightarrow \prod_{(i,j) \in I \times I} F(U_{ij}),
\]

where $U_{ij} = U_i \times_X U_j$ denotes the fibered product.

Now let $C$ be a 2-category, and view a site $S$ as a 2-category with only identity 2-arrows. Then a stack on $S$ is a 2-functor $F : S^{op} \to C$ satisfying the stack axioms. The stack axioms are encoded in the following statement: for every cover $\{U_i \to X\}_{i \in I}$ in $S$, $F(X)$ is an equalizer, or more correctly a 2-categorical limit, of the diagram:

\[
\prod F(U_i) \Rightarrow \prod F(U_{ij}) \Rightarrow \prod F(U_{ijk}).
\]

Now we will go through the definitions of site, 2-category, 2-functor, and 2-categorical limit, which are all used in the definition of a stack. Afterwards there will be geometric examples.

A site is a category with just enough extra structure for the sheaf axiom to be phrased. More precisely, a site $S = (S, J)$ is a category $S$ equipped with a collection $J$ of covers of objects of $S$.

A cover of an object $X$ is a family of arrows $\{U_i \to X\}_{i \in I}$ of $S$. The collection $J$ must satisfy:

1. Every isomorphism $\{Y \to X\}$ is a cover.
2. A cover of a cover is a cover, that is, if $\{U_i \to X\}$ is a cover and $\{V_j \to U_i\}$ are covers for each $i$ then the composite $\{V_{ij} \to U_i \to X\}$ is also a cover.
3. Pullbacks of covers are covers, that is, if $Y \to X$ is an arrow in $S$ and $\{U_i \to X\}$ is a cover then $\{U_i \times_X Y \to Y\}$ is a cover as well.

When the collection of covers is understood we refer to the site and its underlying category interchangeably. For convenience we write $U_{ij} \equiv U_i \times_X U_j$ and refer to this as the intersection of $U_i$ and $U_j$. This notation still treats the indices as an ordered pair, so that $U_{ij} \neq U_{ji}$.

**Proposition 0.14.** The following are sites:

1. Top$_{el} = \text{topological spaces, with covers being open covers.}$
2. Top = \text{topological spaces, with covers being open surjections.}$

---

4This is the approach taken, for instance, in the final chapters of [KS].
5We assume that for any arrow $Y \to X$ and any member $U_i$ of a cover $\{U_i \to X\}$, the fibered product $Y \times_X U_i$ exists. This requirement can be circumvented by using the language of covering sieves, and does not result in any loss of generality. We also assume all sites are subcanonical, meaning that the Yoneda functors $\text{Hom}_S( , X) : S \to \text{Sets}$ are not just presheaves, but sheaves. All of the sites mentioned below are subcanonical.
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(3) Man_{et} = smooth manifolds, with covers being open covers.
(4) Man = smooth manifolds, with covers being submersive surjections.
(5) Zar = algebraic varieties with Zariski-open covers.
(6) Aff_k = affine k-schemes with fpqc covers.

And for any site S and X ∈ S,

(1) S/X = the site of objects over X. Covers of Y → X are the pullbacks of covers of X.
(2) Cov X = the smallest sub-site of S that contains X (this is the site one implicitly refers to when speaking of a “sheaf on X”). Thus the objects and arrows of Cov X are precisely those which partake in some S-cover of X.

Exercise: Prove that Top and Man are sites.

Equalizers and limits. Let C be a category and f, g ∈ Hom_C(A, B). Recall that an equalizer of f and g is an object E ∈ C and an arrow E \overset{\alpha}{\to} A such that:

1. f \circ \alpha = g \circ \alpha
2. Any other arrow E' \overset{\alpha'}{\to} A satisfying f \circ \alpha' = g \circ \alpha' determines an unique arrow E' \overset{h}{\to} E satisfying (E' \overset{\alpha'}{\to} A) = (E' \overset{h}{\to} E \overset{\alpha}{\to} A).

An equalizer, if it exists, is unique up to unique isomorphism, which makes us comfortable saying “the” equalizer instead of “an” equalizer. Analogous usage of the word “the” for limits and 2-limits will be in force without mention.

In Sets, the equalizer of two functions f, g : A → B is the subset \{a ∈ A \mid f(a) = g(a) \} ⊂ A.

Thus for a presheaf of sets F, the sheaf axiom [0.3] states that for any cover \{U_i \to X\},

F(X) \simeq \{ f = (f_i) \in \prod F(U_i) \mid \rho_1 f = \rho_2 f \}

where \rho_1 is induced by omitting the second factor of an intersection U_{ij} → U_i, and \rho_2 corresponding to omitting the first factor U_{ij} → U_j. (For example (\rho_1 f)_{ij} = f_i|U_{ij}).

The notions of site and equalizer are all one needs to define sheaves. For the rest of this section we describe limits and express the sheaf axiom in terms of limits, in a couple of ways. The main purpose for doing this is just to develop notation that will be used for the stack axiom.

A limit of a functor F : I → C, is an object c ∈ C (often denoted lim_I F) together with an arrow c \overset{\alpha}{\to} F(x) for each x ∈ I, such that

1. For each γ ∈ Hom_I(x, y) the following diagram is commutative:

   \[
   \begin{array}{ccc}
   F(x) & \xrightarrow{\alpha_x} & F(x) \\
   \downarrow{\alpha_y} & & \downarrow{\alpha_y} \\
   F(y) & \xrightarrow{F(\gamma)} & F(y)
   \end{array}
   \]

2. Any other such data \{c', c' \overset{\alpha'}{\to} F(x)\} determines an unique arrow c' \to c making all possible composite diagrams commute.

The image of F (that is, all objects of the form F(x) and all arrows of the form F(x) \overset{F(\gamma)}{\to} F(y)) forms what is called a diagram in C, and in this case a limit of a diagram is a synonym for the limit of the functor.
An equalizer is a special case of a limit, indeed let $\Delta^2 := ([1] \xrightarrow{a, b} [2])$ be the category with two objects and two non-identity arrows, then the equalizer of two arrows $f, g \in \text{Hom}_C(A, B)$ is the same as $\lim_{\Delta^2} F$, where $F : \Delta^2 \to C$ is defined by:

$$F([1]) = A, \quad F([2]) = B, \quad F(a) = f, \quad F(b) = g.$$ 

In particular, the sheaf axiom can be interpreted as a limit. For a presheaf $F : S^{\text{op}} \to C$ and a cover $\mathcal{U} = \{U_i \to X\}$, define $F_\mathcal{U} : \Delta^2 \to C$ by

$$F_\mathcal{U}([1]) = \prod F(U_i), \quad F_\mathcal{U}([2]) = \prod F(U_{ij}), \quad F_\mathcal{U}(a) = \rho_1, \quad F_\mathcal{U}(b) = \rho_2.$$ 

Then $F$ is a sheaf if and only if for all such covers

$$F(X) \simeq \lim_{\Delta^2} F_\mathcal{U}.$$ 

There is a "bigger" limit one can form from the presheaf $F$. Let $\Delta$ be the category whose objects are $\{[1], [2], [3], \ldots\}$ and whose arrows are

$$\text{Hom}_\Delta([\ell], [k]) := \text{Order preserving injections } \{1, \ldots, \ell\} \to \{1, \ldots, k\}.$$ 

so that there are $k$-choose-$\ell$ arrows from $[\ell]$ to $[k]$. Composition is given by composition of functions (note that distinct pairs of arrows can compose to the same arrow).

Let $\Delta_n$ denote the full subcategory whose objects are $\{[1], \ldots, [n]\}$. Then $F_\mathcal{U}$ extends to $F_\mathcal{U} : \Delta_n \to C$ and to $F_\mathcal{U} : \Delta \to C$, by the formulas

$$[k] \mapsto F_\mathcal{U}([k]) := \prod F(U_{i_1 \ldots i_k})$$

$$\text{Hom}_\Delta([k-1], [k]) \ni (j^{\text{th}} \text{ arrow}) \mapsto \rho_{1\ldots j-1j+1\ldots k} \in \text{Hom}_C(F_\mathcal{U}([k-1]), F_\mathcal{U}([k]))$$

where $\rho_{1\ldots j-1j+1\ldots k}$ comes from the arrows $U_{i_1 \ldots i_k} \to U_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_k}$ omitting the $j^{\text{th}}$ factor of a $k$-fold intersection.

By virtue of $F$ being a functor, the arrows

$$\lim_{\Delta} F_\mathcal{U} \to \cdots \to \lim_{\Delta^3} F_\mathcal{U} \to \lim_{\Delta^2} F_\mathcal{U}$$

are all isomorphisms, so the sheaf axiom is also equivalent to $F(X) \simeq \lim_{\Delta^2} F_\mathcal{U}$. This phrasing of the axiom is in some sense the most natural, and translates directly into a stack axiom when $C$ is replaced by a 2-category.

2-Categories. A 2-category $\mathcal{C}$ is the following data:

1. A collection of objects $A, B, \ldots$ of $\mathcal{C}$.
2. A category $\mathcal{C}(A, B)$ for every pair of objects.
3. An associative composition functor $\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ for every triplet of objects.

Objects of $\mathcal{C}(A, B)$ are called 1-arrows while morphisms in $\mathcal{C}(A, B)$ are called 2-arrows. These must satisfy the following properties:

- For each $A$, the category $\mathcal{C}(A; A)$ has an identity object which acts as a unit for the composition functor.
- The interchange law holds, that is, for composeable 2-arrows $\alpha, \beta \in \text{Mor} \mathcal{C}(B, C)$ and $\eta, \tau \in \text{Mor} \mathcal{C}(A, B)$, we have $(\alpha \cdot \beta) \circ (\eta \cdot \tau) \equiv (\alpha \circ \eta) \cdot (\beta \circ \tau)$. 

The standard example to keep in mind is the 2-category $\text{Cat}$ in which objects are small categories, 1-arrows are functors, and 2-arrows are natural transformations. Thus $\text{Cat}(A,B)$ is the category of functors from $A$ to $B$ and natural transformations between them. In this case the interchange law is a relation automatically satisfied by natural transformations betweencomposeable functors.

Remark 0.15. What we have defined here is a strict 2-category, in contrast to the more general notion of weak 2-category (also called a bicategory). On the other hand, the 2-functors between strict 2-categories are not “strict”.

2-Functors. A 2-functor $F : C \to D$ between 2-categories is an assignment

$\text{Objects}(C) \ni A \mapsto FA \in \text{Objects}(D)$

together with functors

$C(A, B) \xrightarrow{F_{A,B}} D(FA, FB)$

that preserve identity objects and intertwine the compositions of $C$ and $D$ up to coherent natural isomorphisms (these isomorphisms are part of the data of the 2-functor. Look up the term pseudofunctor for the detailed definition.) Note that every category can be viewed as a 2-category with only identity 2-arrows, so it makes sense to speak of a 2-functor $C \to D$ even when $C$ or $D$ is only a 1-category.

Now we are ready to understand the definition of a stack:

Stacks. Let $S$ be a site and let $C$ be a 2-category. A prestack is a 2-functor $F : S^{\text{op}} \to C$. A stack is a prestack satisfying the stack axiom, which says that for every object $X$ of $S$ and every cover $\{U_i \to X\}$, $F(X)$ is an equalizer, or more correctly a 2-categorical limit, of the diagram:

$$
\prod F(U_i) \xrightarrow{\eta U} \prod F(U_{ij}) \xrightarrow{\eta U_{ijk}} \prod F(U_{ijk}).
$$

A morphism of prestacks (and of stacks) $F, G : S \to C$ is a natural transformation $\eta : F \Rightarrow G$ of the underlying 2-functors. This is the data of a 1-arrow $F(U) \xrightarrow{\eta_U} G(U)$ for each $U \in S$, as well as a compatible collection of 2-arrows $\eta_f$ for each $f \in \text{Hom}_S(V,U)$ making the following diagram 2-commutative:

$$
\begin{array}{ccc}
F(U) & \xrightarrow{\eta_U} & G(U) \\
F(f) \downarrow & \xrightarrow{\eta_f} & \downarrow G(f) \\
F(V) & \xrightarrow{\eta_V} & G(V).
\end{array}
$$

We say $\eta$ is a monomorphism (resp. an epimorphism, resp. an equivalence) if for each $U \in S$ and each $A \in C$, the functor

$C(A, F(U)) \xrightarrow{\eta_U} C(A, G(U))$

is fully faithful (resp. essentially surjective, resp. an equivalence of categories). In the case that $C = \text{Cat}$, these notions are equivalent to saying that $\eta_U$ itself is fully faithful (resp. essentially surjective, resp. an equivalence).

The last thing we need to do is define the 2-categorical limits that are used in the stack axiom:

Categorical concepts in 2-categories. A general prescription for giving the 2-categorial analogue of a categorical concept (that is, a concept phrased in terms of objects and arrows) is to replace arrows with 1-arrows and equality signs with invertible 2-arrows. The fact that this is
a well-founded prescription follows immediately from the enriched category interpretation of a 2-category.

Performing the replacements turns a commutative diagram into a 2-commutative diagram, which is by definition a diagram of objects and 1-arrows which commutes up to an invertible 2-arrow. For example the 2-commutative square:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow{\eta} & \nearrow{\theta} & \downarrow{\eta} \\
X & \xrightarrow{g} & Z
\end{array}
\]

corresponds to the equation \( \eta : f \circ i \Rightarrow g \circ j \), where \( \eta \) is an invertible 2-arrow.

2-Limits. To obtain the definition of a 2-limit, we follow the prescription, that is we take the commutative diagrams which define a limit and replace them with 2-commutative diagrams. Actually we will restrict to the slightly simpler limit of a 2-functor whose domain is a 1-category, which is all that is needed for stacks.

Let \( F : I \to C \) be a 2-functor, and suppose that \( I \) is a 1-category. A 2-limit of \( F \), is an object \( c \in C \) together with a 1-arrow \( c \xrightarrow{\alpha_x} F(x) \) for each \( x \in I \), and a 2-arrow \( \alpha : F(\tau)\alpha_x \Rightarrow \alpha_y \) for each \( \tau \in \text{Hom}_I(x,y) \) such that:

1. The diagrams are 2-commutative:

\[
\begin{array}{ccc}
c & \xrightarrow{\alpha_x} & F(x) \\
\downarrow{\alpha_y} & \nearrow{\theta_x} & \downarrow{\alpha_y} \\
F(y) & & F(y)
\end{array}
\]

2. For each composition of arrows \( x \xrightarrow{\tau} y \xrightarrow{\eta} z \) in \( I \) we have \( \alpha_{y} \alpha_{\tau} = \alpha_{\eta \tau} \).

3. Any other such data

\[
c' \xrightarrow{\alpha'_x} F(x), \quad F(\tau)\alpha'_x \xrightarrow{\alpha'_y} \alpha'_y
\]

determines an essentially unique 1-arrow \( c' \to c \) making all possible composite diagrams 2-commute.

It makes sense to refer to a 2-limit as a 2-limit of a diagram, where the diagram is just the image of the 2-functor.

Thus for a 2-functor \( F : S^{\text{op}} \to C \) and cover \( \mathcal{U} = \{ U_i \to X \} \), the diagram

\[
\prod F(U_i) \xrightarrow{\rho_2} \prod F(U_{ij}) \xrightarrow{\rho_{12}\rho_{23}} \prod F(U_{ijk}).
\]

is the image of \( F_{\mathcal{U}} : \Delta_3 \to C \). Its 2-limit should be an object \( A_{\mathcal{U}} \in C \), together with 1-arrows

\[
A_{\mathcal{U}} \xrightarrow{a} \prod F(U_i), \quad A_{\mathcal{U}} \xrightarrow{b} \prod F(U_{ij}), \quad A_{\mathcal{U}} \xrightarrow{c} \prod F(U_{ijk})
\]

and invertible 2-arrows

\[
\rho_1 a \Rightarrow b, \quad \rho_2 a \Rightarrow b, \quad \rho_{12} b \Rightarrow c, \quad \rho_{13} b \Rightarrow c, \quad \rho_{23} b \Rightarrow c.
\]

Actually, one should also specify 2-arrows to \( c \) such as \( \rho_{12}\rho_1 a \Rightarrow c \) but to do so is redundant because necessarily, \( (\rho_{12}\rho_1 a \Rightarrow c) = (\rho_{12}(\rho_1 a) \Rightarrow \rho_{12} b \Rightarrow c) \).
Because $F$ is a 2-functor, $\lim_{\Delta} F_{\mathfrak{U}} \simeq \lim_{\Delta_{\mathfrak{U}}} F_{\mathfrak{U}}$, which leads to the aesthetically pleasing statement of the stack axioms, that $F(X) \simeq \lim_{\Delta} F_{\mathfrak{U}}$ for all $X$ and $\mathfrak{U}$.

For a general 2-category $C$ we have said all that can be said about $A_{\mathfrak{U}}$. But for $C = \text{Cat}$ the 2-limit exists and can be constructed as follows.

- The objects of $A_{\mathfrak{U}}$ are:
  \[
  \{ (P = (P_i) \in \prod F(U_i) , \rho_2 P \xrightarrow{\phi} \rho_1 P) \mid \rho_{12}(\phi_P) \circ \rho_{23}(\phi_P) = \rho_{13}(\phi_P) \}
  \]

  The defining equation $\rho_{12}(\phi) \circ \rho_{23}(\phi) = \rho_{13}(\phi)$ only makes sense because of the equalities
  \[
  \rho_{12}\rho_1 = \rho_{13}\rho_1, \quad \rho_{23}\rho_1 = \rho_{12}\rho_2, \quad \rho_{13}\rho_2 = \rho_{23}\rho_2
  \]
  (which are a consequence of the relations in $\Delta$).

- The morphisms in $A_{\mathfrak{U}}$ are:
  \[
  \text{Hom}_{A_{\mathfrak{U}}}(P, \phi_P), (Q, \phi_Q) = \{ P \xrightarrow{\alpha} Q \mid \phi_Q\rho_2(\alpha) = \rho_1(\alpha)\phi_P \}.
  \]

- The arrow $A_{\mathfrak{U}} \xrightarrow{\alpha} \prod F(U_i)$ is just the projection $(P, \rho_2 P \xrightarrow{\phi} \rho_1 P) \mapsto P$.

- The arrow $A_{\mathfrak{U}} \xrightarrow{b} \prod F(U_{ij})$ is $b := \rho_1 a$, and similarly $c := \rho_{13}\rho_1 a = \rho_{13} b$.

- The arrows from Equation (0.7) are somewhat obvious: $\rho_1 a \Rightarrow b$ is the identity, $\rho_2 a \Rightarrow b$ is the assignment $(P, \rho_2 P \xrightarrow{\phi} \rho_1 P) \mapsto \phi$, which the reader may verify is a natural transformation, and $\rho_{12} b = \rho_{12} \rho_1 a \Rightarrow \rho_{12} \rho_1 a = c$ is the identity, $\rho_{13} b = \rho_{13} \rho_1 a \Rightarrow c$ is also the identity, and $\rho_{23} b = \rho_{23} \rho_1 a \Rightarrow \rho_{23} \rho_1 a = c \Rightarrow \rho_{12} \rho_1 a = c$ is equivalent to the arrow $\rho_2 a \Rightarrow \rho_1 a$.

- The fact that the composite $\rho_{13} \rho_2 a \Rightarrow \rho_{23} b \Rightarrow c$ equals $\rho_{13} \rho_2 a \Rightarrow c$ is equivalent to the conditions $\rho_{12}(\phi) \circ \rho_{23}(\phi) = \rho_{13}(\phi)$.

- Finally, any other such data $(A', a', b', \rho_1 a' \xrightarrow{\eta} b', \ldots)$ determines an arrow $A' \rightarrow A_{\mathfrak{U}}$ by the formulas:
  \[
  A' \ni x \mapsto (a'(x), \eta(x)) \in A_{\mathfrak{U}} \quad \text{Mor} A' \ni \gamma \mapsto a'\gamma \in \text{Mor} A_{\mathfrak{U}}.
  \]

We leave it to the reader to check that the remaining details about the fact that this is indeed the 2-limit.

This category, $A_{\mathfrak{U}}$, is called descent data in the algebraic geometry literature. The functor $F(X) \rightarrow A_{\mathfrak{U}}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective, and taking these two conditions separately reproduces the more common phrasing of the stack axioms (for groupoid or category valued stacks) which are:

1. $(F(X) \rightarrow A_{\mathfrak{U}}$ is essentially surjective) Given a collection of objects $P_i \in F(U_i)$ on a cover $\{U_i \rightarrow X\}$ and isomorphisms $P_j|_{U_{ij}} \xrightarrow{\phi_{ij}} P_i|_{U_{ij}}$ satisfying $\phi_{ij}\phi_{jk} = \phi_{ik}$ on triple intersections $U_{ijk}$, there exists a “glued together” object $P \in F(X)$ and isomorphisms $P|_{U_i} \xrightarrow{s_i} P_i$ satisfying $s_i = \phi_{ij}s_j$.

2. $(F(X) \rightarrow A_{\mathfrak{U}}$ is fully faithful) Given a collection of arrows $\gamma_i \in \text{Mor} F(U_i)$ such that $\gamma_i|_{U_{ij}} = \gamma_j|_{U_{ij}}$, there exists a unique “glued together” arrow $\gamma \in \text{Mor} F(X)$ such that $\gamma|_{U_i} = \gamma_i$ for all $i$.

**Exercise:** Prove that for a group object $G$ in a site $S$, the 2-functor

$$\text{Prin}_G : S^{op} \rightarrow \text{Grpd}$$
which assigns to each $T \in S$ the groupoid of principal $G$-modules is a stack.

0.9. **Sheafification and stackification.** One way to come about examples of sheaves is to start with a presheaf $F : S^{op} \to \mathcal{C}$, and define a new functor (which is automatically a sheaf) by the formula:

$$F^{sh} : S^{op} \to \mathcal{C}, \quad F^{sh}(X) := \text{colim}_{\Delta} \lim_{\Delta} F_{\mathcal{U}}.$$

Recall that for a cover $\mathcal{U} = \{U_i \to X\}$, $\lim_{\Delta} F_{\mathcal{U}}$ is simply the equalizer of $\prod F(U_i) \rightrightarrows \prod F(U_{ij})$. To form the colimit over $\text{Cov} X$ we are viewing the covers of $X$ as a set directed by refinement. $F^{sh}$ is of course called the **sheafification** of $F$; sheafification has nice properties which can be encoded by saying that it is a functor and that it is left adjoint to the inclusion $\text{Sheaves}(S) \hookrightarrow \text{Presheaves}(S)$.

One obtains examples of stacks by an analogous operation. Given a 2-category $\mathcal{C}$ and a 2-functor $F : S^{op} \to \mathcal{C}$, the **stackification** $F^{st}$ of $F$ is the stack defined by Equation (0.8), where now the limit and colimit are interpreted in the 2-categorical sense.

Stackification can be counterintuitive, because while a sheaf with values in a 1-category is already a special case of a stack (since the sheaf and stack axioms coincide on a 1-category), a sheaf $F : S^{op} \to \mathcal{C}$ with values in a 2-category is dramatically changed by stackification. To give an explicit example, consider the sheaf

$$G = \text{Hom}(\ , G) : S \to \text{Groups} \subset \text{Groupoids}$$

for a group object $G \in S$. It is very different from its stackification, which we denote

$$G^{st} : S \to \text{Groupoids}$$

Let us see precisely what $G^{st}(X)$ is.

For a fixed cover $\mathcal{U} = \{U_i \to X\}$, $\lim_{\Delta} G_{\mathcal{U}} = A_{\mathcal{U}}$ (defined in the previous section) is the groupoid whose objects are pairs

$$(P_i) \in \prod \text{Hom}_{\mathcal{S}}(U_i, G), \quad (P_i|_{U_{ij}} \xrightarrow{\phi_{ij}} P_j|_{U_{ij}})$$

satisfying $\phi_{ij}\phi_{jk} = \phi_{ik}$ on $U_{ijk}$. Note that $\phi_{ij}$ can be viewed as left multiplication by maps $\phi_{ij} \in \text{Hom}_{\mathcal{S}}(U_{ij}, G)$. An arrow of $A_{\mathcal{U}}$ from $(\phi_{ij})$ to $(\psi_{ij})$ is a collection $(\eta_i) \in \prod \text{Hom}_{\mathcal{S}}(U_i, G)$ satisfying $\eta_i\phi_{ij} = \psi_{ij}\eta_j$.

Then an object of the colimit $G^{st}(X) = \text{colim}_{\mathcal{U}} A_{\mathcal{U}}$ is represented by an object of $A_{\mathcal{U}}$ for some cover $\mathcal{U}$, and two representatives are equivalent if there is a common refinement of covers upon which they are isomorphic (i.e. connected by an arrow). An arrow of $G^{st}(X)$ is represented by an arrow of $A_{\mathcal{U}}$ for some cover $\mathcal{U}$, and two arrows are equivalent if there is a common refinement of covers upon which the two arrows are equal.

**Exercise** Repeat this procedure for a groupoid object $\mathcal{G}$ in $S$, to make an associated stack $G^{st}$.

**Remark** 0.16. The full subcategory $A^0_{\mathcal{U}}$ of $A_{\mathcal{U}}$ whose objects are pairs $((P_i), \phi_{ij})$ with $P_i = e_{U_i} \in \text{Hom}(U_i, G)$ (i.e. the unit map) is equivalent to $A_{\mathcal{U}}$, and $G^{st}(X)$ is equivalent to the colimit over these subcategories, so whenever convenient we assume $P_i = e_{U_i}$ and refer to an object simply as $(\phi_{ij})$. (For the groupoid minded reader, this gives a simpler characterization than $A_{\mathcal{U}}$, because $A^0_{\mathcal{U}}$ is easily seen as equivalent to the functor category $\text{Fun}(\mathcal{U}, G)$ where $\mathcal{U}$ denotes the Čech groupoid of the cover $\mathcal{U}$.)
Lecture 5: Five interpretations of stacks

Last time we finished the definition of a stack and stackification. There are numerous interpretations of stacks, and today we will go through five of them.

Interpretation 1: A stack is the 2-categorical analogue of a sheaf.

This is the perspective we have taken from the beginning. Continuing Grothendieck’s “functor of points” analogy (which says that a sheaf is a generalized space and a map $T \to F$ from a space to a sheaf is a generalized point of $F$), one can view maps from spaces to a stack $F$ as the generalized points of the stack. This can be compared to representations (or maybe $C^*$-modules) of a $C^*$-algebra being viewed as generalized points of a noncommutative space. The viewpoint that stacks are generalizations of spaces is strengthened by the following definitions:

Definition 0.17. A morphism $\mathcal{X} \to \mathcal{Y}$ of $S$-stacks is called **representable** if for every map from a space $T \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a space. Let $P$ be a property of maps in $S$ which is preserved under pullback. Then a representable morphism

$$\mathcal{X} \to \mathcal{Y}$$

is said to have property $P$ if for every $T \to \mathcal{X}$, the pullback map of spaces

$$\mathcal{X} \times_{\mathcal{Y}} T \to T$$

has property $P$.

For example, we have seen that the pullback of an open map is open, so a representable morphism $\mathcal{X} \to \mathcal{Y}$ is called open if and only if for every topological space $T$ mapping to $\mathcal{X}$, the pullback map $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is an open map of topological spaces. One might say a morphism is open when it is open at all of its generalized points.

Remark 0.18. One thing to take note of is that objects of $S$ are implicitly viewed as sheaves, thus a map $X \to Y$ is called an **epimorphism** when the morphism of sheaves is an epimorphism, that is, when there exists a cover $\tilde{Y} \to Y$ and “local sections” $\tilde{Y} \to X$. Another subtle point is that not even every map between spaces is representable! However, this is only due to a failure of fiber products to exist in the category $S$ (In particular, it depends only on the underlying category, not on the covers.). For example in [Met] Lemma (71) it is shown that a map of manifolds is representable if and only if it is a submersion. (In other words, $X \times_Y Z$ is a manifold for every $Z \to Y$ if and only if $X \to Y$ is a submersion.)

Interpretation 3: A stack is something equivalent to the category of principal $G$-modules (also called $G$-torsors).

To see how this works, first let us specialize to a concrete situation: the stack $G^{st}$ associated to a group object $G$ in $S = Top$. I claim this stack is equivalent to the stack of principal $G$-bundles. Indeed, write $G^{st}(X) = \text{colim}_U A_U$ as in Lecture (4). Then an object $(\phi_{ij})$ of $A_U$ can be viewed as transition functions defining a topological principal $G$-bundle. The bundle is:

$$P(\phi) := \coprod U_i \times G / \sim \phi_{ij}$$
It is essentially the definition of a principal bundle that the assignment \((\phi_{ij}) \mapsto P(\phi)\) induces an equivalence of groupoids, for each \(X\),

\[G^{st}(X) \rightarrow \text{Prin}_G(X).\]

Now we will generalize to an arbitrary site \(S\) and to the stack \(G^{st}\) associated to a groupoid object \(G \in S\). By definition, this interpretation is only valid for stacks equivalent to ones of the form \(\text{Prin}_G\), which are called \textbf{representable}.

\textit{Remark 0.19.} There are several ways in which a stack can fail to be representable. Of course, stacks whose target 2-category is not \(\text{Grpd}\) are not representable, this includes for example \(T \rightarrow \text{Sheaves}(T)\) and \(T \rightarrow \text{Vect}(T)\). Another obvious way a stack fails to be representable is if the “wrong” site is chosen, for example if \(G\) is a topological groupoid which admits no manifold structure, then

\[\text{Man} \hookrightarrow \text{Top} \xrightarrow{\text{Prin}_G} \text{Grpd}\]

is a stack on Manifolds which may not be representable (though it is still possible to have a different groupoid in Man representing the stack). Lastly, most constant sheaves are not representable in any useful “big” site, thus they are examples of non-representable stacks.

\textbf{Interpretation 4:} A stack is a canonical solution to a moduli problem, or in other words, a stack classifies things.

This interpretation is a consequence of the 2-Yoneda Lemma, which we leave as an exercise:

\textbf{Exercise} For any site \(S\), any \(T \in S\), and stack \(\mathcal{X} : S^{op} \rightarrow \text{Grpd}\), there is a canonical isomorphism:

\[\text{Hom}_{\text{Stacks}}(T, \mathcal{X}) \simeq \mathcal{X}(T).\]

(Actually this holds for any prestack \(\mathcal{X} : S^{op} \rightarrow C\))

So if you want to classify families of things (parameterized by objects of a site \(S\)), up to isomorphism, simply define a 2-functor which assigns to each \(T\) the groupoid whose objects are such families and whose arrows are the isomorphisms between them. Then, tautologically, this is the classifying stack for such things (assuming that it is indeed a stack).

For example \(\text{Prin}_G\) is the classifying stack for \(G\)-bundles, so it is sometimes denoted \(BG\). The most famous example from algebraic geometry is \(M_{g,n}\), the stack defined by the property that a map \(T \rightarrow M_{g,n}\) is a family of curves of genus \(g\) with \(n\) marked points.

\textbf{Interpretation 5:} A (representable) stack is a Morita equivalence class of groupoids.

This brings us full circle to Morita equivalence for groupoids. To make everything precise, we will define, for any site \(S\), a 2-category \(M^1\text{Grd}(S)\) whose objects are \(S\)-groupoids, arrows are Morita morphisms, and 2-arrows are isomorphisms of Morita morphisms. (The \(M^{-1}\) stands for inverting Morita equivalences.) Everything is designed so that the following theorem is true:

\textbf{Theorem 0.20.} The 2-functor

\[M^1\text{Grd}(S) \rightarrow S - \text{Stacks}, \quad G \mapsto \text{Prin}_G\]

establishes an equivalence between \(M^1\text{Grd}(S)\) and the full sub-category of representable \(S\)-stacks. A quasi-inverse is given by:

\[
\{\text{Representable } S\text{-stacks} \} \rightarrow M^1\text{Grd}(S), \quad \mathcal{X} \mapsto T \times_{\mathcal{X}} T
\]

where \(T \in S\) and \(T \rightarrow \mathcal{X}\) is a choice of representable epimorphism.

Before sketching the proof, we need to carefully define \(M^1\text{Grd}:\)
Definition 0.21. An S-groupoid is a groupoid object \((\mathcal{G}, \iota, \mu, s, t, e)\) in a site \(S\), such that all of the structure maps are representable.

A \(\mathcal{G}\)-module is an arrow \(P \to \mathcal{G}_0\) in \(S\) together with an action morphism \(P \times_{\mathcal{G}_0} \mathcal{G} \xrightarrow{a} P\). That is, a morphism such that the following diagram is commutative

\[
P \times \mathcal{G}_0 \xrightarrow{id \times \mu} P \times \mathcal{G}_0 \xrightarrow{a} P
\]

and such that units act trivially, meaning

\[
(P \simeq P \times_{\mathcal{G}_0} \mathcal{G}_0 \xrightarrow{id \times e} P \times_{\mathcal{G}_0} \mathcal{G} \xrightarrow{a} P) = (P \xrightarrow{id} P).
\]

A morphism of \(\mathcal{G}\)-modules is a \(\mathcal{G}\)-equivariant map. It is called an isomorphism if the underlying map is an isomorphism in \(S\). The base of \(P\) (if it exists) is the co-equalizer of

\[
(id \times t), a : P \times_{\mathcal{G}_0} \mathcal{G} \rightrightarrows P
\]

\(P\) is called a principal \(\mathcal{G}\)-module, or a \(\mathcal{G}\)-torsor, if there exists a cover \(X' \to B\) of the base of \(P\) (thus the base is required to exist), a map \(X' \to \mathcal{G}_0\), and an isomorphism of \(\mathcal{G}\)-modules:

\[
X' \times_B P \xrightarrow{\alpha} X' \times_{\mathcal{G}_0} \mathcal{G}.
\]

The \(\mathcal{G}\)-torsors of the form \(X' \times_{\mathcal{G}_0} \mathcal{G}\) are called trivial, and we say that \(\alpha\) is a trivialization of \(P\) over \(X'\).

A Morita morphism \(\mathcal{G} \xrightarrow{P} \mathcal{H}\) is a right principal \(\mathcal{G}\)-\(\mathcal{H}\)-bimodule such that \(P/\mathcal{H} \simeq \mathcal{G}_0\). An isomorphism of bimodules is a bi-invariant isomorphism. We denote by \(M^1\text{-Grd}(S)\) the 2-category of \(S\)-groupoids, Morita morphisms, and bimodule isomorphisms.

Proof (Sketch) of Theorem (0.20). First, observe that the essential image of \(\text{Prin}\) consists of stacks for which there exists a representable epimorphism. Indeed, the map \(\mathcal{G}_0 \to \text{Prin}_{\mathcal{G}}\) is a representable epimorphism and conversely, if \(T \to X'\) is a representable epimorphism then

\[
T \times_{X'} T \rightrightarrows T
\]

can be given the structure of a groupoid object in \(S\).

The second step is to show that the 2-functor \(\text{Prin} : M^1\text{-Grd} \to \text{Stacks}\) is a monomorphism, that is, for every pair of groupoids \(\mathcal{G}, \mathcal{H}\),

\[
\text{HOM}_{M^1\text{-Grd}}(\mathcal{G}, \mathcal{H}) \to \text{HOM}_{\text{Stacks}}(\text{Prin}_{\mathcal{G}}, \text{Prin}_{\mathcal{H}})
\]

is an equivalence of categories. To see that it is essentially surjective, observe that if \(\text{Prin}_{\mathcal{G}} \xrightarrow{\phi} \text{Prin}_{\mathcal{H}}\) is a morphism of stacks, then the trivial right \(\mathcal{G}\)-bundle \(\mathcal{G} \in \text{Prin}_{\mathcal{G}}(\mathcal{G}_0)\) is sent to a right principal \(\mathcal{H}\)-bundle \(\phi(\mathcal{G})\) over \(\mathcal{G}_0\). But \(\phi(\mathcal{G})\) also has a left \(\mathcal{G}\)-action, indeed the source map \(\mathcal{G} \xrightarrow{\phi} \mathcal{G}_0\) makes \(\mathcal{G}_2 \simeq s^*(\mathcal{G})\), and pullbacks commute (up to isomorphism) with \(\phi\), so we obtain a \(\phi(\mathcal{G})\)-action as the composition:

\[
\mathcal{G} \times_{\mathcal{G}_0} \phi(\mathcal{G}) = s^*\phi(\mathcal{G}_0) \simeq \phi(s^*\mathcal{G}) = \phi(\mathcal{G}_2) \xrightarrow{\phi(\mu)} \phi(\mathcal{G}).
\]

(The last map is \(\phi\) applied to the multiplication map \(\mathcal{G}_2 \xrightarrow{\mu} \mathcal{G}\), which is a right \(\mathcal{G}\)-module morphism.) This action commutes with the right action of \(\mathcal{H}\) since the multiplication map \(\mathcal{G}_2 \xrightarrow{\mu} \mathcal{G}\) commutes with the right action of \(\mathcal{G}\) on itself.
Now we want to check fully faithfulness, meaning that for every two Morita morphisms $G^P \to Q$ in $\mathcal{H}$, Prin induces a bijection
\[
\{ \text{Bi-invariant isomorphisms } P \to Q \} \to \{ \text{transformations } (-) \ast P \Rightarrow (-) \ast Q \}
\]
\[(P \xrightarrow{\eta} Q) \mapsto ((-) \ast P \xrightarrow{id(-) \ast \eta} (-) \ast Q)\]

To see that it is a bijection, note that it has an inverse, given by sending a transformation $\eta : (-) \ast P \Rightarrow (-) \ast Q$ to the isomorphism of principal $H$-modules
\[
P \simeq (G) \ast P \xrightarrow{\eta G} (G) \ast Q \simeq Q.
\]

\[\square\]

**Proposition 0.22.** Let $S$ be a site for which a map is an isomorphism if and only if its pullback via some covering map is an isomorphism, and let $G$ be an $S$-groupoid. Then every morphism $P \xrightarrow{\phi} Q$ of principal $G$-modules which induces the identity on the base is an isomorphism.

**Proof.** First, if $P$ admits a trivialization over a cover $X_P$ and $Q$ admits a trivialization over $X_Q$, then both $P$ and $Q$ admit trivializations over the common refinement $X' := X_P \times_B X_Q$ of the two covers. Then the trivializations of $P$ and $Q$ over $X'$ induce an isomorphism $X' \times_B P \to X' \times_B Q$ of $G$-torsors.

Thus we have a pullback diagram
\[
\begin{array}{ccc}
X' \times_B P & \to & P \\
\downarrow & & \downarrow \phi \\
X' \times_B Q & \to & Q
\end{array}
\]
in which the horizontal arrows are covers and the left vertical arrow is an isomorphism. Then by hypothesis, $\phi$ must be an isomorphism. \[\square\]

**Exercise:** Which of the sites that we have discussed satisfy the hypotheses of the proposition?

**Proposition 0.23.** Every $G$-torsor admits a trivialization over its base map. That is, if $P \to B$ is a $G$-torsor, then there is an isomorphism of $G$-torsors
\[
P \times_B P \to P \times G_0 G
\]
which is the identity on the first factor.

**Proof.** Let
\[
X \times_B P \to X \times G_0 G, \quad (x, p) \mapsto (x, \alpha(x, p))
\]
be a trivialization over some cover $X \to B$. Then the map
\[
X \times_B P \times_B P \to X \times_B P \times G_0 G, \quad (x, p, q) \mapsto (x, p, \alpha(x, p)^{-1} \alpha(x, q))
\]
has the property that $\alpha(x, p)^{-1} \alpha(x, q) = \alpha(x', p)^{-1} \alpha(x', q)$ whenever $x$ and $x'$ lie in the same fiber over $B$. Thus it descends to a map
\[
P \times_B P \xrightarrow{\phi} P \times G_0 G.
\]
The action map induces an inverse $(p, g) \mapsto (p, pg)$, so $\phi$ is an isomorphism. \[\square\]

**Proposition 0.24.** In the site $\text{Top}$, a $G$-module is a torsor if and only if it admits a trivialization over its base map.
Proof. We only need to prove the ‘if’ direction. If there is an isomorphism
\[ P \times_B P \rightarrow P \times_{G_0} G, \]
then the base map \( P \rightarrow B \) is open because it is isomorphic to the target map of \( G \), thus \( P \rightarrow B \) is a cover in \( \text{Top} \) over which \( P \) admits a trivialization. \( \square \)

Remark 0.25. I would like to answer a question brought up by RL, which is roughly, “are all stacks colimits of representable stacks?”

First we should note that all sheaves (of sets) are colimits of representable presheaves. In fact, let \( C \) be any category and write \( \hat{C} \) for the category of presheaves of sets on \( C \). Let \( C \xrightarrow{y} \hat{C} \) denote the Yoneda embedding and write \( y(C) \) for its image. For any \( F \in \hat{C} \), let \( y(C)/F \) denote the category of objects over \( F \). We have a forgetful functor
\[ \phi : y(C)/F \rightarrow y(C) \hookrightarrow \hat{C}, \quad (y(T) \rightarrow F) \mapsto y(T) \]
whose image consists of representable presheaves. The reader should verify immediately that:
\[ F = \text{colim}(\phi), \]
thus we see that every presheaf on \( C \) can be expressed canonically as a colimit of representable presheaves. For this reason \( \hat{C} \) is referred to as a co-completion of \( C \). Furthermore, if \( C \) is a subcanonical site, then every presheaf (and in particular every sheaf) is a colimit of representable sheaves.

So let us adapt the above construction to prestacks. For any category \( C \), let \( \hat{C}^2 \) denote the 2-category of 2-functors \( C^{op} \rightarrow \text{Grpd} \), that is, the prestacks on \( C \). Then we have a Yoneda functor
\[ C \xrightarrow{\hat{y}^2} \hat{C}^2, \quad T \mapsto \text{HOM}(-, T). \]

There is a Yoneda lemma says that for any \( X \in \hat{C}^2 \) and \( T \in C \),
\[ \text{HOM}(y^2(T), X) \simeq X(T). \]
Let \( y^2(C) \) denote the image of the 2-Yoneda functor, and let \( y^2(C)/X \) be the category of objects over \( X \). There is again a forgetful functor
\[ \phi : y^2(C)/X \rightarrow y^2(C) \hookrightarrow \hat{C}^2 \]
whose image consists of representable prestacks. Finally, we claim that
\[ X \simeq 2 - \text{colim}(\phi). \]
To see that this is true, first note that, tautologically, \( X \) is the recipient of a coherent system of arrows with indexing category \( y^2(C)/X \), thus to show that \( X \) is the 2-colimit of \( \phi \), it suffices to check that any other such recipient, \( Y \), will also be a recipient of an arrow \( X \rightarrow Y \) making all possible composite diagrams commute. Well, an arrow \( X \rightarrow Y \) is given by the data of an arrow \( X(T) \rightarrow Y(T) \) for each \( T \in C \), but a coherent system of arrows to \( Y \) indexed by \( y^2(C)/X \) is precisely a map (natural in \( T \))
\[ \text{HOM}(y^2(T), X) \rightarrow \text{HOM}(y^2(T), Y). \]
(The left side is viewed as the indexing category restricted to \( T \), while the right denotes the arrows from \( \phi(y^2(T) \rightarrow X) \equiv y^2(T) \rightarrow Y \). Applying the Yoneda lemma gives the desired result.

Note that this actually expresses every pre-stack as a colimit of spaces. This is kind of surprising to me; I would have expected that every stack is a co-limit of functors of the form \( \text{HOM}(-, G) \) for groupoid objects in \( C \). It seems reasonable to assume that there exists an \( n \)-co-completion of a category for each \( n = 1, 2, \ldots, \infty \), but I don’t have an application of this result in mind.
Lecture 6: Topological T-duality

Concepts: The groupoid approach, the Mathai-Rosenberg approach, the Bunke-Schick approach

Lecture 7: T-duality in physics

Concepts: Free bosonic string theory, D-branes

Lecture 8: Geometric aspects of T-duality

Concepts: K-theory and differential K-theory isomorphisms, twisted deRham isomorphisms, GC-geometric equivalences

Lecture 9: Derived geometric aspects of T-duality

Concepts: Derived categories, DG-categories, Mukai-Transform.

Lecture 10: Towards Mirror symmetry

Concepts: Is homological mirror symmetry a form of T-duality?
Here is an annotated list of references. You can find some more references in my papers which are on my website. Sometimes only the author is given, with no specific article. I will fix this at some point, but it is usually easy to just look on the arxiv and find the article.

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